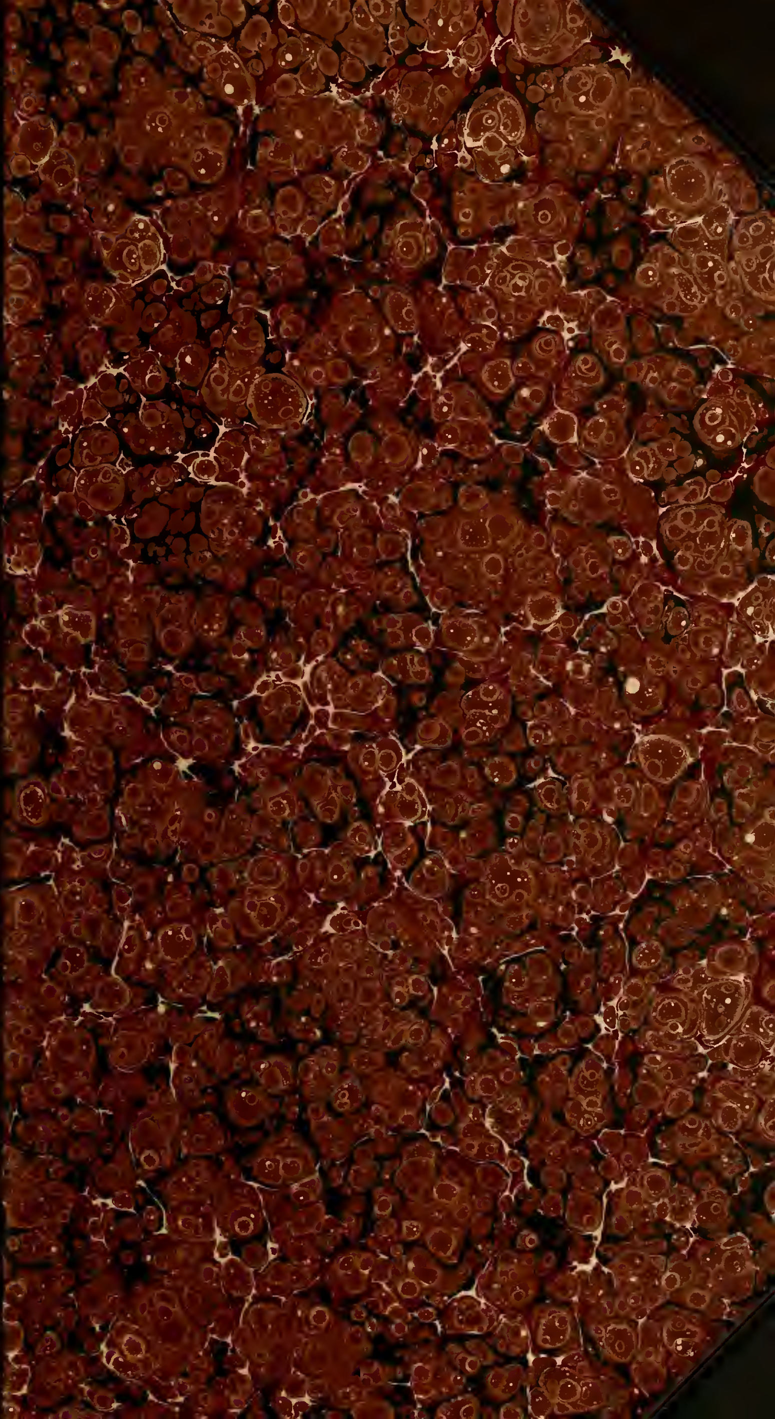
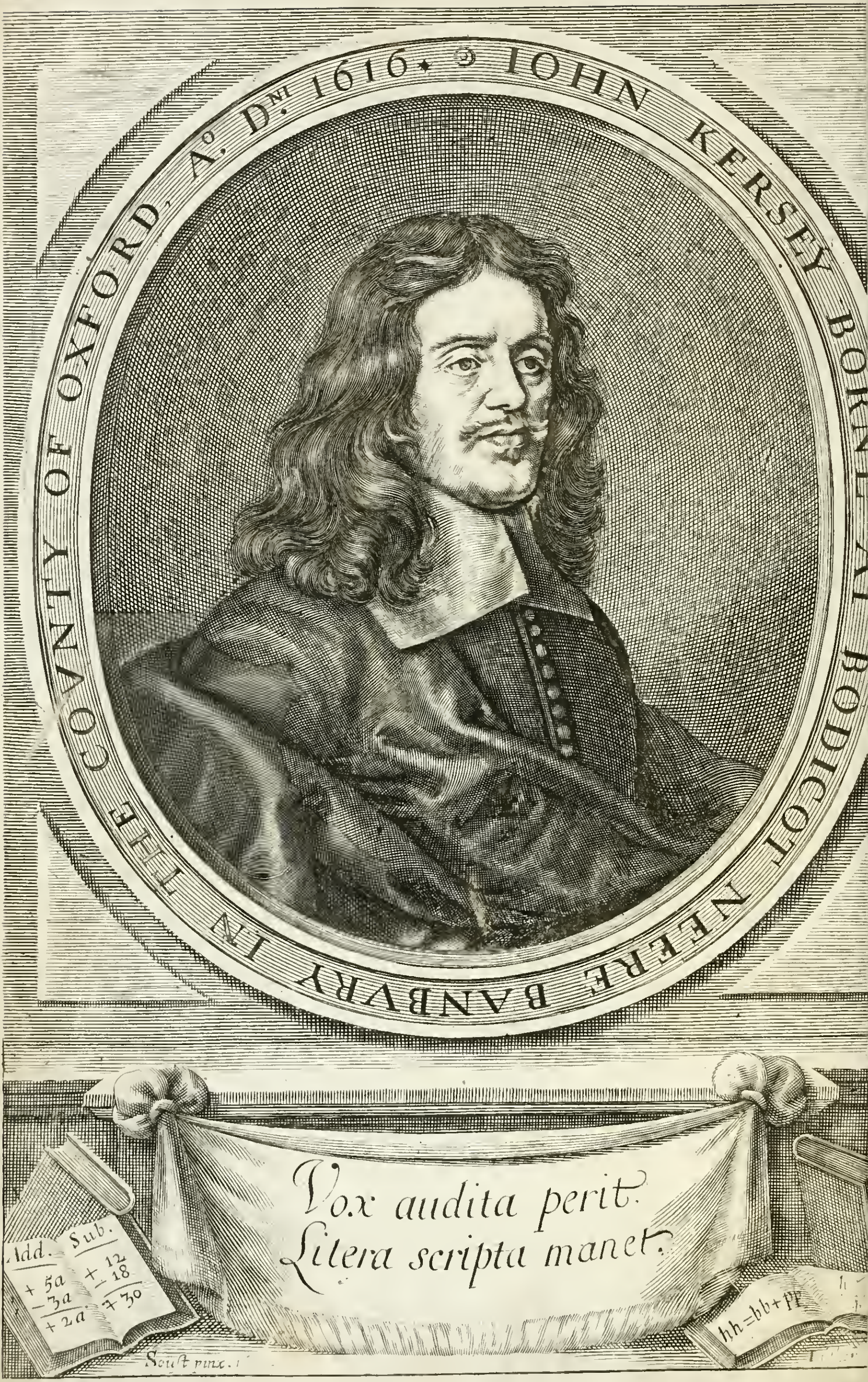


D. S. a. 1. 33.

Digitized by the Internet Archive
in 2009

http://www.archive.org/details/elementsofthatma00kers


IOHN KERSEY BORNE AT BODICOT NEERE BANBVRY IN THE COVNTY OF OXFORD, Aº. Dni. 1616.
Vox audita perit.
Litera scripta manet.
Add.
Sub.
+ 5a
− 3a
+ 2a
+ 12
− 18
+ 30
hh=bb+pp
Soust pinx.

THE
ELEMENTS
OF THAT
Mathematical Art
Commonly Called
ALGEBRA.

By *JOHN KERSEY.*

LONDON,
Printed in the Year, M.DCC.X.

EX BIBL. UNIV. EDINBURGEN.

A
TREATISE
OF THE
ELEMENTS
OF THE
Algebraical ART.

BOOK I.

CHAP. I.

Concerning the Nature, Scope, and Kinds of ALGEBRA: *The Conſtruction of* Coſſic *Quantities, or* Powers; *with the manner of expreſſing them by Alphabetical Letters*: *The ſignification of* Characters *uſed in the Firſt Book.*

I. THE Mathematical Arts or Sciences are exercis'd about *Quantity*, which is compris'd under Numbers, Lines, Superficies, and Solids: Theſe if they be conſidered abſtractively, and ſeparate from all kind of Matter, are the proper Objects of *Arithmetic* and *Geometry*, which are called *Pure Mathematics.*

II. The *Method* which Mathematicians are wont to uſe in ſearching out Truth about Quantity, is twofold; *viz.* 1. *Synthetical*, or by way of Compoſition: 2. *Analytical*, or by way of Reſolution.

III. Mathematical Compoſition, or the *Synthetical Method*, argues altogether with known Quantities to ſearch out unknown; and then demonſtrates that the Quantity found out will ſatisfie the Propoſition.

IV. Mathematical Reſolution, or the *Analytical Art*, commonly call'd *Algebra*, is a way of reaſoning which aſſumes or takes the Quantity ſought as if it were known or granted; and then with the help of one or more Quantities given or known, proceeds by Conſequences, until at length the Quantity firſt only aſſumed or feigned to be known, is found equal to ſome Quantity or Quantities certainly known, and is therefore likewiſe known.

V. The Scope, Drift or Office of the Analytic or Algebraic Art, is to ſearch out three kinds of Truths; *viz.*

1. *Theorems*; which are nothing elſe but Declarations, or Affirmations of certain Properties, Proportions, or Equalities, juſtly inferr'd from ſome Suppoſitions or Conceſſions about Quantity: Which Theorems are to be reſerved in ſtore, as ready helps to find out new, and to confirm old Truths. This kind of Reſolution when it reſts in a bare Invention of Truth, is called *Contemplative*, or *Notional.*

2. *Canons*, or infallible Rules, to direct how to ſolve knotty Queſtions, by the help of Quantities given or known; this kind of Reſolution is called *Problematical.*

3. *Demonſtrations*, or evident and indubitable Proofs, to manifeſt the Truth of ſuch Theorems and Canons as are Analytically found out.

VI. *Algebra* is by late Writers divided into two kinds; to wit, *Numeral* and *Literal* (or *Specious.*)

VII. *Numeral Algebra* is ſo called, becauſe in this Method of reſolving a Queſtion, the Quantity ſought or unknown is ſolely deſign'd or repreſented by ſome Alphabetical Letter, or other Character taken at Pleaſure, but all the Quantities given are expreſt by Numbers.

VIII. *Literal*, or *Specious Algebra* is ſo called, becauſe in this Method of reſolving a Queſtion, as well the given or known Quantities, as the unknown are all ſeverally expreſſed or repreſented by Alphabetical Letters. Whence it comes to paſs, that at the end of the Reſolution of a Queſtion, every Quantity appearing diſtinct under the ſame Letter or Form by which it was at firſt expreſſed, a *Canon* is diſcovered to direct how the Queſtion propos'd may be ſolved, not only by the quantities firſt given, but by any other whatſoever that are capable of ſolving the Queſtion. In this Reſpect therefore *Literal Algebra* far excels the *Numeral*; for this latter ſerves only to ſolve *Arithmetical Queſtions*, and produces not a *Canon* without much difficulty, in regard the Numbers firſt given, by reiterated Multiplications, Diviſions, and other Arithmetical Operations, will for the moſt part be ſo confounded and interwoven, that their Foot-ſteps can hardly be traced out: But *Literal* or *Specious Algebra* is applicable to the ſolving of *Geometrical Problems*, as well as *Arithmetical.*

IX The *Doctrine* of *Algebra* is principally grounded upon the Knowlege of certain Quantities called by ſome Authors *Coſſic Quantities*, by others, *Powers*; the Conſtruction whereof is explain'd in ſix Sections next following.

X. Numbers are ſaid to be *in Geometrical Proportion continued*, when as the firſt is to the ſecond, ſo is the ſecond to the third, and ſo is the third to the fourth, *&c.* As, for Example, theſe Numbers, 1, 2, 4, 8, 16, 32, *&c.* are Continual Proportionals; for, as the firſt Term 1, is the half of the ſecond Term 2; ſo is the ſecond Term 2, the half of the third Term 4; and ſo is 4 the half of 8, *&c.* Likewiſe theſe Numbers, 3, 9, 27, 81, 243, *&c.* are in Geometrical Proportion continued; For as the firſt Term 3 is a third part of the ſecond Term 9, ſo is the ſecond Term 9 a third part of the third Term 27; and ſo is 27 one third of 81, *&c.* Alſo, theſe numbers are continual Proportionals, to wit, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, *&c.* for as the firſt Term 1, is the double of the ſecond Term $\frac{1}{2}$, ſo is $\frac{1}{2}$ the double of $\frac{1}{4}$, and $\frac{1}{4}$ the double of $\frac{1}{8}$, *&c.*

XI. In any ſeries or rank of Numbers proceeding from Unity in a continued Geometrical proportion, whether aſcending or deſcending, all the Numbers or Terms except the firſt, which is ſuppoſed to be 1, (to wit, Unity,) are called *Coſſic Numbers*, or *Powers*; *viz.* the ſecond Term or Proportional is called the *Root*, or firſt Power; the third Proportional is called the *Square*, or ſecond Power; the fourth Proportional is called the *Cube*, or third Power; the fifth Proportional is called the *Biquadrate*, or fourth Power, the ſixth Proportional, the fifth Power, *&c.* As for Example, in this rank of Continual Poportionals, 1, 2, 4, 8, 16, 32, *&c.* the ſecond Term 2 is the Root; the third Term 4 is the ſecond Power, or the Square of the Root 2; the fourth Term 8 is the third Power, or the Cube of the Root 2; the fifth Term 16 is the Biquadrate or fourth Power of the ſame Root 2, *&c.*

In like manner in this rank of continual Proportionals deſcending from 1, to wit, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, *&c.* the ſecond Term $\frac{1}{2}$ is the Root; the third Term $\frac{1}{4}$ is the ſecond Power; the fourth Term $\frac{1}{8}$ is the third Power, *&c.* The like is to be underſtood of any other Rank of Numbers in a continued Geometrical Proportion, whoſe firſt Term or Proportional is Unity.

XII. From the two laſt preceding Sections, (which are grounded upon 10. Prop. 8. Elem. *Euclid.*) it is evident that any Number whatſoever being propoſed for a *Root*, the ſecond Power, or the *Square*, is produced by the Multiplication of the Root by it ſelf; the third Power, or the *Cube*, is produced by the Multiplication of the ſecond Power by the Root; the fourth Power is produced by the Multiplication of the third Power by the Root, *&c.*

As, for Example, if 2 be given for the *Root*, this 2 multiplied by it ſelf, produces 4 for the ſecond Power, to wit, the Square of the Root 2: Again, 4 the ſecond Power being multiplied by the Root 2 gives 8 the third Power, or the Cube; which third Power multiplied by the Root 2, produces the fourth Power 16, &c.

In like manner, if this Fraction $\frac{2}{3}$ be preſcribed for a Root, by multiplying $\frac{2}{3}$ by it ſelf, there comes forth $\frac{4}{9}$ for the ſecond Power, or the Square of the Root $\frac{2}{3}$; Again, the ſecond Power $\frac{4}{9}$ multiplied by the Root $\frac{2}{3}$ produces the third Power $\frac{8}{27}$, or the Cube of the Root $\frac{2}{3}$; and the third Power $\frac{8}{27}$ multiplied by the Root $\frac{2}{3}$ gives the fourth Power $\frac{16}{81}$, &c.

But when the *Root* is 1, to wit, Unity, every one of its Powers will alſo be 1; for multiplication by 1 makes no alteration. All which will be further illuſtrated by the Scales of Coſſic numbers or Powers in the following Table, which ſhews that if the Root be 5, the Square is 25, the Cube 125, the Biquadrate or fourth Power 625, the fifth Power 3125, &c.

A Table of Powers *in Numbers.*

The *Root* or firſt Power.	1	2	3	4	5
The *Square* or ſecond Power.	1	4	9	16	25
The *Cube* or third Power.	1	8	27	64	125
The *Biquadrate* or fourth Power.	1	16	81	256	625
The fifth Power.	1	32	243	1024	3125
The ſixth Power.	1	64	729	4096	15625
The ſeventh Power.	1	128	2187	16384	78125
The eighth Power, &c.	1	256	6561	65536	390625

XIII. The *Root* or firſt Power being given, the third, fifth, eighth, or any other Power may be found out without reſpect to the intermediate Power or Powers, in this manner; *viz.* Suppoſe the number 3 be preſcribed for the Root, and that the fifth Power be deſired; firſt write down the Root 3 five times thus, 3, 3, 3, 3, 3; then multiply theſe five equal numbers one into another according to the Rule of continual Multiplication, ſo the laſt Product 243 ſhall be the deſired fifth Power raiſed from the Root 3.

In like manner, if the eighth Power of the Root 2 be deſired, you may write the Root 2 eight times thus, 2, 2, 2, 2, 2, 2, 2, 2, theſe multiplied continually produce 256, which is the eighth Power of the Root 2. After the ſame manner you may find out any other Power from a number given for the Root.

XIV. If over or under any Series or Rank of Coſſic numbers or Algebraic Powers, conſtituted according to the three laſt foregoing Sections, there be placed a rank of Numbers beginning with Unity, and proceeding according to the natural order of numbers, as 1, 2, 3, 4, 5, 6, 7, 8, 9, &c. theſe numbers ſo placed are uſually called the *Indices*, or *Exponents* of thoſe Powers, as well becauſe they ſhew the order, ſeat, or place of each Power, as alſo its number of Degrees or Dimenſions; that is, how many times the Root is involved or multiplied in producing each Power reſpectively: As for Example, let there be a Rank or Scale of Algebraic powers raiſed from the root 3, as 3, 9, 27, 81, 243, 729, 2187, &c. and over them let there be ſo many numbers placed in an Arithmetical progreſſion, beginning with 1, and proceeding according to the natural order of Numbers, as here you ſee:

INDICES.	1	2	3	4	5	6	7	8	&c.
POWERS.	3	9	27	81	243	729	2187	6561	&c.

I ſay the Index 4 in the Arithmetical progreſſion, ſhews that the fourth Power 81, which ſtands under 4, is produced by the multiplication of the Root 3 four times into it ſelf, *viz.* theſe four numbers 3, 3, 3, 3, multiplied continually will produce 81; likewiſe the Index 7 in the Arithmetical progreſſion ſhews, that the ſeventh Power 2187, which ſtands under 7, is produced by the multiplication of the Root 3 ſeven times into it ſelf; *viz.* theſe ſeven equal numbers 3, 3, 3, 3, 3, 3, 3, multiplied continually produce 2187. And ſo of others.

To that uſe of *Indices*, this may be added; *viz.* If any two or more Indices be added together, the ſum will be an Index ſhewing what power will be produced by the multiplication of thoſe Powers one into another which anſwer to the Indices that were added together: As for Example, if the Indices 3 and 5 be added together, the ſum is the Index 8; which ſhews, that if the third and fifth Powers be multiplied one by the other, the eighth Power will be produced: As in the rank of Powers in the preceding Tabulet, if the third power 27 be multiplied by the fifth Power 243, the Product will give the eighth Power 6561. In like manner, for as much as the Indices 2 and 6 added together make the Index 8; therefore the ſecond Power 9 multiplied by the ſixth Power 729 will alſo produce the eighth Power 6561: Again becauſe the Indices 1, 2, and 5 added together make the Index 8; therefore the firſt, ſecond and fifth Powers, to wit, 3, 9, and 243 multiplied continually will likewiſe produce the eighth Power 6561. And as the Index 3 added to it ſelf makes the Index 6, ſo the third Power 27 multiplyed by it ſelf, or ſquared, will produce the ſixth Power 729.

And as the Addition of Indices anſwers to the Multiplication of their correſpondent Powers, ſo the ſubtraction of Indices anſwers to the diviſion of their correſpondent Powers: As, for Example, becauſe the Index 8 leſſened by the Index 5, leaves for a Remainder the Index 3; therefore the eighth Power 6561 divided by the fifth Power 243 gives in the Quotient the third Power 27. Likewiſe, as the Index 7 leſſened by the Index 3 leaves the Index 4; ſo the ſeventh Power 2187 divided by the third Power 27, gives the fourth Power 81.

XV. From the premiſſes it is evident, that upon an Arithmetical foundation, a Scale or Rank of Algebraic Powers may be raiſed and continued as far as you pleaſe; the three firſt of which have an affinity with, and may be expounded by Geometrical dimenſions: For firſt, we may conceive any terminated Right-line, to be divided into a number of equal parts at pleaſure, ſuppoſe 12; then this number 12, or that Right-line, may be eſteemed as a Root: Secondly, the ſaid 12 multiplied by it ſelf produces 144 the ſecond Power, which is equal to the Area of a ſquare Superficies whoſe ſide is 12: Thirdly, the ſaid ſecond Power 144 multiplied by the Root 12 produces the third Power 1728, which is equal to the Solid content of a Cube, (to wit, a Solid in the form of a Dye) whoſe ſide is 12.

But none of the reſt of the Algebraic powers can properly be explain'd by any Geometrical quantity, in regard there are but three dimenſions in Geometry, to wit, Length, Breadth, and Depth (or Thickneſs.)

XVI. In ſearching out the ſolution of a Queſtion by the Algebraic Art, the number or line ſought is uſually called a *Root*, which ſo long as it remains unknown cannot be really expreſt, and therefore it muſt be deſign'd or repreſented by ſome Symbol or Character, at the will of the Artiſt; alſo the Powers which may be imagined to proceed from the ſaid Root in ſuch manner as has before bin declared are likewiſe to be repreſented by Symbols or Characters; concerning which there is much diverſity among *Algebraical* Writers, every one pleaſing his fancy in the choice of Characters: But in this matter I ſhall imitate Mr. *Thomas Harriot* in his *Ars Analytica*, and *Renates des Cartes* in his *Geometry*, but chiefly the former; whoſe method of expreſſing Quantities by Alphabetical Letters, I conceive to be the plaineſt for Learners, *viz.*

To deſign or repreſent the Root ſought, whether it be a number or a Line in a Queſtion propoſed, we may aſſume any Letter of the Alphabet, as *a*, *b*, or *c*, &c. but for the better diſtinguiſhing of known quantities from unknown, ſome *Analyſts* are wont to aſſume one of the five Vowels, as, *a*, or *e*, &c. to repreſent the quantity ſought; and Conſonants, as, *b*, *c*, *d*, &c. to repreſent quantities known or given: Now if the letter *a* be aſſumed to repreſent the Root ſought, then (according to Mr. *Harriot*) the ſecond Power, or the Square raiſed from that Root, may be repreſented by *aa*; the third Power, or the Cube, by *aaa*; the fourth Power by *aaaa*; the fifth Power by *aaaaa*; and after the ſame manner any higher Power of the Root or number a may be repreſented: For ſo many Dimenſions or Degrees as are in the Power, ſo many times the Letter which at firſt was aſſumed for the Root is to be repeated.

Or after the manner of *Renates des Cartes*, if the letter a be aſſumed to repreſent the Root, the Square may be deſigned thus, a^2. the Cube thus, a^3. the fourth Power thus, a^4. the fifth Power thus, a^5. And ſo any other power may be expreſt by writing the Index or Exponent of the Power in a ſmall figure next after, and near the head of the letter aſſumed to repreſent the Root. Both which ways will be further illuſtrated by the following Table.

A Table ſhewing two ways (now moſt in uſe) to expreſs ſimple Powers *by Alphabetical Letters.*

The *Root* or firſt Power,	a.	a
The *Square* or ſecond Power,	aa.	a^2
The *Cube* or third Power,	aaa.	a^3
The fourth Power,	$aaaa$.	a^4
The fifth Power,	$aaaaa$.	a^5
The ſixth Power,	$aaaaaa$.	a^6
The ſeventh Power,	$aaaaaaa$.	a^7
The eighth Power,	$aaaaaaaa$.	a^8

After the ſame manner, known Quantities and their Powers may be repreſented by Conſonants; as, b may be put for any known number in a Queſtion, and then its Square may be ſignified by bb, the Cube by bbb, the fourth Power by $bbbb$, the fifth Power by $bbbbb$, the ſixth by $bbbbbb$, and ſo forwards: Or the Square of the Root b may be expreſt thus, b^2. the Cube thus, b^3. the fourth Power thus, b^4. the fifth Power thus, b^5. the ſixth Power thus, b^6. and ſo forward.

XVII. Numbers ſet before, that is, on the left hand of quantities expreſt by letters are called Numbers prefixt; but if no number be prefixt to the letter, then 1 or unity muſt be imagined to be prefixt: As, in theſe quantities a, (or $1a$,) $2a$, $3a$, $\frac{1}{2}a$, $\frac{2}{3}a$, $5bbb$ (or $5b^3$) the numbers prefixt are (as you ſee) 1, 2, 3, $\frac{1}{2}$, $\frac{2}{3}$, and 5, every one of which numbers (and the like ſo prefixt) ſhews how often the quantity repreſented by the letter or letters immediately following the number is taken; ſo a or $1a$ ſignifies ſome number or line once taken, alſo $2a$ repreſents the double, $\frac{1}{2}a$ the half, and $\frac{2}{3}a$ two third parts of the number or line repreſented by a. In like manner $5bbb$, or $5b^3$, ſignifies that the Cube of the number or line repreſented by b is taken five times.

XVIII. All numbers expreſt by figures and cyphers (as in vulgar Arithmetic) not having any letter or letters annexed to them, are for diſtinction ſake called Abſolute numbers; as theſe numbers, 5, 20, 105, $\frac{1}{2}$, $\frac{2}{3}$, and all others when they be not prefixt or annext to any letter or letters are called abſolute numbers.

XIX. All *Algebraical Operations* are perform'd in an Arithmetical manner, partly in the vulgar way by numbers, and partly by Alphabetical letters in all the parts of Arithmetic, to wit, Addition, Subtraction, Multiplication, Diviſion, and the Extraction of Roots: But ſince letters cannot be diſpoſed like numbers to perform thoſe operations, ſome Characters muſt of neceſſity be uſed to ſignifie ſuch operations. The Characters uſed in this firſt Book are explained in the following Sections.

XX. This Character + is a ſign of *Affirmation*, as alſo of *Addition*, and always belongs to the quantity that follows the ſign; as, $+a$ affirms the quantity denoted by a to be real, or greater than nothing; the like may be ſaid of $+b$, and $+2c$, &c.

When no ſign is prefixt before a quantity, the ſign + is always to be underſtood, and muſt be imagined to be prefixt; ſo a implies $+a$, likewiſe $2b$ ſignifies the ſame thing with $+2b$; the like of others.

But when the ſign + is placed between two quantities, it imports as much as the word *plus*, or *more*, and ſignifies that thoſe quantities are added or to be added together: As $3+4$ (or 3 more 4) ſignifies the ſum of 3 and 4; or it hints that 4 is to be added to 3. In like manner $a+b$ ſignifies the ſum of numbers or quantities repreſented by a and b; and $a+b+c$ ſignifies the ſum of quantities denoted by a, b, and c.

XXI. This Character — is a ſign of *Negation*, as alſo of *Subtraction*, and always belongs to the following quantity; as for Example, — 5 is a fictitious number leſs than nothing by 5; *viz.* as + 5 *l.* may repreſent five pounds in money, or the Eſtate of ſome perſon who is clearly worth five pounds; ſo — 5 *l.* may repreſent a Debt of five pounds owing by ſome perſon who is worſe than nothing by five pounds.

But when the ſign — is placed between two quantities, it imports as much as the word *minus*, or *leſs*; and intimates that the number or quantity following that ſign is ſubtracted or to be ſubtracted from the number or quantity that ſtands next before the ſame ſign: As 8—3 (or 8 leſs 3) ſignifies that 3 is ſubtracted or to be ſubtracted from 8; or 8—3 denotes the exceſs of 8 above 3, to wit, 5.

In like manner $a-b$ (or a leſs b) ſignifies that the quantity denoted by b is ſubtracted or to be ſubtracted from the quantity a; or $a-b$ may ſignifie the exceſs of the quantity a above the quantity b.

XXII. This Character ∽ ſignifies the *Difference* of two quantities, to wit, the exceſs of the greater above the leſs, when 'tis not determin'd or known in which of thoſe quantities the exceſs lyes; ſo $a \backsim b$ ſignifies the difference of two quantities repreſented by a and b when 'tis not known whether a be greater or leſs than b.

XXIII. This Character × is a ſign of *Multiplication*, and is put for the word *into*, or *by*; *viz.* when 'tis ſet between two quantities it ſignifies that they are multiplied, or to be multiplied mutually one by the other: As, 6×3 (or 6 into or by 3) imports the Product of the multiplication of 6 by 3, to wit, 18.

In like manner $a\times b$ ſignifies that the quantity repreſented by a is multiplied or to be multiplied by the quantity b: alſo $a\times b\times c$ ſignifies the Product made by the continual multiplication of the quantities a, b, and c, one into another.

But for the moſt part the Multiplication of quantities denoted by letters is ſignified by the joyning of letters together, like letters in a word; as ab ſignifies the Product of the multiplication of the quantity a by the quantity b. Alſo abc ſignifies the Product of the continual multiplication of the quantities a, b and c one into another: All which will be further illuſtrated in Chap. 4.

XXIV. Quantities deſign'd or repreſented by letters are either Simple or Compound.

XXV. A Simple quantity is deſigned or expreſſed either by a ſingle letter or by two or more letters joyned together like letters in a word: As a (or $+a$) is a ſimple quantity; likewiſe $2aa$, $3abc$, and $dddd$ are ſimple quantities.

XXVI. A Compound quantity conſiſts of two or more ſimple quantities connected or joyned one to another by + or —; ſo $a+b$ is a compound quantity, likewiſe $a-c$, alſo $a+b+c$, and $a+b-c$ are compound quantities.

XXVII. Every one of theſe four Characters, to wit, +, —, ∽, and ×, (before defined in Sect. 20, 21, 22, and 23.) may ſometimes have reference to ſuch a Compound quantity as follows the ſign, and has a line drawn over every member of it. As, for Example, by $a+\overline{b \backsim c}$, you are to underſtand that the difference of the quantities b and c (whether the Exceſs be in b or in c) is added or to be added to the quantity a.

In like manner, $a-\overline{b+c}$ ſhews that the Compound quantity $b+c$ is ſubtracted or to be ſubtracted from the quantity a; where in regard of the line drawn over $b+c$, the ſign— hath reference to the ſubtraction of c as well as b from the quantity a. But if that line were omitted, then the ſign — would only refer to the next following ſimple quantity: As, $a-b+c$, (or $a+c-b$) ſignifies the ſubtraction of b only from $a+c$.

Moreover, $a \backsim \overline{b+c}$ ſignifies the difference between the ſimple quantity a, and the compound quantity $b+c$.

And $a\times\overline{b-c}$ ſignifies that the quantity a is multiplied or to be multiplied by the exceſs of the quantity b above the quantity c.

XXVIII. This Character √ is called a radical ſign, and ſignifies that the Square root of the number or quantity that ſtands next after the ſaid ſign √, is extracted, or to be extracted; as √25 ſignifies the ſquare root of 25, to wit, 5; and √36 ſignifies the ſquare root of 36, to wit, 6.

Likewiſe $\surd ab$ ſignifies the ſquare root of the quantity *ab*. So that when a number or quantity immediately follows the ſaid radical ſign $\surd$, the ſquare root of that number or quantity is thereby denoted.

But to deſign or repreſent the Root of a Power higher than a Square, ſome *Algebraical* Writers (whom in this matter I ſhall follow) are wont to write the Index of the Power within a Circle next after the ſign $\surd$; As for Example, $\surd(3)27$ ſignifies the Cubic root of 27, to wit, 3. Likewiſe, $\surd(4)16$ denotes the Biquadrate root of 16, to wit, 2; that is, the root from whence 16 conſidered as the fourth Power is produced. Again, $\surd(5)243$ ſignifies the root from whence 243 conſider'd as the fifth Power is raiſed, which Root is 3. And if you pleaſe you may write $\surd(2)81$ to denote the ſquare root of 81, to wit, 9.

Likewiſe $\surd(3)a$ ſignifies the Cubic root of ſome number or quantity repreſented by *a*. Alſo $\surd(4)bc$ ſignifies the Biquadrate root of the Quantity *bc*.

Sometimes the Radical Sign belongs to as many of the following Quantities as have a Line drawn over them; as $\surd:\overline{b+c}:$ or, $\surd(2):\overline{b+c}:$ ſignifies the Square root of the ſum of the Quantities *b* and *c*. Likewiſe $\surd:\overline{bb-c}:$ imports the Square root of the Remainder when the quantity *c* is ſubtracted from the Square of the quantity *b*. Which Roots, and ſuch like, are called *Univerſal Roots*.

Again, $d+\surd:\overline{bb-c}:$ ſignifies that the Quantity *c* is firſt to be ſubtracted from the Square *bb*, and then the Square root of the Remainder is to be added to the quantity *d*. But that the Learner may the better perceive my meaning in the three laſt Examples concerning Univerſal Roots, let *b* ſignifie 4; *bb*, 16; *c*, 12; and *d*, 23. Then $\surd:\overline{b+c}:$ ſignifies $\surd:\overline{4+12}:$ that is, $\surd 16$, to wit, 4. Alſo $\surd:\overline{bb-c}:$ ſignifies $\surd:\overline{16-12}:$ that is, $\surd 4$, to wit, 2. And $d+\surd:\overline{bb-c}:$ ſignifies 23+2, that is, 25. After the ſame manner the Univerſal Square root of $d+\surd:\overline{bb-c}:$ may be expreſt thus;

$$\surd:\overline{d+\surd bb-c}:\ \text{that is, } 5.$$

XXIX. Four points ſet in this form :: are always in the middle of four Geometrical Proportionals, as, for Example, theſe four Numbers 2 . 4 :: 6 . 12 are Geometrical Proportionals, and to be read thus; As 2 is to 4, ſo is 6 to 12; or, (in the Phraſe of *The Rule of Three*) If 2 give 4, then 6 will give 12.

In like manner theſe four Quantities, $b.d::c.a$ are to be read thus; As *b* is to *d*, ſo *c* to *a*, that is, look what proportion *b* has to *d*, the ſame proportion has *c* to *a*.

Alſo theſe four Quantities, $b+c.d-a::f.g$ do intimate that the ſum of *b* and *c* has ſuch proportion to the Exceſs of *d* above *a*, as *f* has to *g*. The like is be underſtood of others.

XXX. This Character $\div$ ſet at the end of three or more Quantities, imports that they are Continual Proportionals Geometrical; ſo by $2.4.8.16.32 \div$ it is ſignified that ſuch proportion as 2 has to 4, the ſame has 4 to 8, 8 to 16, and 16 to 32.

Likewiſe by theſe $a.b.c\div$ you are to underſtand that the quantity *a* has the ſame proportion to the quantity *b*, as *b* to *c*.

XXXI. This Character = is the ſign of an Equation or Equality, and imports as much as the Word *Equal*; as $8+4=7+5$ ſignifies that the ſum of 8 and 4 is equal to the ſum of 7 and 5. Likewiſe $8=12-4$ that 8 is equal to 12 leſs 4, to wit, the exceſs of 12 above 4.

Again, $8\times 3=4\times 6$ denotes the Product of 8 multiplied by 3 to be equal to the Product of 4 into 6.

So alſo $a+b=c+d$ ſignifies that the ſum of the quantities *a* and *b* is equal to the ſum of the quantities *c* and *d*. This will be farther explained in the XI. Chapter.

XXXII. This Character $\sqsubset$ ſtands for the Word *Greater*, *viz.* it ſignifies that the Quantity which ſtands before, that is, on the left hand of the ſaid Character is greater than the quantity following the ſame; ſo $5\sqsubset 4$ muſt be read thus, 5 is greater than 4. Likewiſe $a+b\sqsubset c$ ſignifies that the Compound quantity $a+b$ is greater than the Simple quantity *c*. And $d\sqsubset a+c$ ſignifies that the quantity *d* is greater than $a+c$.

XXXIII. This Character $\sqsupset$ ſignifies that the quantity ſtanding before the Character is leſs than the quantity following the ſame; as $4\sqsupset 5$ muſt be read thus, 4 is leſs than 5. Likewiſe, $a+b\sqsupset c+d$ ſignifies that the compound quantity $a+b$ is leſs than the compound quantity $c+d$.

XXXIV. Quantities, whether they be Simple or Compound, which are expreſt either wholly by Letters, or partly by Letters and partly by Numbers written upon one Line, are called Algebraical Integers, or whole Quantities; as theſe, a, ab, $cd+ff$, $a+3$, &c. But theſe quantities, $\frac{b}{c}$, $\frac{aa+bb}{a+c}$, $\frac{a+3}{b}$, and others ſo written, are called Algebraical Fractions, becauſe each of them like a Fraction in vulgar *Arithmetic* conſiſts of a Numerator placed above a Line, and a Denominator underneath.

CHAP. II.

Addition of Algebraical Integers.

I. ALgebraical *Addition* finds out the Sum or Aggregate of two or more Quantities expreſt either wholly by Letters, or partly by Letters and partly by Numbers.

II. The Operations in Algebraic Addition depend principally upon a diligent obſervation of three things, *viz.*

Firſt, You muſt obſerve whether the Quantities to be added be Like or Unlike.

Like Quantities are thoſe which are expreſt by the ſame Letters equally repeated in every one of the Quantities; ſuch are theſe, a, $5a$, $-2a$, each of which is expreſt by the ſingle letter a. Alſo theſe are like quantities, $3aa$, aa, $-2aa$, each of which is expreſt by a double a, to wit, aa. Likewiſe theſe, $2ab$, $3ab$, $-ab$ are called Like quantities becauſe every one of them is expreſt by the ſame Letters, to wit, ab.

Unlike Quantities are thoſe which are expreſt by different Letters, or elſe by the ſame letters unequally repeated; as, for Example, b and c are unlike quantities, becauſe they are expreſt by different letters; alſo $2abc$ and $2ab$ are unlike quantities, becauſe the letter c is in the one, but not in the other. Again, a and aa are unlike quantities, in regard the letter a is not equally repeated in both. The like is to be underſtood of others.

Secondly, You muſt obſerve whether the Signs (to wit, + and —) belonging to like quantities given to be added be Like or Unlike: As, for Example, theſe quantities $+2a$ and $+3a$ have like ſigns, the ſame ſign + being prefixt before each quantity. Alſo theſe quantities, $-2a$ and $-3a$ have like ſigns, the ſame ſign — being prefixt to each quantity; but theſe quantities $+2a$ and $-3a$ have unlike or different ſigns prefixt.

Thirdly, The Numbers prefixed before the Letters muſt be diligently obſerved, for their ſum or difference will be concern'd in Algebraical Addition, as will be manifeſt by the following Rules.

III. When two or more ſimple Algebraical Integers (or whole quantities) propos'd to be added or collected into one Sum are like, and have like ſigns, Firſt collect the numbers prefixt into one Sum; then to that Sum annex the letter or letters by which any one of the quantities propos'd is expreſt; laſtly, prefix the given ſign whether it be + or —, ſo ſhall this new quantity be the Sum deſired. As, for Example, if it be deſired to add a to a, or $+1a$ to $+1a$, the Sum will be $2a$ or $+2a$; for (according to the Rule) the Sum of the prefixed Numbers 1 and 1 is 2, to which I annex a and prefix + (or imagine it to be prefixed,) ſo $2a$ or $+2a$ is the Sum deſired.

$$\begin{array}{lr|r} \text{Add} & \left\{\begin{matrix} a \\ a \end{matrix}\right. & \begin{matrix} +1a \\ +1a \end{matrix} \\ \text{Sum} & 2a & +2a \end{array}$$

In like manner, if to $-2b$ you would add $-b$, the Sum will be $-3b$. For the numbers prefixt are 2 and 1, which added together make 3, to which annexing b, and prefixing the given ſign —, there ariſes $-3b$, the Sum deſired.

$$\begin{array}{ll} \text{Add} & \left\{\begin{matrix} -2b \\ -\ b \end{matrix}\right. \\ \hline \text{Sum} & -3b \end{array}$$

More Examples of the Rule of Addition in the foregoing Sect. III.

To be added,	$5a$ $3a$	$-5aa$ $-2aa$	$+7ab$ $+13ab$
The Sum,	$8a$	$-7aa$	$+20ab$

To be added,	ac $2ac$ $3ac$	$-3bcd$ $-bcd$ $-6bcd$	$+3a^3$ $+2a^3$ $+7a^3$
The Sum,	$6ac$	$-10bcd$	$+12a^3$

IV. When two ſimple Quantities propos'd to be added together be like, and have equal Numbers prefix'd, but unlike or contrary Signs, the Sum will be 0, or nothing; for the affirmative Quantity will deſtroy or extinguiſh the Negative: As for Example, if it be required to add c, or $+c$, to $-c$, the Sum will be 0, to wit, nothing. For ſuppoſing $-c$, or $-1c$ to be a Debt of one Crown that I owe; and $+c$, or $+1c$ to be one Crown in my Purſe, it is evident that one Crown in ready Money will diſcharge or ſtrike off a Debt of one Crown; and ſo that Debt and Credit being added or compared together, the Sum amounts to 0.

Add,	$-c$ $+c$
Sum,	0

In like manner, if it be deſired to add $-6\ l.$ to $+6\ l.$ the Sum will be 0; for if my whole Eſtate be worth but 6 Pounds, and I owe a Debt of 6 Pounds, it is manifeſt that my clear Eſtate is worth or amounts to juſt nothing.

Add,	$+6\ l.$ $-6\ l.$
Sum,	0

More Examples of the Rule of Addition in the preceding Sect. IV.

To be added,	$+3a$ $-3a$	$-5abc$ $+5abc$	$+7ddd$ $-7ddd$
The Sum,	0	0	0

V. When two ſimple Quantities propos'd to be added together be like, but their Signs unlike, and the prefixed Numbers unequal between themſelves; firſt ſubtract the leſſer Number prefixed from the greater, then to the Remainder annex the Letter or Letters by which either of the Quantities propoſed is expreſt; laſtly, before the ſaid Remainder ſet the Sign which ſtands before the greater Number prefix'd, ſo ſhall this new Quantity be the Sum deſired.

As for Example, if it be deſired to add $-2a$ to $+3a$, the Sum will be a. For firſt Subtracting 2 from 3 the Remainder is 1, to which annexing a and prefixing $+$ (becauſe $+$ belongs to that Quantity which has the greater Number prefix'd) there ariſes $+1a$, or $+a$ for the Sum ſought.

Add,	$+3a$ $-2a$
Sum,	$+1a$, or, $+a$

Again, to add $+b$ to $-3b$, I ſubtract 1 the leſſer Number prefix'd, from 3 the greater, and to the Remainder 2 annexing b and prefixing $-$, (becauſe $-$ belongs to $3b$ whoſe prefix'd Number 3 is greater than that of $+b$ or $+1b$) I find $-2b$ for the Sum deſired.

Add,	$-3b$ $+b$
Sum,	$-2b$

Thus you ſee that this laſt Rule of Addition is performed by Subtraction, and may eaſily be underſtood under the Notion of diſcharging or paying off a Debt, or at leaſt part of a Debt by ſo much ready Money or Credit, and then obſerving what Debt remains unpaid, or what Money or Credit remains as an overplus: So in the firſt of the two laſt Examples, you may conceive $+3a$ to be three Pounds in ready Caſh, and $-2a$ to be a Debt of two Pounds; then comparing the ſaid ready Money and Debt together, you will find by Subtraction that the clear Money remaining after the Debt is pay'd, will be one Pound, to wit, $+1a$ or a which is the Sum of the Quantities $+3a$ and $-2a$. Likewiſe in the latter Example, if $-3b$ be conceived to repreſent a Debt of three Pounds, and $+b$ or $+1b$ one Pound in ready Money; 'tis evident that this will ſtrike off one Pound of that Debt, and ſo the Debt remaining will be two Pounds, to wit, $-2b$, which is the Sum of $-3b$ and $+b$.

More Examples of the Rule of Addition in the preceding Sect. V.

To be added,	$+5aa$ $-7aa$	$+6abcd$ $-4abcd$	$-8f^4$ $+3f^4$
The Sum,	$-2aa$	$+2abcd$	$-5f^4$

VI. When three or more ſimple Quantities propos'd to be added be like, but have unlike Signs; Firſt, (by the Rule in *Sect.* III. of this *Chap.*) collect the Affirmative quantities into one Sum, and the Negative quantities into another; then (by *Sect.* IV. or V.) add thoſe two Sums into one, ſo this laſt Sum ſhall be that which is ſought.

As, for Example, If the Sum of theſe four Quantities, $7a$, $2a$, $-3a$, $-5a$ be deſired; Firſt, (by *Sect.* III.) the Sum of $7a$ and $2a$ is $+9a$; alſo the Sum of $-3a$ and $-5a$ is $-8a$; laſtly (by *Sect.* V.) $+9a$ added to $-8a$ makes $+a$, that is, a, which is the Sum deſired.

More Examples of the Rule of Addition in Sect. VI.

To be added,	$+5a$ $+3a$ $-8a$	$-2bc$ $+3ba$ $-4bc$	$+4d^5$ $+3d^5$ $-5d^5$
The Sum,	0	$-3bc$	$+2d^5$

To be added,	$+5ee$ $+2ee$ $-ee$ $-4ee$	$-4fff$ $-3fff$ $-2fff$ $+8fff$	$+4ggbb$ $-3ggbb$ $+2ggbb$ $-ggbb$
The Sum,	$+2ee$	$-fff$	$+2ggbb$

VII. When two or more Simple quantities given to be added be unlike, write them down one after another without altering their Signs; as, if the Number (or Line) a be to be added to the Number (or Line) b; I write $a+b$, or, $b+a$ for the Sum.

In like manner the Sum of theſe Quantities, a, b, c, may be written thus, $a+b+c$; or thus, $a+c+b$; or thus, $b+a+c$.

More Examples of the Rule of Addition in Sect. VII.

To be added,	$+3a$ $+2d$	$+aa$ $-bb$
The Sum,	$3a+2d$	$+aa-bb$

Again,

To be added,	$+ab$ $-ac$ $+ad$	$+5ddd$ $-3dd$ $-4d$
The Sum,	$+ab-ac+ad$	$+5ddd-3dd-4d$

Addition of Compound Algebraical Integers.

VIII. The Addition of Compound whole Quantities may eaſily be diſpatch'd by the help of the Rules in the preceding *Sections* of this *Chapter*, as will appear by the following Examples.

Firſt then, If this Compound quantity $a+b$ be to be added to $a+2b$, their Sum is $a+b+a+2b$, that is $2a+3b$; for $a+a$ makes $2a$; and $+b+2b$ makes $+3b$.

Again, The Sum of theſe two Compound quantities $3b+5a$ and $2b-2a$ is $3b+5a+2b-2a$, that is, $5b+3a$; for $3b+2b$ makes $5b$; and (by *Sect.* V.) $+5a-2a$ makes $+3a$.

Likewiſe, The Sum of theſe two Compound quantities $5ee+3f-8$ and $3ee-2f+6$ will be found $8ee+f-2$: For $5ee$ added to $3ee$ makes $8ee$; alſo $+3f$ added to $-2f$ gives $+f$, and -8 added to $+6$ makes -2.

After the ſame manner, $3a-8$ added to $10-a$ makes $2a+2$; (for $+3a$ added to $-a$ makes $+2a$, and -8 added to $+10$ gives $+2$.)

Again, The Sum of theſe two Compound quantities $a+b$ and $c-d$ is $a+b+c-d$, which Sum admits of no Contraction, in regard all the Simple quantities are unlike.

More Examples of the Addition of Compound whole Quantities.

To be added,	$a+b$ $a-b$	$aa+2a-3$ $aa+a-6$
The Sum,	$2a$	$2aa+3a-9$

To be added,	$aa-2ab$ $aa+ab$	$4c-d+3$ $-4c+2d-2$
The Sum,	$2aa-ab$	$d+1$

To be added,	$2ee+3ef-ff$ $-3ee+5ef$	$a^3-abc+6$ $+3abc-6$
The Sum,	$-ee+8ef-ff$	a^3+2abc

To be added,	$-aaa+2bba$ $8aaa+4bba$ $6aaa-6bba$	$aa-5a+24$ $aa+a-17$ $-2aa+2a+12$
The Sum,	$13aaa$	$-2a+19$

To be added,	$a+b$ $c-d$ $e+f$	$5b^3+24$ $-2b^3+40$ $6b^3-64$
The Sum,	$a+b+c-d+e+f$	$9b^3$, or, $9bbb$

CHAP. III.

Subtraction in Algebraic Integers.

I. A*lgebraical Subtraction* takes one Quantity, whether it be expreſs'd by a Letter or Letters, or partly by Letters and partly by Number, out of, or from anther, in ſuch manner, that if the Remainder be added (according to the Rules of Algebraic Addition) to the Quantity ſubtracted, the Sum will be always equal to the ſaid other Quantity.

II. A general Rule to find out the Remainder in all caſes of Algebraical Subtraction is this: Firſt, joyn both the given Quantities together, by writing one after the other; but with this caution, that every Sign of the Quantity given to be ſubtracted, be ever changed into the contrary Sign, *viz.* + into — and — into +; then ſhall the Sum of both Quantities ſo connected be the Remainder ſought, which is to be contracted (when it may be done) into the feweſt and ſmalleſt Terms, by the Rules of Algebraical Addition.

As for Example, If from $5a$ it be deſired to ſubtract $3a$, firſt, I write down $5a$, then next after the ſame I write $-3a$; (where obſerve, that according to the Rule above given I change +, the Sign belonging to $3a$ the Quantity given to be ſubtracted, into —,) ſo there ariſes $5a-3a$, which being contracted (by the Rule of Addition in *Sect.* V. *Chap.* II.) makes $2a$ the Remainder ſought.

Out of	$5a$
Subtract	$3a$
Remainder,	$5a-3a$
Remainder contracted,	$2a$

Likewiſe, if from $3b$ it be deſired to ſubtract $-2b$, I firſt write down $3b$, and next after the ſame I write $+2b$; ſo $3b+2b$, that is, $5b$ is the Remainder ſought; where obſerve (as before) that I change the Sign —, which belongs to $2b$ the Quantity propos'd, to be taken out of $3b$, into the contrary Sign +. But that the ſaid $5b$ is a true Remainder, we may prove by Addition; for $+5b$ added to $-2b$ the Quantity ſubtracted, makes $+3b$, which is the Quantity out of which the ſaid $-2b$ was ſubtracted.

Out of	$+3b$
Subtract	$-2b$
Remainder,	$3b+2b$
Remainder contracted,	$5b$

Moreover, if a be to be ſubtracted from a, the Remainder will be $a-a$, that is, 0 or nothing. And if from $2b$ there be ſubtracted $-4b$, the Remainder will be $2b+4b$, that is, $6b$.

Likewiſe, if from $-2m$ it be required to ſubtract $-m$, the Remainder will be found $-2m+m$, that is, $-m$. In every one of which Examples you may obſerve that the Sign of the Quantity propos'd to be ſubtracted is changed into the contrary Sign.

Again, if from $2bc$, it be deſired to ſubtract $2ab$, the Remainder will be $2bc-2ab$ which, becauſe it conſiſts of unlike Quantiries, cannot be contracted into fewer or leſſer Terms, by any of the Rules of Algebraical Addition, But according to the definition of Subtraction, the ſaid $2bc-2ab$ is a true Remainder, for if it be added to $2ab$ the Quantity ſubtracted, the Sum is $2bc$, which is the Quantity out of which the ſaid $2ab$ was ſubtracted.

Out of	$2bc$
Subtract	$2ab$
Remainder,	$2bc-2ab$

More Examples of Subtraction in Simple Algebraic Integers.

Out of	$2b$	$+3c$	$-2n$
Subtract	b	$-c$	$-n$
Remainder,	$2b-b$	$+3c+c$	$-2n+n$
Remainder contracted,	b	$+4c$	$-n$

Again,

		Again,	
Out of	$3a$	$-8d$	$-a$
Subtract	$5a$	$-10d$	$+a$
Remainder,	$3a-5a$	$-8d+10d$	$-a-a$
Remainder contracted,	$-2a$	$+2d$	$-2a$

Out of	$-bcd$	$-4rs$	$+4abc$
Subtract	$-bcd$	$+9rs$	$-abc$
Remainder,	$-bcd+bcd$	$-4rs-9rs$	$+4abc+abc$
Remainder contracted,	0	$-13rs$	$+5abc$

From	d	$-2b$	$+a^3$
Subtract	e	$-3a$	$-3a$
Remainder,	$d-e$	$-2b+3a$	$+a^3+3a$

From	$8bbd$	$+3abcd$
Subtract	$7bbb$	$-7aa$
Remainder,	$8bbd-7bbb$	$+3abcd+7aa$

Nor will the Operation be otherwiſe in the Subtraction of Compound Algebraic Integers; as for Example, if from this Compound quantity $3a+2b$, it be deſired to ſubtract $a+3b$. Firſt I write down $3a+2b$, then next after the ſame I write $-a-3b$, where obſerve, that the Sign $+$ which belongs to a, and alſo to $3b$, in the Quantity propos'd to be ſubtracted, is changed into the contrary Sign $-$ (according to the Rule of Subtraction before given; (ſo the Remainder ſought is $3a+2b-a-3b$, that is, $2a-b$, (by *Sect.* V. *Chap.* II.)

From	$3a+2b$
Subtract	$a+3b$
Remainder,	$3a+2b-a-3b$
Remainder contracted,	$2a-b$

Again, If from $2a+b$, it be deſired to ſubtract $5a-6b$, the Remainder will be $2a+b-5a+6b$, that is, $7b-3a$ for (according to the Rule of Algebraical Subtraction) I joyn together the two given Quantities, changing only the Signs of $+5a-6b$ (the Quantity to be Subtracted) into the contrary Signs, ſo there ariſes $2a+b-5a+6b$ which contracted (by the Rules of Addition in *Sect.* III. and V. of *Chap.* II.) make $7b-3a$, which is the Remainder ſought, as will eaſily appear by the Proof.

Out of	$2a+b$
Subtract	$5a-6b$
Remainder,	$2a+b-5a+6b$
Remainder contracted,	$7b-3a$

Likewiſe, to ſubtract $c-d$ from $a+b$, I change the Signs of $c-d$ into the contrary Signs; *viz.* inſtead of $c-d$, I take $-c+d$, which added to $a+b$ makes $a+b-c+d$, which becauſe it conſiſts altogether of unlike Quantities, cannot be contracted into fewer Terms, and therefore the ſaid $a+b-c+d$ is the Remainder ſought, to wit, that which ariſes by ſubtracting $c-d$ from $a+b$.

From	$a+b$
Subtract	$c-d$
Remainder,	$a+b-c+d$

After the ſame manner, $cd+36$ ſubtracted from $3aa+bc+24$ leaves $3aa+bc+24-cd-36$, that is, $3aa+bc-cd-12$.

More Examples of Subtraction in Compound Algebraic Integers.

Out of	$a+b$	$3c-8$
Subtract	$a-b$	$c+5$
Remainder,	$a+b-a+b$	$3c-8-c-5$
Remainder contracted,	$+2b$	$2c-13$

Out of	$5a-4b$	$29e$
Subtract	$3a-3b$	$-3e+7$
Remainder,	$5a-4b-3a+3b$	$29e+3e-7$
Remainder contracted,	$2a-b$	$32e-7$

Out of	$aa+2ba+bb$	$-2cd+6$
Subtract	$+4ba$	$+cd-2$
Remainder,	$aa+2ba+bb-4ba$	$-2cd+6-cd+2$
Remainder contracted,	$aa-2ba+bb$	$-3cd+8$

Out of	$5a^3+27$	$3aa+6$
Subtract	$-8+3a^3$	$-3dd$
Remainder,	$5a^3+27+8-3a^3$	$3aa+6+3dd$
Remainder contracted,	$2a^3+35$	

From	$a+b$	$aa-bb$
Subtract	$c-d$	$-cc+dd$
Remainder,	$a+b-c+d$	$aa-bb+cc-dd$

III. The reaſon of changing the Signs of the Quantity to be ſubtracted into their contraries, to wit + into —, and — into + (according to the Rule before given) will be manifeſt from a ſerious Conſideration of the definition of Subtraction, which requires that the Sum of the Quantity ſubtracted and the Remainder be equal to the quantity from which the Subtraction is made: for firſt, (according to the ſaid Rule) the Remainder is always compos'd of both the quantities propos'd for Subtraction, with this Caution, that the Signs + and — in the quantity to be Subtracted be changed into the contrary Signs; Secondly, (according to Algebraical Addition) the quantity to be ſubtracted with its own ſigns being added to it ſelf with contrary ſigns, will deſtroy or extinguiſh it ſelf; therefore the Sum of the Remainder and the Quantity to be Subtracted will neceſſarily be equal to the Quantity from which the Subtraction was made: And therefore the certainty of the ſaid Rule of Algebraical Subtraction, and the Reaſon of changing the Signs of the Quantity to be ſubtracted into their contraries, to wit, + into —, and — into +, is manifeſt: So if from $a+b$ there be ſubtracted $a-b$, the Remainder (according to the Rule of Algebraical Subtraction before given) will be $a+b-a+b$, to which if $a-b$ (the quantity ſubtracted) be added, it is evident that $a-b$ will deſtroy $-a+b$, and ſo the Sum will be $a+b$, to wit, the quantity from which $a-b$ was ſubtracted.

CHAP. IV.

Multiplication in Algebraic Integers.

I. A*Lgebraical Multiplication* does by two Quantities, whether they be expreſs'd by Letters wholly, or partly by Letters and partly by Numbers, find out a third Quantity, which is called the Product, the Fact, or the Rectangle.

The Quantities given to be multiplied one by the other are called Factors; or (as in vulgar Arithmetic) either of them may be called the Multiplicand, and the other the Multiplicator or Multiplier.

II. When two Simple (or ſingle) Quantities expreſs'd by Letters, whether like or unlike, be to be multiplied by one another, and have no Numbers prefix'd to them, join the Letters of both Quantities together, like Letters in a Word, it matters not in what order they be written; then the new Quantity repreſented by the Letters ſo ſet together is the Product ſought.

As for Example, If the Number or Line a be to be multiplied by it ſelf, to wit, by a, I write aa for the Product: So alſo to multiply a by b, I write ab or ba for the Product; in like manner if I would multiply abc by bc, I write $abcbc$, or $abbcc$, or $accbb$, &c. for the Product.

And if a, b, and c, be to be multiplied one into another; firſt a multiplied by b produces ab, then ab multiplied by c produces abc, or bac, or bca; to wit, the Product made by the continual Multiplication of the three Quantities a, b, and c.

Again, if aa be to be multiplied by ba, the Product will be $aaab$; which may alſo be written thus, a^3b; where the Learner muſt diligently note that the Figure 3 which ſtands next after but a little higher than a, muſt not be taken as a Number prefix'd to b, but as an Index to ſhew the number of Dimenſions in a^3, or aaa, (as before has been ſaid in *Sect.* XVI. and XVII. *Chap.* I.)

Likewiſe, if aaa be to be multiplied by aaa, or a^3 by a^3, the Product will be $aaaaaa$, or a^6, in which latter way of expreſſing the Product, the Index 6 ſtanding at the Head of a is the Sum of 3 and 3 the Indices of the Quantities a^3 and a^3 propos'd to be multiplied.

So the Product made by the Multiplication of $bbbb$ by bbb or b^4 by b^3 will be $bbbbbbb$, or b^7 (7 being the Sum of the Indices 4 and 3.)

Likewiſe if theſe three Quantities be to be multiplied continually, to wit, $aaaaa$, $bbbb$ and ccc, the Product may be expreſs'd thus, $aaaaabbbbccc$, or compendiouſly thus, $a^5b^4c^3$: and ſo of others.

More Examples of Multiplication in ſimple Algebraic Integers, according to the preceding Sect. II.

Multiplicand,	b	d	ac	ccc
Multiplicator,	c	d	d	cc
Product,	bc	dd	acd	$ccccc$

Multiplicand,	$aabc$	def	$aabbcc$
Multiplicator,	bca	abc	$aabbcc$
Product,	$aaabbcc$	$abcdef$	$a^4b^4c^4$

III. If two ſimple Quantities, whether like or unlike, having Numbers prefix'd before them, be to be multiplied one by the other; firſt multiply the Numbers prefix'd, one into the other, then to this Product annex the Letters of both Quantities, by ſetting them immediately one after another, (as before in *Sect.* II.) ſo this new Quantity ſhall be the Product ſought.

As, for Example, if it be deſired to multiply $2a$ by $3b$; firſt I multiply 2 by 3, and the Product is 6; to which annexing ab, (to wit, the Letters found in both Quantities given to be multiplyed) there ariſes $6ab$ the Product ſought; which ſhews that ſix times the Product of the Multiplication of any two Numbers, or Right-lines, a and b, is equal to the Product made by the Multiplication of the Double of a by the Triple of b.

Multiply	$2a$
by	$3b$
Product,	$6ab$

In like manner, if $2b$ be multiplyed be c the Product will be $2bc$, or $2cb$; for 2 which is prefix'd to b in the Multiplicand, being multiplied by 1, which is ſuppos'd to be prefix'd to the Multiplier c, makes 2, to which annexing bc, there is found $2bc$ for the Product ſought.

Multiply	$2b$
by	c
Product,	$2bc$

More Examples of Multiplication in Simple Algebraic Integers, according to Sect. III.

Multiply	$4b$	$12ac$	$5ddfg$
by	$2a$	$3d$	dgh
Product,	$8ab$	$36acd$	$5d^3fggh$
Multiply	aaa	$3a^3$	$16aab$
by	$3bbb$	b^3	4
Product,	$3aaabbb$	$3a^3b^3$	$64aab$

IV. The Multiplication of Compound quantities depends upon the precedent Rules of multiplying ſimple quantities; for when a Compound quantity is to be multiplied by a ſimple (or ſingle) quantity, every Member of that muſt be multiplied by this; alſo, when two compound quantities are to be mutually multiplied, every Member of the one muſt be multiplied into every Member of the other. It matters not whether you begin to multiply at the right Hand or the left, nor in what order the particular Products be ſet; (for quantities expreſs'd by Letters retain their peculiar and unaltered values whereſoever they ſtand;) but due regard muſt be had to the Signs + and —, one of which always belongs to every particular Product, and may be diſcovered by this Rule, *viz.* + multiplied by +, or — by —, makes + in the Product; but + multiplied by —, or — by +, makes — in the Product; laſtly, all the particular Products added together (according to the Rules in the preceding *Chap.* 2.) make the total Product ſought: All which will be made manifeſt by the following Examples.

Firſt, if a Compound quantity, as $a+b$, be to be multiplied by a ſimple quantity, as c, I begin at the left Hand, and multiplying $+a$ by $+c$ the Product is $+ac$, (for + multiplied by gives +;) likewiſe $+b$ multiplied by $+c$ produces $+bc$; which two Products added together make $ac+bc$, which is the Product of the Multiplication of $a+b$ by c.

Multiply	$a+b$
by	c
Product,	$ac+bc$

So if $a-b$ be to be multiplied by c, the Product will be $ac-bc$. For $+a$ multiplied by $+c$ produces $+ac$; and $-b$ multiplied by $+c$ produces $-bc$; (for according to the Rule, — multiplied by + gives —:) Therefore $+ac-bc$ or $ac-bc$ is the Product ſought.

Multiply	$a-b$
by	c
Product,	$ac-bc$

After the ſame manner, if it be deſired to multiply $a+b$ by $c+d$, the Product will be found $ac+bc+ad+bd$. For, firſt $a+b$ being multiplied by c, (as in the firſt Example) produces $+ac+bc$; likewiſe $a+b$ again multiplied by d, produces $+ad+b$; then adding thoſe Products together, the Sum is $ac+bc+ad+bd$, which is the required Product of $a+b$ multiplied by $c+d$.

$$\begin{array}{lr} \text{Multiply} & a+b \\ \text{by} & c+d \\ \hline & +ac+bc \\ & +ad+bd \\ \hline \text{Product,} & +ac+bc+ad+bd \end{array}$$

Again, if $a-b$ be multiplied by $c-d$ the Product will be $ac-bc-ad+bd$: For Firſt, $a-b$ multiplied by c produces $ac-bc$, (as in the laſt Example but one;) then $a-b$ again multiplied by $-d$ produces $-ad+bd$; (for according to the Rule, $+a$ multiplied by $-d$ produces $-ad$, and $-b$ by $-d$ produces $+bd$.) Laſtly, thoſe particular Products added together make $ac-bc-ad+bd$, which is the Product of $a-b$ multiplied by $b-c$.

$$\begin{array}{lr} \text{Multiply} & a-b \\ \text{by} & c-d \\ \hline \text{Product,} & ac-bc-ad+bd \end{array}$$

Likewiſe, if $a+b$ be multiplied by $a-b$, the Product will be $aa+bb$: For firſt, $a+b$ multiplied by a produces $aa+ba$; then $a+b$ multiplied by $-b$ produces $-ba-bb$; laſtly, the ſaid Products $aa+ba$ and $-ba-bb$ added together make $aa-bb$; (for $+ba$ and $-ba$ by Addition do quite vaniſh;) Therefore $aa-bb$ is the Product of $a+b$ multiplied by $a-b$.

$$\begin{array}{lr} \text{Multiply} & a+b \\ \text{by} & a-b \\ \hline & aa+ba \qquad \\ & -ba-bb \\ \hline \text{Product,} & aa \qquad -bb \end{array}$$

Moreover, if $aa-ab+bb$ be multiplied by $a+b$, the Product will be only $aaa+bbb$; for the reſt of the particular Products will vaniſh by Addition.

And if $a+b$ be multiplied by it ſelf, to wit, by $a+b$, the Product will be $aa+2ab+bb$, which is the Square of $a+b$.

Likewiſe the Square of $a-b$ will be found $aa-2ab+bb$.

Nor will the Operation be otherwiſe when Numbers are prefixed to compound Quantities propoſed to be multiplied, reſpect being had to the Third *Sect.* of this *Chap.* as, for Example, to multiply $3a-2e$ by $3a-2e$; Firſt, $3a-2e$ multiplied by $3a$ produces $9aa-6ae$, and $3a-2e$ again multiplied by $-2e$ produces $-6ae+4ee$; which particular Products added together make $9aa-12ae+4ee$ which is the Square of $3a-2e$.

$$\begin{array}{lr} \text{Multiply} & 3a-2e \qquad \\ \text{by} & 3a-2e \qquad \\ \hline & +9aa-6ae \qquad \\ & -6ae+4ee \\ \hline \text{Product,} & 9aa-12ae+4ee \end{array}$$

When abſolute Numbers are members of Quantities to be multiplied, the Rules of Multiplication in vulgar Arithmetic and thoſe before given muſt be mixtly obſerved; as,

$$\begin{array}{lr} \text{If it be deſired to multiply} \ldots\ldots\ldots\ldots\ldots & 3a+6 \\ \text{By the abſolute Number} \ldots\ldots\ldots\ldots\ldots & 5 \\ \hline \text{The Product will be} \ldots\ldots\ldots\ldots\ldots & 15a+30 \end{array}$$

For five times $3a$ makes $15a$, and five times 6 makes 30.

Likewiſe, if $2aa-3$ be multiplied by $a-6$, the Product will be $2aaa-12aa-3a+18$, and the work will ſtand as here you ſee;

$$\begin{array}{ll} \text{Multipicand,} & 2aa-3 \\ \text{Multiplicator,} & \quad a-6 \\ \hline & +2aaa-3a \\ & \qquad -12aa+18 \\ \hline \text{Product,} & 2aaa-12aa-3a+18 \end{array}$$

For further illuſtration of the Multiplication of Algebraic Integers, the Learner may peruſe the following Examples; in every one of which, as alſo in thoſe afore-going, I begin to multiply at the left Hand, becauſe in Algebraical Multiplication it being a thing indifferent to begin the work either at the right Hand or the left, it will be eaſier to write forward than backward. And as to the placing of the particular Products, there is no neceſſity of obſerving any Order therein; for whether they be written upon one, two, or more Lines, they retain the ſame values, and muſt by Algebraical Addition be collected into one Sum, to make the total Product: And therefore you may either write the particular Products all upon one Line when there is room, or elſe upon ſo many ſeveral Lines as there be particular Multipliers, ſetting like Products (when they happen) under one another to facilitate their Addition; or otherwiſe, as you ſhall find it moſt convenient.

More Examples of Multiplication in Compound Algebraic Integers, according to Sect. IV.

Multiplicand,	$a+e$	$2b-3d$	$5g-8$
Multiplicator,	d	f	6
Product,	$da+de$	$2bf-3fd$	$30g-48$

Multiplicand,	$5a+3c$	$2b+3$
Multiplicator,	$3a-2c$	$4b-6$
	$+15aa+9ca$	$8bb+12b$
	$-10ca-6cc$	$-12b-18$
Product,	$15aa+9ca-10ca-6cc$	$8bb+12b-12b-18$
Product contracted,	$15aa-ca-6cc$	$8bb-18$

Multiplicand,	$3dd+4de+ee$
Multiplicator,	$3dd-ee$
	$+9dddd+12ddde+3ddee$
	$-3ddee-4deee-eeee$
Product,	$9dddd+12ddde+3ddee-3ddee-4deee-eeee$
Product contracted,	$9d^4+12d^3e-4de^3-e^4$

Multiplicand,	$a+e$	$a+e$
Multiplicator,	$a+e$	$a-e$
	$aa+ac$	$aa+ae$
	$+ae+ee$	$-ae-ee$
Product,	$aa+2ae+ee$	$aa-ee$

Multiplicand,	$4aaa+3aa-2a+1$
Multiplicator,	$aa-5a+6$
	$4aaaaa+3aaaa-2aaa+aa$
	$-20aaaa-15aaa+10aa-5a$
	$+24aaa+18aa-12a+6$
Product,	$4aaaaa-17aaaa+7aaa+29aa-17a+6$

Again,

$$\begin{array}{ll} \text{Multiplicand,} & 2aa+3ba-bc \\ \text{Multiplicator,} & 3aa-2ba-cc \\ \hline & 6aaaa+9baaa-3bcaa \\ & \qquad -4baaa-6bbaa+2bbca \\ & \qquad\qquad -2ccaa-3bcca+bccc \\ \hline \text{Prod.} & 6aaaa+5baaa\left.\begin{array}{l}-3bc\\-6bb\\-2cc\end{array}\right\}aa\left.\begin{array}{l}+2bbc\\-3bcc\end{array}\right\}a+bccc \end{array}$$

V. Sometimes when Compound quantities be to be multiplied one by the other, it will be very commodious to omit the Operation, and to ſet only the word *into*, or × (the Sign of Multiplication) between the Quantities to be multiplied, to ſignifie the Product of their Multiplication: But in ſuch Caſe, to avoid Miſtake, it will be convenient to draw a Line over each Compound quantity, to ſhew that every Member of the one is to be multiplied by every Member of the other.

As to multiply $4aaa+3aa-2a+1$ by $aa-5a+6$, I write

$\overline{4aaa+3aa-2a+1}$ into $\overline{aa-5a+6}$

Or, $\overline{4aaa+3aa-2a+1} \times \overline{aa-5a+6}$

But that + multiplied by —, or — by + makes —; alſo, that — multiplied by — makes + in the Multiplication of compound Quantities, I ſhall hereafter make manifeſt in the laſt *Sect.* of *Chap.* XI.

CHAP. V.

Diviſion in Algebraic Integers.

I. *ALgebraical Diviſion* does by two Quantities, (whether they be expreſs'd wholly by Letters, or partly by Letters and partly by Numbers,) whereof one is called the Dividend, and the other the Diviſor, find out a Third called the Quotient; to wit, ſuch a Quantity, that if it be multiplied by the Diviſor, the Product will be equal to the Dividend.

II. The Nature of Diviſion is to reſolve or undo that which is compoſed or done by Multiplication; for the Dividend always repreſents the Fact or Product in Multiplication, the Diviſor one of the two Factors or Multipliers, and the Quotient the other. As, if 12 be to be divided by 2, the Dividend 12 repreſents the Fact or Product made by the Multiplication of two Numbers, one of which is the Diviſor 2, and the other is the Quotient ſought, to wit, 6.

III. Every Fraction is equal to the Quotient of the Numerator divided by the Denominator: So $\frac{3}{4}$ is the Quotient of 3 divided by 4; for, according to the Proof of Diviſion, if the Quotient $\frac{3}{4}$ be multiplied by the Diviſor 4, the Product will be equal to the Dividend 3. Upon this ground, Diviſion in Algebraic Integers, whether Simple or Compound is moſt commonly performed; *viz.* by ſetting the Dividend as the Numerator of a Fraction, and the Diviſor as a Denominator; for this Fraction is equal to the Quotient ſought.

As for Example, to divide the Quantity a by b, I write $\frac{a}{b}$, which ſignifies that that a is divided by b; or $\frac{a}{b}$ is equal to the Quotient of the Quantity a divided by the Quantity b.

In like manner, if b be propos'd to be divided by ac, I write $\frac{b}{ac}$ to repreſent the Quotient; alſo, if ac be to be divided by b, I write $\frac{ac}{b}$ to ſignifie the Quotient.

Again, If $2ab$ be given to be divided by $3cd$, the Quotient will be $\frac{2ab}{3cd}$; and if a be to be divided by 5, I write for the Quotient $\frac{a}{5}$; alſo to divide 1 by a, I write $\frac{1}{a}$ to ſignifie the Quotient.

So alſo, if $a+b$ be given to be divided by c, the Quotient may be repreſented by $\frac{a+b}{c}$; and if $3a$ be to be divided by $2b-c$, the Quotient is $\frac{3a}{2b-c}$.

More Examples of Diviſion in Algebraic Integers, according to the foregoing Sect. III.

Dividend,	bb	$2de$	$3abc$	$a+b$
Diviſor,	a	fg	$2dd$	$2d^3$
Quotient,	$\frac{bb}{a}$	$\frac{2de}{fg}$	$\frac{3abc}{2dd}$	$\frac{a+b}{2d^3}$

Dividend,	$aa+bb$	$2ab-3bd$	aaa
Diviſor,	c	$d+e$	$a+b-c$
Quotient,	$\frac{aa+bb}{c}$	$\frac{2ab-3bd}{d+e}$	$\frac{aaa}{a+b-c}$

Dividend,	$4aa$	$2cc+5dd$
Diviſor,	3	3
Quotient,	$\frac{4aa}{3}$, or $\frac{4}{3}aa$	$\frac{2cc+5dd}{3}$, or, $\frac{2}{3}cc+\frac{5}{3}dd$.

IV. When the Dividend is equal to the Diviſor, the Quotient is 1; for every Quantity contains it ſelf once, and therefore being divided by it ſelf gives 1 in the Quotient: As to divide 4 by 4 the Quotient is 1; likewiſe, a divided by a gives 1 for the Quotient; alſo, if $a+b$ be divided by $a+b$ the Quotient is 1; and if $3a+2cd$ be divided by $3a+2cd$ the Quotient is 1. The like is to be underſtood of others.

V. When the Quotient is expreſſed Fraction-wiſe, (according to *Sect* III) if the ſame letter or letters be found equally repeated in every member of the Numerator and Denominator, caſt away thoſe letters, ſo the remaining Quantities ſhall ſignifie the Quotient.

As, for Example, If ab be to be divided by a, the Quotient expreſt Fraction-wiſe will be $\frac{ab}{a}$; But becauſe the letter a is found in the Numerator and Denominator, I caſt away a out of both, ſo b only is left, which is the Quotient of ab divided by a.

Likewiſe, If aa be divided by a the Quotient is $\frac{aa}{a}$, that is, a; (by caſting away a out of the Numerator and Denominator.)

Again, If aaa be to be divided by aa, the Quotient will be $\frac{aaa}{aa}$, that is, a; by caſting away aa out of the Numerator and Denominator. And if abc be to be divided by ab, the Quotient expreſt Fraction-wiſe will be $\frac{abc}{ab}$, that is, c, after ab is caſt out of the Numerator and Denominator.

After the ſame manner, if a^5 be propos'd to be divided by a^3, (that is, $aaaaa$ by aaa) the Quotient will be a^2, or aa, by expunging a^3 (or aaa) out of the Dividend and Diviſor.

This Contraction of Diviſion is like to the reducing of a Fraction expreſt by large numbers to more ſimple Terms, by dividing the Numerator and alſo the Denominator by a common Diviſor.

Again, If $ab+ac$ be to be divided by $ad-af$, the Quotient expreſt Fraction-wiſe according to the preceding *Sect.* III. will ſtand thus, $\frac{ab+ac}{ad-af}$, where becauſe the letter a is found in every member of the Numerator and Denominator, it may be quite ſtruck out, and then the new Quotient will be $\frac{b+c}{d-f}$, which Fraction is equal to the former, and expreſt by more ſimple Terms.

Dividend,	$ab+ac$
Diviſor,	$ad-af$
Quotient,	$\left\{\frac{ab+ac}{ad-af}\right.$
Quotient contracted,	$\left\{\frac{b+c}{d-f}\right.$

Likewiſe, If $ab+a$ be divided by a, the Quotient (according to *Sect.* III.) will be $\frac{ab+a}{a}$, that is, $b+1$; for by caſting away a, there will remain $\frac{b+1}{1}$, that is, $b+1$; (for $\frac{b}{1}$ is but b; and $\frac{1}{1}$ is 1;) but that $b+1$ is the true Quotient it will appear by the proof of Diviſion, for $b+1$ Multiplied by the Diviſor a will produce the Dividend $ab+a$.

So alſo to divide $3bc-2b$ by $2bb+b$, I write $\frac{3c-2}{2b+1}$ for the Quotient; where obſerve, that altho' the letter b be caſt out of every Member of the given Dividend and Diviſor, yet the number prefixt to the letter caſt out muſt ſtand ſtill in the new Quotient.

But note diligently, That in this kind of Diviſion of Compound Algebraic Integers, a letter cannot be cancell'd or caſt away, unleſs it be found in every Member of the Dividend and Diviſor; and therefore this Quotient $\frac{bc+cd}{c+f}$ cannot be contracted by caſting away any letter.

More Examples of Contractions in Algebraic Diviſion, according to the Preceding Sect. V.

Dividend,	aab	$ddef$	abc	a^7
Diviſor,	aa	ef	b	a^3
Quotient,	$\frac{aab}{aa}$	$\frac{ddef}{ef}$	$\frac{abc}{b}$	$\frac{a^7}{a^3}$
Quotient contracted,	b	dd	ac	a^4

Dividend,	$ab+ac-a$	$ab-2a$
Diviſor,	a	$3a$
Quotient,	$\frac{ab+ac-a}{a}$	$\frac{ab-2a}{3a}$
Quotient contracted,	$b+c-1$	$\frac{b-2}{3}$, or, $\frac{1}{3}b-\frac{2}{3}$.

Dividend,	$2abd+3bd$	$2ba^3+caa-3aa$
Diviſor,	$3bb-b$	$baa-daa+aa$
Quotient,	$\frac{2ad+3d}{3b-1}$	$\frac{2ba+c-3}{b-d+1}$

VI. If an Algebraic Integer, whether Simple or Compound, be to be divided by a ſimple Quantity, and there be ſuch numbers prefix'd to the letters in the Dividend and Diviſor as may all be ſeverally divided by ſome number as a common Diviſor without leaving a Remainder, ſet the Quotients ariſing by the Diviſion of thoſe numbers by their common Diviſor, before the letters reſpectively, inſtead of the numbers that were firſt prefix'd: As, for Example, if $8a$ be to be divided by $6b$; Firſt, the Quotient expreſt Fraction-wiſe (according to *Section* III. of this *Chap.*) will be $\frac{8a}{6b}$, then dividing the prefixed Numbers 8 and 6 by their common Diviſor 2, I ſet the Quotients 4 and 3 inſtead of 8 and 6 before a and b; ſo the Quotient ſought is $\frac{4a}{3b}$.

In like manner, $6abc-3dbc$ Divided by $9fbc$ gives the Quotient $\frac{2a-d}{3f}$;

Dividend,	$6abc-3dbc$
Diviſor,	$9fbc$
Quotient,	$\frac{6abc-3dbc}{9fbc}$
Quotient contracted	$\frac{2a-d}{3f}$

For firſt, the Dividend and Diviſor being ſet Fraction-wiſe will ſtand thus, $\frac{6abc-3dbc,}{9fbc}$; then, (according to *Sect.* V.) bc is to be caſt out of the Numerator and Denominator; laſtly, the prefixed numbers 6, 3, and 9 being divided by their common Diviſor 3, give 2, 1, and 3, which being ſet before the remaining letters a, d and f reſpectively, give the contracted Quotient $\frac{2a-1d}{3f}$ or $\frac{2a-d}{3f}$.

More Examples of Contractions in Diviſion, according to Sect V. *and* VI.

Dividend,	$4cd$	$27ab$	$16gb$
Diviſor,	$2c$	$9ad$	$8gb$
Quotient,	$\frac{4cd}{2c}$	$\frac{27ab}{9ad}$	$\frac{16gb}{8gb}$
Quotient contracted,	$2d$	$\frac{3b}{d}$	2

Dividend,	$18aaaa$	$30b^5c^4dd$
Diviſor,	$6aa$	$5bbccd$
Quotient,	$\frac{18aaaa}{6aa}$	$\frac{30b^5c^4dd}{5bbccd}$
Quotient contracted	$3aa$	$6b^3ccd$

Dividend,	$28bbc+16bbd$
Diviſor,	$20bb$
Quotient,	$\frac{28bbc+16bbd}{20bb}$
Quotient contracted,	$\frac{7c+4d}{5}$, or, $\frac{7}{5}c+\frac{4}{5}d$.

VII. If every Member of a Compound quantity be multiplied by one and the ſame ſimple quantity, it is evident from the Nature of Multiplication and Diviſion, that if the Product of that Multiplication be divided by the ſaid Compound Quantity, the Quotient will be the ſimple Quantity.

As, for Example, If $b+c$ be multiplied by a the Product will be $ba+ca$, and therefore $ba+ca$ divided by the Factor $b+c$ will give the other Factor a. And for the ſame Reaſon, $2bca+a$, that is $2bca+1a$, divided by $2bc+1$ will give the Quotient a.

Likewiſe, If $6a+5a-a$ (that is $10a$) be divided by $6+5-1$ (that is, 10,) the Quotient will be a.

Again, If $2ba+2ca+2da$ be divided by $b+c+d$, the Quotient will be $2a$; and if $2baa+caa-daa-aa$ be divided by $2b+c-d-1$, the Quotient will be aa.

More Examples of Contractions in Diviſion, according to the preceding Sect. VII.

Dividend,	$2da+3ca$	$23b+18b+1b$
Diviſor,	$2d+3c$	$23+18+1$
Quotient,	a	b

Dividend,	$2baa-3caa$	$2af-2bf+2cf-6f$
Diviſor,	$2b-3c$	$a-b+c-3$
Quotient	aa	$2f$

VIII. When the Dividend and Diviſor are Compound whole Quantities, the precedent Rules of Algebraical Diviſion will not always give the Quotient in the leaſt Terms; but the ſimpleſt Quotient may be found out by one of theſe two ways, *viz.*

1. When the Dividend and Diviſor are Algebraic Integers, and there is a poſſibility of expreſſing the Quotient by an Algebraical Integer, it may be found out by the general Method of Diviſion handled in the next following *Section*, which way is like that of dividing whole Numbers in Vulgar Arithmetic; but if the Learner find it difficult, he may wave it until he has proceeded as far as the 8. *Chapter* of the 2. *Book.*

2. The Quotient, whether it happen to be an Algebraic Integer, or a Fraction, may be found out in its leaſt Terms by the Method hereafter delivered in *Sect.* 7. *Chap.* 8. of the Second Book; where the manner of finding out all the *Aliquot Parts* or juſt Diviſors, every one of which will divide the Dividend and Diviſor propos'd without any Remainder is exhibited.

IX. In this *Section* a general Method of Diviſion in Algebraical Integers is handled. As to the order of the Work, it agrees with that form of Diviſion in whole Numbers which I have explained in Mr. *Wingate*'s Arithmetic, but the Work it ſelf depends upon the preceding Rules of Algebraical *Diviſion*, *Multiplication* and *Subtraction*, as alſo upon this Rule for diſcovering the due Sign belonging to every particular Quotient, *viz.* + divided by +, or — by —, gives + in the Quotient; but + divided by —, or — by +, gives — in the Quotient. Whether the Operation be begun at the Right Hand or the Left, it matters not; but becauſe 'tis eaſier to Write forwards than backwards, I ſhall (as in Vulgar Arithmetic) begin to Divide at the Left Hand, and proceed towards the Right.

Example 1. Let it be required to divide $ac+ad+bc+bd$ by $c+d$.

Having placed the Dividend and Diviſor in ſuch order as you ſee in the next Page, firſt I divide $+ac$ by $+c$, (according to *Sect.* 5. of this *Chap.*) and there ariſes $+a$, ($+a$, becauſe + divided by + gives +,) therefore I write $+a$ or a in the Quotient; then Multiplying the whole Diviſor $c+d$ by the ſaid Quotient a, I write the Product $ac+ad$ under the two firſt Members of the Dividend towards the Left Hand, to wit, under $ac+ad$; that done, drawing a Line under the ſaid Product $ac+ad$, I ſubtract the ſame from $ac+ad$, (the two firſt Members of the Dividend) and there remains 0, which I ſet under the Line, as you may ſee in the Page following.

Diviſor.

$$\begin{array}{rll}
\text{Diviſor.} & \text{Dividend} & \text{Quotient.} \\
c+d) & ac+ad+bc+bd & (a+b \\
 & \underline{ac+ad\qquad\quad} & \\
 & 0\quad 0\quad +bc+bd & \\
 & \qquad\quad\ \underline{+bc+bd} & \\
 & \qquad\qquad 0\quad 0 &
\end{array}$$

Then there remains to be divided $+bc+bd$ which I bring down to the Remainder 0, and renew the Work, *viz.* I divide $+bc$ by $+c$, and there ariſes $+b$ which I write in the Quotient next after a; then multiplying the whole Diviſor $c+d$ by the ſaid Quotient b, the Product is $bc+bd$, which being ſubſcribed, and ſubtracted from that which remained to be divided, there remains 0. So the Diviſion is finiſhed, and the Quotient is found $a+b$; but that it is a true Quotient the Proof will make manifeſt; for $a+b$ multiplied by the Diviſor $c+d$ produces the Dividend $ac+ad+bc+bd$.

Example 2. In like manner, if $aa-bb$ be to be divided by $a+b$ the Quotient will be found $a-b$; For firſt, aa divided by a gives a in the Quotient, by which multiplying the whole Diviſor $a+b$ the Product is $aa+ab$, which ſubtracted from the Dividend $aa-bb$, there remains to be divided $-bb-ab$. Now I renew the Work, and divide $-bb$ by its correſpondent Diviſor $+b$, (not by a, becauſe the Quotient will be a Fraction, which is to be avoided when there is a poſſibility) and there ariſes $-b$ to be written next after a in the Quotient, I ſay $-b$, not $+b$; for according to the Rule before given, − divided by + gives − in the Quotient; then multiplying the whole Diviſor $a+b$ by $-b$ (laſt ſet in the Quotient) the Product is $-ab-bb$, or $-bb-ab$, which ſubtracted from $-bb-ab$ that remained to be divided, there remains 0; ſo the Diviſion is finiſh'd and the Quotient is found $a-b$, to wit, ſuch a Quantity that if it be multiplied by the Diviſor $a+b$, it will produce the Dividend $aa-bb$.

$$\begin{array}{rl}
a+b) & aa-bb\ (a-b \\
 & \underline{aa+ab\qquad\quad} \\
 & \quad -bb-ab \\
 & \quad \underline{-bb-ab} \\
 & \quad\ \ 0\qquad 0
\end{array}$$

Example 3. Again, If it be deſired to divide $aaa+bbb$ by $aa-ba+bb$, the Quotient will be found $a+b$, and the Work will ſtand thus:

$$\begin{array}{rl}
aa-ba+bb) & aaa+bbb\ \ldots\ldots\ldots\ (a+b \\
 & \underline{aaa-baa+bba\qquad\qquad} \\
 & \quad +bbb+baa-bba \\
 & \quad \underline{+bbb+baa-bba} \\
 & \qquad 0\qquad 0\qquad 0
\end{array}$$

In which Example, firſt (as before) I begin at the firſt Term of the Dividend towards the Left Hand, and dividing aaa by aa, (not by $-ba$ nor by $+bb$, becauſe each of theſe will give a Fraction in the Quotient) there ariſes a, which I ſet in the Quotient; then Multiplying the whole Diviſor $aa-ba+bb$ by the ſaid Quotient a, the Product is $aaa-baa+bba$, which I ſubtract from the Dividend $aaa+bbb$; ſo there remains to be yet divided $+bbb+baa-bba$.

Now I renew the Work, and divide $+bbb$ by its correſpondent Diviſor $+bb$, (not by $+aa$, nor by $-ba$, becauſe each of theſe gives a Fraction) and there ariſes $+b$, which I write next after a in the Quotient; then multiplying the whole Diviſor $aa-ba+bb$ by the ſaid Quotient $+b$, the Product is $bbb+baa-bba$, which I ſet under, and ſubtract from the Quantity that remained to be divided, ſo there remains 0, and the Quotient ſought is $a+b$: But that it is a true Quotient the Proof will diſcover; for if the Diviſor $aa-ba+bb$ be multiplied by the Quotient $a+b$, it will produce the Dividend $aaa+bbb$.

Example 4. In like manner, if $aaa-bbb$ be divided by $aa+ba+bb$, the Quotient will be $a-b$, and the work will ſtand thus;

$$\begin{array}{rll}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.}\\
aa+ba+bb) & aaa-bbb \ldots\ldots\ldots & (a-b\\
 & aaa+baa+bba & \\
\hline
 & -bbb-baa-bba & \\
 & -bbb-baa-bba & \\
\hline
 & 0 \quad 0 \quad 0 &
\end{array}$$

Example 5. Again, If $9dddd+12ddde-4deee-eeee$ be to be divided by $3dd-ee$, the Quotient will be found $3dd+4de+ee$, as will be manifeſt by the ſubſequent Operation.

$$\begin{array}{rl}
3dd-ee) & 9dddd+12ddde-4deee-eeee \ (3dd+4de+ee\\
 & 9dddd-3ddee\\
\hline
 & +12ddde+3ddee-4deee\\
 & +12ddde \qquad\quad -4deee\\
\hline
 & +3ddee-eeee\\
 & +3ddee-eeee\\
\hline
 & 0 \quad 0
\end{array}$$

In which Example, firſt I divide $9dddd$ by $3dd$, and it gives $3dd$, which I write in the Quotient; then multiplying the whole Diviſor $3dd-ee$ by the ſaid Quotient $3dd$, the Product is $9dddd-3ddee$, which I write under the two firſt Members of the Dividend, and ſubtract the ſame from the ſaid two Members, ſo there remains $+12ddde+3ddee$; to which I bring down $-4deee$ (the next Member of the Dividend) and it makes $+12ddde+3ddee-4deee$ which comes now to be divided; therefore I renew the work, and dividing $+12ddde$ by $+3dd$, it gives $+4de$, which I ſet in the Quotient next after $3dd$, then multiplying the whole Diviſor, $3dd-ee$ by the ſaid Quotient $+4de$, the Product is $+12ddde-4deee$, which I write under $+12ddde+3ddee-4deee$ (the Quantity laſt ſet apart to be divided;) and having drawn a Line under the ſaid Product I ſubtract it from the ſaid particular Dividend, ſo there remains $+3ddee$ which I write underneath the Line; that done, to the ſaid Remainder $+3ddee$ I bring down $-eeee$, (the laſt Member of the total Dividend) and it makes $+3ddee-eeee$ which is yet to be divided: Therefore I renew the Work, and dividing $+3ddee$ by $+3dd$, it gives $+ee$ which I ſet in the Quotient next after $+4de$; (or I might here divide $+3ddee$ by $-ee$ in regard it will give an Algebraical Integer in the Quotient, as I ſhall ſhew in the next Example :) then multiplying the Diviſor $3dd-ee$ by $+ee$, (laſt ſet in the Quotient,) and ſubtracting the Product $+3ddee-eeee$ from the Quantity that remained to be divided, there now remains 0. So the Diviſion is finiſhed without any Quantity remaining, and the entire Quotient is $+3dd+4de+ee$.

Note. By this general Method of Diviſion the Quotient may oftentimes be found out and expreſs'd various ways, both as to the Order and Multitude of Members in the Quotient, but yet the entire Quotient in each Form will have one and the ſame value, as will appear by the following manner of Dividing the two Quantities propos'd in the laſt Example.

Let it therefore be again propos'd to divide $9dddd+12ddde-4deee-eeee$ by $3dd-ee$. Firſt, I work as before in the laſt Example to find out the two firſt Members in the Quotient, to wit, $3dd+4de$, and then there remains to be divided $+3ddee-eeec$ which you ſee ſtands at this Mark * in the following Operation: Now becauſe $+3ddee$ divided by $-ee$ gives an Algebraic Integer for the Quotient, to wit, $-3dd$, therefore I write $-3dd$ in the Quotient; then multiplying the whole Diviſor $3dd-ee$ by $-3dd$ (laſt ſet in the Quotient) I ſubtract the Product $+3ddee-9dddd$ from $+3ddee-eeee$ which remained to be divided; ſo there remains to be yet divided $-eeee+9dddd$.

$$\begin{array}{llllll}
3dd-ee) & 9dddd+12ddde-4deee-eeee & & & & (\ \ 3dd+4de \\
 & 9dddd-3ddee & & & & (-3dd+ee+3dd \\
\cline{2-2}
 & \quad +12ddde+3ddee-4deee & & & & \\
 & \quad +12ddde \qquad\quad -4deee & & & & \\
\cline{2-2}
 & \qquad * & +3ddee-eeee & & & \\
 & & +3ddee-9dddd & & & \\
\cline{3-5}
 & & & -eeee+9dddd & & \\
 & & & -eeee+3ddee & & \\
\cline{4-5}
 & & & & +9dddd-3ddee & \\
 & & & & +9dddd-3ddee & \\
\cline{5-5}
 & & & & \quad 0 \qquad 0 &
\end{array}$$

Then I divide $-eeee$ (which ſtands immediately under the third black Line) by its correſpondent Diviſor $-ee$, (for it cannot be divided by $3dd$ ſo as to give an Integer in the Quotient,) and there ariſes $+ee$, which I ſet in the Quotient; then multiplying the whole Diviſor $3dd-ee$ by the ſaid Quotient $+ce$ the Product is $-eeee+3ddee$, which ſubtracted from $-eeee+9dddd$ (to wit, the Quantity that remained to be divided) there remains to be yet divided $+9dddd-3ddee$, (which ſtands immediately under the laſt black Line but one;) therefore I divide $+9dddd$ by $+3dd$ and it gives $+3dd$ to be ſet in the Quotient; then multiplying the whole Diviſor $3dd-ee$ by the ſaid $+3dd$, it makes $+9dddd-3ddee$, which ſubtracted from $+9dddd-3ddee$ (the Quantity that remained to be divided) leaves 0; ſo the Diviſion is finiſhed without any Quantity remaining, and the Quotient is found $3dd+4de-3dd+ee+3dd$, that is, $3dd+4de+ee$: So that the Quotient found out by the latter Operation, after it is contracted by Algebraical Addition, is the ſame found out by the former way of dividing the Quantities given in the fifth Example.

Example 6. Again, If $yyyyyy-8yyyy-124yy-64$ be divided by $yy-16$, the Quotient will be found $yyyy+8yy+4$, and the Work will ſtand thus:

$$\begin{array}{llll}
\text{Diviſor.} & \text{Dividend.} & & \text{Quotient.} \\
yy-16) & yyyyyy-8yyyy-124yy-64 & & (yyyy+8yy+4 \\
 & yyyyyy-16yyyy & & \\
\cline{2-2}
 & \qquad +8yyyy-124yy & & \\
 & \qquad +8yyyy-128yy & & \\
\cline{2-3}
 & & +4yy-64 & \\
 & & +4yy-64 & \\
\cline{3-3}
 & & \quad 0 \qquad 0 &
\end{array}$$

If the Powers of the Root y in the laſt Example be expreſſed according to *Carteſius* his way, the work will ſtand thus:

$$\begin{array}{llll}
yy-16) & y^6-8y^4-124yy-64 & & (\ y^4+8yy+4 \\
 & y^6-16y^4 & & \\
\cline{2-2}
 & \qquad +8y^4-124yy & & \\
 & \qquad +8y^4-128yy & & \\
\cline{2-3}
 & & +4yy-64 & \\
 & & +4yy-64 & \\
\cline{3-3}
 & & \quad 0 \qquad 0 &
\end{array}$$

But *Carteſius* in dividing the Quantities propos'd in the laſt Example works backwards, *viz.* from the right Hand of the Dividend towards the left, as you here ſee in the following Operation.

$$\begin{array}{rl}
yy-16\,) & y^6-8y^4-124yy-64\ (4+8yy+y^4 \\
 & \qquad\qquad +\ \ 4yy-64 \\ \hline
 & \qquad -8y^4-128yy \\
 & \qquad +8y^4-128yy \\ \hline
 & y^6-16y^4 \\
 & y^6-16y^4 \\ \hline
 & 0 \quad 0
\end{array}$$

More Examples are here added for the fuller exerciſe and illuſtration of Diviſion in compound Algebraic Integers, according to the general Method in Sect. IX. *of this* Chapter.

$$\begin{array}{lll}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.} \\
2c-3d) & 6ca-9da-8bc+12db & (3a-4b \\
 & 6ca-9da & \\ \hline
 & 0 \quad\ 0\ -8bc+12db & \\
 & \qquad\quad -8bc+12db & \\ \hline
 & \qquad\qquad 0 \quad\ 0 &
\end{array}$$

$$\begin{array}{lll}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.} \\
a-b) & aaa-3aab+3abb-bbb & (aa-2ab+bb \\
 & aaa-\ aab & \\ \hline
 & \quad -2aab+3abb & \\
 & \quad -2aab+2abb & \\ \hline
 & \qquad\quad +\ abb-bbb & \\
 & \qquad\quad +\ abb-bbb & \\ \hline
 & \qquad\qquad 0 \quad\ 0 &
\end{array}$$

$$\begin{array}{lll}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.} \\
2aa+3bb) & 4aaaa+12aabb+9bbbb & (2aa+3bb \\
 & 4aaaa+\ 6aabb & \\ \hline
 & \qquad +\ 6aabb+9bbbb & \\
 & \qquad +\ 6aabb+9bbbb & \\ \hline
 & \qquad\quad 0 \qquad\ 0 &
\end{array}$$

$$\begin{array}{ll}
a+b) & aaa-abb\ (aa-bb-ab+bb,\ \text{that is, } aa-ab \\
 & aaa+aab \\ \hline
 & \quad -abb-aab \\
 & \quad -abb-bbb \\ \hline
 & \qquad\quad -aab+bbb \\
 & \qquad\quad -aab-abb \\ \hline
 & \qquad\qquad\quad +bbb+abb \\
 & \qquad\qquad\quad +bbb+abb \\ \hline
 & \qquad\qquad\qquad 0 \quad\ 0
\end{array}$$

Again,

$$\begin{array}{lll}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.}\\
ab-aa) & aab^3+a^4b-2a^5 & (abb+a^3+aab+a^3\\
 & aab^3-a^3bb & \\
\cline{2-2}
 & +a^4b+a^3bb-2a^5 & \\
 & +a^4b \qquad -a^5 & \\
\cline{2-2}
 & +a^3bb-a^5 & \\
 & +a^3bb-a^4b & \\
\cline{2-2}
 & -a^5+a^4b & \\
 & -a^5+a^4b & \\
\cline{2-2}
 & 0 \quad 0 &
\end{array}$$

$$\begin{array}{lll}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.}\\
\frac{2}{3}ab-\frac{1}{2}aa) & \frac{4}{9}aab^3+\frac{13}{12}a^4b-a^5. & (\frac{2}{3}abb+\frac{13}{8}a^3+\frac{1}{2}aab+\frac{3}{8}a^3\\
 & \frac{4}{9}aab^3-\frac{1}{3}a^3bb & (\ \ viz.\ \frac{2}{3}abb+\frac{1}{2}aab+2a^3\\
\cline{2-2}
 & +\frac{13}{12}a^4b+\frac{1}{3}a^3bb-a^5 & \\
 & +\frac{13}{12}a^4b \qquad -\frac{13}{16}a^5 & \\
\cline{2-2}
 & +\frac{1}{3}a^3bb-\frac{3}{16}a^5 & \\
 & +\frac{1}{8}a^3bb-\frac{1}{4}a^4b & \\
\cline{2-2}
 & -\frac{3}{16}a^5+\frac{1}{4}a^4b & \\
 & -\frac{3}{16}a^5+\frac{1}{4}a^4b & \\
\cline{2-2}
 & 0 \quad 0 &
\end{array}$$

If Algebraical Diviſion according to this general Method will not work off juſt without a Remainder, then you may write the Dividend and Diviſor fraction-wiſe, according to *Sect.* III. of this *Chap.* Or ſometimes the Quotient may be expreſs'd partly by Integers, and partly by a Fraction; as if $bb+bd+cc$ be to be divided by $b+d$, the Quotient may be expreſs'd either thus $\frac{bb+bd+cc}{b+d}$; or elſe thus, $b+\frac{cc}{b+d}$, which latter Quotient is found out by the help of the ſaid general Method; for after you have thereby diſcovered as many Integers as can ariſe in the Quotient, you may ſet the Remainder of the Dividend as a Numerator over the Diviſor as a Denominator, ſo this Fraction together with the ſaid Integer or Integers ſhall be equal to the Quotient ſought; as in this following Example.

$$\begin{array}{lll}
\text{Diviſor.} & \text{Dividend.} & \text{Quotient.}\\
a-b) & 2aac+3aaa-2abc-3aab+2cc & (2ac+3aa+\frac{2cc}{a-b}\\
 & 2aac \qquad -2abc & \\
\cline{2-2}
 & +3aaa \qquad -3aab & \\
 & +3aaa \qquad -3aab & \\
\cline{2-2}
 & 0 \qquad 0\ +2cc &
\end{array}$$

CHAP. VI.

Containing the Arithmetic of Algebraical Fractions.

Of the riſe of Algebraic Fractions, and the manner of expreſſing Integers and mixed Quantities fraction-wiſe.

I. THE Operations about Algebraic Fractions are wrought like thoſe of vulgar Fractions, by the help of the Rules of Algebraic Integers before delivered, as will appear by the following Rules of this *Chapter.*

II. From the manner of dividing Quantities according to *Sect.* 3. of the preceding *Chap.* 5. Algebraic Fractions ariſe; as, if a be to be divided by b, the Quotient is repreſented by the Fraction $\frac{a}{b}$: Likewiſe $\frac{a+b}{c-d}$, which imports as much as the Quotient of $a+b$ divided by $c-d$; alſo $\frac{2aa+3cd}{bb}$, and ſuch like, are called Algebraical Fractions.

III. If the Numerator be equal to the Denominator, that Fraction (or Quotient expreſs'd fraction-wiſe) is equal to 1, (to wit, Unity;) as before hath been ſaid in *Sect.* 4. *Chap.* 5.

So $\frac{aa}{aa}=1.$ And $\frac{abc+dd}{abc+dd}=1.$

IV. When an Algebraic Integer is to be expreſs'd fraction-wiſe, make it a Numerator, and ſet 1 for the Denominator; as if theſe Quantities ab and $aa-bb$ be to be ſet in the Form of Fractions they will ſtand thus;

$\frac{ab}{1}.$ And $\frac{aa-bb}{1}.$

V. If an Algebraic Integer, as a, be to be ſet in the Form of a Fraction that ſhall have for its Denominator ſome Algebraical Integer preſcribed; as d, multiply a by the Denominator d, and write the Product ad as a Numerator over the Denominator d, thus, $\frac{ad}{d}$; which Fraction is equal to the Integer a firſt propoſed, and hath for its Denominator the preſcribed Quantity d.

Likewiſe the Quantity a reduced to the Form of a Fraction whoſe Denominator is preſcribed $b+c$ will ſtand thus, $\frac{ab+ac}{b+c}$.

Moreover, if $a+\frac{aa}{d}$ be to be reduced to the Form of a Fraction that ſhall have d for a Denominator; let a be multiplied by the Denominator d, and to the Product ad add the Numerator aa; then ſet that Sum, to wit, $ad+aa$ over the Denominator d, ſo there will be $\frac{ad+aa}{d}$ for the Fraction deſired. More Examples of this Rule are theſe following.

$$\frac{bc}{c}=b. \quad \Big| \quad \frac{aa+ab}{a+b}=a. \quad \Big| \quad \frac{dda}{a}=dd.$$

$$\frac{bc+bb}{c}=b+\frac{bb}{c} \quad \Big| \quad \frac{ab-ac+dd}{b-c}=a+\frac{dd}{b-c}.$$

How

How to reduce Algebraic Fractions to others of the ſame value in more ſimple Terms.

VI. When the ſame Letter or Letters be found in the Numerator and Denominator, let them be caſt out of both; and if the Numbers prefix'd can be abbreviated by ſome common Diviſor ſet the Quotients in the places of thoſe Numbers prefix'd, ſo ſhall the new Fraction be of the ſame value with that firſt propoſed: So this Fraction $\frac{abc}{abd}$ will be reduced to $\frac{c}{d}$; and this $\frac{12ab+8ac}{16ad}$ will be reduced to $\frac{3b+2c}{4d}$. This Rule hath already been explained in *Sect.* 5. and 6. of *Chap.* 5. and may be further illuſtrated by theſe following Examples.

$\frac{ad}{ac}=\frac{d}{c}$.	$\frac{12add}{4abc}=\frac{3dd}{bc}$.
$a+\frac{bcd}{cd}=a+b$.	$\frac{36aa}{4ba+16da}=\frac{9a}{b+4d}$.

VII. The ſearching out of the greateſt common Diviſor, for reducing an Algebraic Fraction to the ſmalleſt Terms, after the manner uſed in vulgar Arithmetic, is for the moſt part a tedious and intricate work, eſpecially when the Numerator and Denominator are compound Quantities conſiſting of many Members; and therefore inſtead of that way of finding out a common Meaſure (or Diviſor,) I ſhall by a clear Method in *Chap.* 8. of the Second *Book*, ſhew how to find out all ſuch Diviſors as will divide the Numerator and Denominator preciſely without leaving a Remainder. But in the mean time I ſhall recommend to the Learners exerciſe the following Examples of Fractions abbreviated by Diviſion according to the general Method in *Sect.* 9. *Chap.* 5. of this Book; which Examples, together with the Rule above-delivered in the 6. *Sect.* will be great helps to reduce Algebraical Fractions to lower terms, when there is a poſſibility.

Examples of Fractions reduced to their ſmalleſt Terms.

$\frac{aa+ab}{a+b}=a$	$\frac{aa-ab}{a-b}=a.$
$\frac{aac-aad}{c-d}=aa$	$\frac{aa+2ba+bb}{a+b}=a+b.$
$\frac{a^4+2b^2a^2+b^4}{aa+bb}=aa+bb$	$\frac{aa-2ba+bb}{a-b}=a-b.$
$\frac{a^4-2b^2a^2+b^4}{aa-bb}=aa-bb$	$\frac{aa-bb}{a+b}=a-b.$
$\frac{aaaa-bbbb}{aa+bb}=aa-bb$	$\frac{aa-bb}{a-b}=a+b.$
$\frac{aaaa-bbbb}{aa-bb}=aa+bb$	$\frac{aaa+bbb}{aa-ba+bb}=a+b.$

$\frac{aaa+bbb}{a+b}=aa-ba+bb$	$\frac{aaa-bbb}{aa+ba+bb}=a-b.$
$\frac{aaa-bbb}{a-b}=aa+ba+bb$	$\frac{aaa-abb}{aa-ab}=a+b.$
$\frac{aaa-abb}{aa+ab}=a-b$	$\frac{aaaa-bbbb}{aaa-aab+abb-bbb}=a+b.$

More Examples of Fractions abbreviated.

$\frac{aa+ab}{ad+bd}=\frac{a}{d}.$ (By the common Diviſor $a+b$)

$\frac{aa-ab}{ac-bc}=\frac{a}{c}.$ (By the common Diviſor $a-b$)

$\frac{aac-aad}{cd-dd}=\frac{aa}{d}.$ (By the common Diviſor $c-d$)

$\frac{aaa-abb}{aa+2ab+bb}=\frac{aa-ab}{a+b}.$ (By $a+b.$)

$\frac{aaa-bbb}{aa-bb}=\frac{aa+ba+bb}{a+b}.$ (By $a-b.$)

$\frac{a^4-b^4}{aa+ab}=\frac{aaa-aab+abb-bbb.}{a}$ (By $a+b.$)

How to find out the ſmalleſt Quantity that can be divided by two or more given Quantities ſeverally without a Remainder.

VIII. Two or more Algebraic Quantities whether Simple or Compound being propoſed, the ſmalleſt Quantity that can be divided by every one of thoſe given, without a Remainder, may be found out by the following Operation, (which is grounded upon 36 *Prop.* 7. *Elem.* Euclid.) and the Uſe thereof will hereafter appear.

As for Example, if it be deſired to find the ſmalleſt Quantity that can be divided by aac and cd; ſet them in the Form of a Fraction thus, $\frac{aac}{cd}$, and reduce the Fraction to its primitive or equivalent Fraction in the ſmalleſt Terms $\frac{aa}{d}$, which being ſet near the former, multiply croſs-wiſe, *viz.* aac by d, or aa by cd, and there will ariſe one and the ſame Product, to wit $aacd$ the Quantity ſought; which is the ſmalleſt Quantity that can be divided ſeverally by aac and cd without leaving any Remainder.

$$\frac{aac}{cd}\times\frac{aa}{d}$$

$$aacd$$

In like manner to find the ſmalleſt Quantity that can be divided by $ab+ac$ and $ad-af$ ſeverally, I ſet them Fraction-wiſe thus, $\frac{ab+ac}{ad-af}$, this reduced to its loweſt Terms gives $\frac{b+c}{d-f}$; then I multiply croſs-wiſe (as before) *viz.* $ab+ac$ by $d-f$ or $ad-af$ by $b+c$, and there ariſes $abd+acd-fab-fac$, which is the ſmalleſt Quantity that can be divided by $ab+ac$ and $ad-af$, ſo as to leave no Remainder.

$$\begin{array}{c}\dfrac{ab+ac}{ad-af} \times \dfrac{b+c}{d-f} \\ \hline abd+acd-fab-fac\end{array}$$

IX. But if the given Quantities cannot be reduced to lower Terms, then multiply them one into another, and their Product is the Quantity deſired: So to find the ſmalleſt Quantity that can be divided by $bb+cc$ and $dd+ff$ ſeverally without leaving a Remainder; becauſe $\frac{bb+cc}{dd+ff}$ cannot be reduced to more ſimple Terms, I multiply $bb+cc$ by $dd+ff$, and there is produced $bbdd+ccdd+bbff+ccff$ the Quantity ſought.

$$\begin{array}{c}\dfrac{bb+cc}{dd+ff} \times \dfrac{bb+cc}{dd+ff} \\ \hline bbdd+ccdd+bbff+ccff\end{array}$$

X. When three or more Quantities are given, the ſmalleſt Quantity that can be divided by them ſeverally without leaving a Remainder may be found out in this manner; *viz.* To find out the leaſt Quantity that can be divided by $aaa-abb$, $aa+2ab+bb$ and $aa-bb$; I firſt ſeek (after the Manner of the ſecond Example in *Sect.* 8.) the ſmalleſt Quantity that can be divided by $aaa-abb$, and $aa+2ab+bb$, ſo I find $aaaa-aabb+aaab-abbb$; and becauſe this Quantity may be alſo divided by $aa-bb$ (the third Quantity propoſed) it is manifeſt that $aaaa-aabb+aaab-abbb$ is the Quantity ſought.

$$\begin{array}{c}\dfrac{aaa-abb}{aa+2ab+bb} \times \dfrac{aa-ab}{a+b} \\ \hline aaaa-aabb+aaab-abbb\end{array}$$

In like manner, if there be given theſe four Quantities, $aaaa-bbbb$; $aa+ab$; $aaaa+aabb$; and $a+b$; Firſt, I find out (as before) the ſmalleſt Quantity $aaaaa-abbbb$ that can be divided by the firſt and ſecond Quantities $aaaa-bbbb$ and $aa+ab$;

$$\begin{array}{c}\dfrac{aaaa-bbbb}{aa+ab} \times \dfrac{aaa-aab+abb-bbb}{a} \\ \hline aaaaa-abbbb\end{array}$$

Then becauſe the ſaid $aaaaa-abbbb$ cannot be divided by the third Quantity $aaaa+aabb$, I ſeek the ſmalleſt Quantity that can be divided by $aaaaa-abbbb$ and $aaaa+aabb$, ſo I find (in like manner as before) $aaaaaa-aabbbb$, which, becauſe it is diviſible by the fourth Quantity propoſed, to wit, by $a+b$ ſhall be the Quantity ſought; *viz.* a^6-aab^4 is the ſmalleſt Quantity that can be divided by every one of theſe four Quantities, a^4-b^4; $aa+ab$; a^4+aabb; and $a+b$. And ſo of others.

$$\begin{array}{c}\dfrac{aaaaa-abbbb}{aaaa+aabb} \times \dfrac{aa-bb}{a} \\ \hline aaaaaa-aabbbb\end{array}$$

How to reduce Algebraical Fractions which have different Denominators, into other Fractions of the ſame value that may have a common Denominator.

XI. When two Fractions having different Denominators are to be reduced into two other Fractions of the ſame Value that ſhall have a common Denominator; multiply the Numerator of the firſt Fraction by the Denominator of the ſecond, and the Product ſhall be a new Numerator correſpondent to the Numerator of that firſt Fraction; Alſo, Multiply the Numerator of the ſecond Fraction by the Denominator of the firſt, and the Product is a new Numerator correſpondent to the Numerator of the ſecond Fraction; laſtly, multiply the Denominators one by the other, and the Product ſhall be a common Denominator to both the new Numerators.

As, for Example, to reduce $\frac{ab}{c}$ and $\frac{bd}{a}$ (whoſe Denominators c and a are unlike) into two other Fractions that may be of the ſame value with thoſe given, and have a common Denominator; Firſt, I multiply croſs-wiſe, *viz.* the Numerator ab by the Denominator a, and the Product is aab for a new Numerator inſtead of ab; likewiſe I multiply the Numerator bd by the Denominator c, and the Product is bdc, for a new Numerator inſtead of bd; laſtly, the Denominators c and a multiplyed one by the other produce ac for a Denominator to each of thoſe new Numerators aab and bdc: So the Fractions $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ are found out which have a common Denominator ac, and are equal in value to the Fractions firſt given, *viz.* $\frac{aab}{ac}$ is equal to $\frac{ab}{c}$, and $\frac{bdc}{ac}$ is equal to $\frac{bd}{a}$, as was required.

$$\begin{array}{ccc} \frac{ab}{c} & \times & \frac{bd}{a} \\ \hline \frac{aab}{ac}, & & \frac{bdc}{ac} \end{array}$$

In like manner $\frac{aa}{7bc}$ and $\frac{2bb}{5d}$ (which have unlike Denominators) will be reduced into $\frac{5daa}{35bcd}$ and $\frac{14bbbc}{35bcd}$ which have a common Denominator.

Alſo, $\frac{12}{a}$ and $\frac{b}{5}$ will be reduced into theſe $\frac{60}{5a}$ and $\frac{ba}{5a}$.

Again, to reduce $\frac{aa+2bb}{c+d}$ and $\frac{3cc-dd}{ff}$ to a common Denominator, I multiply croſs-wiſe (as before,) *viz.* $aa+2bb$ by ff, and $3cc-dd$ by $c+d$; ſo the Products are $aaff+2bbff$, and $3ccc-cdd+3ccd-ddd$ for New Numerators; then multiplying the Denominators $c+d$ and ff one into the other, the Product is $cff+dff$ for a common Denominator, and the Fractions ſought are $\frac{aaff+2bbff}{cff+dff}$ and $\frac{3ccc-cdd+3ccd-ddd}{cff+dff}$.

XII. When three or more Fractions having unlike Denominators are to be reduced into as many other Fractions that may be of the ſame value, and have a common Denominator; multiply the Numerator of each Fraction into all the Denominators except its own, ſo the Products made by that continual Multiplication ſhall be new Numerators; multiply alſo all the Denominators one into another, and the Product ſhall be a Denominator to every one of the new Numerators.

As, for Example, To reduce theſe three Fractions $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$ into three others that may be of the ſame value and have a common Denominator; I multiply the Numerator a into the Denominators d and g, ſo the Product adg is a new Numerator inſtead of a; again, I multiply the Numerator c into the Denominators b and g, and the Product cbg is a Numerator inſtead of c; likewiſe, multiplying the Numerator $2ef$ into the Denominators b and d, the Product $2bdef$ is a Numerator inſtead of $2ef$; laſtly, the Denominators b, d and g multiplied one into another produce bdg for a common Denominator to thoſe three new Numerators, and the three Fractions ſought are $\frac{adg}{bdg}$, $\frac{cbg}{bdg}$ and $\frac{2bdef}{bdg}$.

$$\begin{array}{ccc} \frac{a}{b}, & \frac{c}{d}, & \frac{2ef}{g} \\ \hline \frac{adg}{bdg}, & \frac{cbg}{bdg}, & \frac{2bdef}{bdg} \end{array}$$

In like manner theſe three Fractions $\frac{aa+8}{bb}$, $\frac{9}{aa-8}$, and $\frac{dd}{7}$ will be reduced to to theſe three, to wit, $\frac{7aaaa-448}{7aabb-56bb}$, $\frac{63bb}{7aabb-56bb}$, and $\frac{aaddbb-8ddbb}{7aabb-56bb}$ which have for a common Denominator $7aabb-56bb$.

XIII. But if the Denominators of the given Fractions can be reduced to lower Terms, then thoſe Fractions may oftentimes be reduced more compendiouſly than by the Rules in the two laſt preceding *Sections*, into others in the ſmalleſt Terms that have a common Denominator, in this manner; *viz.* Seek (by the Rules in *Sect* 8. and 10. of this *Chap.*) the ſmalleſt quantity that can be divided by every one of the Denominators without a Remainder, which quantity reſerve for a common Denominator; then for the Numerators divide the common Denominator by the Denominator of the firſt Fraction, and multiply the Quotient by the Numerator of the firſt Fraction, ſo ſhall the Product be a new Numerator inſtead of that firſt Numerator; work in like manner to find out the other Numerators, and ſet every one of them over the common Denominator before found out.

As, for Example, to reduce theſe Fractions $\frac{bbbd}{aac}$ and $\frac{aaa}{cd}$ to a common Denominator; I ſeek firſt of all the ſmalleſt quantity that can be divided by the Denominators aac and cd, and I find that quantity to be $aacd$, which ſhall be the common Denominator; then I divide the ſaid $aacd$ by each of the given Denominators aac and cd, and multiply the Quotients d and aa by the given Numerators $bbbd$ and aaa, ſo the Products $bbbdd$ and $aaaaa$ ſhall be the new Numerators, which being ſeverally ſet over the common Denominator $aacd$, there will ariſe $\frac{bbbdd}{aacd}$ and $\frac{aaaaa}{aacd}$ for the Fractions ſought.

Likewiſe, to reduce $\frac{bbbb}{aac-aad}$ and $\frac{aaa+bbb}{cd-dd}$ to a common Denominator, having firſt found the common Denominator $aacd-aadd$, to wit, the leaſt quantity that can be divided by the given Denominators $aac-aad$ and $cd-dd$, I divide the ſaid common Denominator by the ſaid given Denominators ſeverally, and the Quotients d and aa I multiply by the Numerators $bbbb$ and $aaa+bbb$, and then ſetting the Products ſeverally over the common Denominator, the Fractions ſought will be $\frac{bbbbd}{aacd-aadd}$ and $\frac{aaaaa+aabbb}{aacd-aadd}$.

Again, to reduce theſe three Fractions, to wit, $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$ and $\frac{aa-ab}{aa-bb}$ to a common Denominator; Firſt (as in the firſt Example in *Sect.* 10. of this *Chap.*) I ſeek the ſmalleſt quantity that can be juſt divided by every one of the three given Denominators, and I find $aaaa+aaab-aabb-abbb$, for a common Denominator; then dividing this quantity found by every one of the three given Denominators (according to the general Method in *Sect* 9. *Chap* 5.) the Quotients will be $a+b$, $aa-ab$ and $aa+ab$, that done, I multiply the firſt of thoſe Quotients by the Numerator of the firſt Fraction; alſo the ſecond Quotient by the ſecond Numerator, and the third Quotient by the third Numerator; ſo the Products $aa-bb$, $aabb-abbb$ and $aaaa-aabb$ ſhall be new Numerators, which being ſeverally ſet over the common Denominator firſt found, will give the Fractions ſought, to wit, theſe:

$$\frac{aa-bb}{aaaa+aaab-aabb-abbb},$$

$$\frac{aabb-abbb}{aaaa+aaab-aabb-abbb},$$

$$\frac{aaaa-aabb}{aaaa+aaab-aabb-abbb},$$

Nor will the Operation be otherwiſe to reduce theſe four Fractions to wit, $\frac{a^5}{a^4-b^4}$, $\frac{a^3-a^2b}{a^2+ab}$, $\frac{a^5-b^5}{a^4+a^2b^2}$ and $\frac{a^2+ab+b^2}{a+b}$, into theſe four following Fractions having a common Denominator.

1. $\frac{a^7}{a^6-a^2b^4}$;
2. $\frac{a^7-2a^6b+2a^5b^2-2a^4b^3+a^3b^4}{a^6-a^2b^4}$;
3. $\frac{a^7-a^5b^2-a^2b^5+b^7}{a^6-a^2b^4}$;
4. $\frac{a^7+a^5b^2-a^4b^3-a^2b^5}{a^6-a^2b^4}$.

For firſt by the help of the given Denominators, the ſmalleſt common Denominator a^6-aab^4 is found out by the Operation in the laſt Example of the preceding *Sect.* 10. (of this *Chap.*) then the ſaid common Denominator being divided ſeverally by the given Denominators a^4-b^4, $aa+ab$, a^4+aabb, and $a+b$; the Quotients are aa, $a^4-a^3b+aabb-ab^3$, $aa-bb$, and $a^5-a^4b+a^3bb-aab^3$; which multiplied reſpectively by the given Numerators a^5, a^3-aab, a^5-b^5, and $aa+ab+bb$, will produce thoſe new Numerators which are before ſet over the common Denominator a^6-aab^4.

Addition of Algebraical Fractions.

XIV. If two or more Fractions to be added have one common Denominator, add the Numerators together, and ſet their Sum as a new Numerator over the common Denominator, ſo ſhall this new Fraction be the Sum of the Fractions given to be added. As, for Example, to add $\frac{aa}{c}$ to $\frac{bb}{c}$, the Sum will be $\frac{aa+bb}{c}$.

So alſo, $\frac{2ab}{c+d}$ added to $\frac{3bb}{c+d}$ makes $\frac{2ab+3bb}{c+d}$.

Likewiſe the Sum of $\frac{5a-3b}{c+d}$ and $\frac{2b-3a}{c+d}$ will be found $\frac{2a-b}{c+d}$; (For the given Numerators $5a-3b$ and $2b-3a$ added together make $2a-b$.)

Again, the Sum of $\frac{a-b+24}{c+5}$, $\frac{a+b-24}{c+5}$ and $\frac{4a}{c+5}$ will be found $\frac{6a}{c+5}$.

And if theſe be added, to wit, $\frac{3ab}{b+c+d}$, $a+\frac{3ac}{b+c+d}$, and $\frac{3ad}{b+c+d}$; the Sum will be $a+\frac{3ab+3ac+3ad}{b+c+d}$; that is, $4a$. (For by Diviſion, $\frac{3ab+3ac+3ad}{b+c+d}=3a$.)

XV. But if the Fractions propos'd to be added together have unlike Denominators, firſt reduce them to a common Denominator, and then add them as before; as to add $\frac{ab}{c}$ to $\frac{bd}{a}$, firſt I reduce them to $\frac{aab}{ac}$ and $\frac{bdc}{ac}$ which have the ſame Denominator ac; then ſetting the Sum of the Numerators aab and bdc over the common Denominator ac, there will be $\frac{aab+bdc}{ac}$ for the Sum required.

So alſo to add $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{2ef}{g}$, their Sum will be found $\frac{adg+cbg+2bdef}{bdg}$

Likewiſe, to add theſe three Fractions $\frac{a-b}{aaa-abb}$, $\frac{bb}{aa+2ab+bb}$, and $\frac{aa-ab}{aa-bb}$; firſt I reduce them to three others of the ſame value under a common Denominator, (as in the third Example of the preceding 13. *Sect.*) and then ſetting the Sum of the three new Numerators over the common Denominator, I find the Sum of the given Fractions to be $\frac{aaaa+aa-abbb-bb}{aaaa+aaab-aabb-abbb}$.

XVI. When mixed quantities are to be added together, collect the Fractions into one Sum, and the Integers into another, then thoſe two Sums added together give the Sum deſired; as for Example:

To add theſe mixed quantities . . . $\frac{aa}{b}-a$ and $\frac{dd}{c}+d$.

The Sum of the Fractions, after they are reduced to a common Denominator, is . . . . $\frac{caa+bdd}{bc}$

To which Sum adding the Integers in the mix'd quantities propoſed, the Sum deſir'd will be $\frac{caa+bdd}{bc}-a+d$.

Or, when mixed quantities are to added together, you may reduce them to improper Fractions, (by *Sect.* 5. of this *Chap.*) and then add theſe together as in the preceding Examples; as,

To add thoſe mixed quantities in the laſt Example, to wit, . . . $\frac{aa}{b}-a$ and $\frac{dd}{c}+d$;

I Firſt reduce them to theſe Fractions . . . . $\frac{aa-ba}{b}$ and $\frac{dd+cd}{c}$;

Which reduced to a common Denominator produce theſe . . $\frac{caa-cba}{bc}$ and $\frac{bdd+bcd}{bc}$

Which two laſt Fractions added together give the Sum required, to wit, ... $\frac{caa-cba+bdd+bcd}{bc}$

Which is equal to the Sum before found, to wit, ... $\frac{caa+bdd}{bc}-a+d$.

Subtraction of Algebraical Fractions.

XVII. If the two Fractions given have the ſame Denominator, ſubtract the Numerator of the Fraction preſcribed to be ſubtracted, from the other Numerator, and ſet the Remainder as a new Numerator over the common Denominator, ſo ſhall this new Fraction be the remainder ſought.

As, for Example, If from $\frac{aa}{c}$ you deſire to ſubtract $\frac{bb}{c}$, take bb from aa, and ſet the Remainder $aa-bb$ as a Numerator over the common Denominator c; ſo $\frac{aa-bb}{c}$ ſhall be the Remainder ſought.

In like manner, If from $\frac{2ab}{b-c}$ you would ſubtract $\frac{2ac}{b-c}$, the Remainder will be $\frac{2ab-2ac}{b-c}$, that is, (by Diviſion) $2a$.

Again, if from $\frac{8aa-7b+6}{a+b}$ it be deſired to ſubtract $\frac{3aa+12b-18}{a+b}$, the

Remainder will be found $\frac{5aa-19b+24}{a+b}$. (For $3aa+12b-18$ ſubtracted from $8aa-7b+6$, leaves $5aa-19b+24$.)

So alſo, from $d+\frac{bb}{b+d}$ ſubtracting $\frac{bd}{b+d}$, there remains $\frac{dd+bb}{b+d}$. For, (by *Sect.* 5. of this *Chap.*) $d+\frac{bb}{b+d}$ will be reduced to $\frac{db+dd+bb}{b+d}$; from which ſubtracting $\frac{bd}{b+d}$, the Remainder is $\frac{dd+bb}{b+d}$.

XVIII. But if the two Fractions given have different Denominators, firſt reduce them to a common Denominator, and then ſubtract as before; ſo as from $\frac{dd}{c}$ it be deſired to ſubtract $\frac{aa}{b}$, I reduce them to $\frac{ddb}{cb}$ and $\frac{aac}{cb}$, which have the ſame Denominator cb; then from $\frac{ddb}{cb}$ ſubtracting $\frac{aac}{cb}$, there remains $\frac{ddb-aac}{cb}$, which is the Remainder ſought.

After the ſame manner, If from $\frac{aa+d}{b-c}$ you would take away $\frac{aa}{b}$, there will remain $\frac{db+aac}{bb-bc}$.

Likewiſe from $\frac{aaa+bbb}{cd-dd}$ to take away $\frac{bbbb}{aac-aad}$, I firſt reduce theſe given Fractions to a common Denominator, (as in the ſecond Example of *Sect.* 13. of this *Chap.*) and ſo I find $\frac{aaaaa+aabbb}{aacd-aadd}$ and $\frac{bbbbd}{aacd-aadd}$, which latter Fraction ſubtracted from the former there remains $\frac{daaaa+aabbb-bbbbd}{aacd-aadd}$.

Again, If from a it be deſired to ſubtract $\frac{aa-ab}{a+b}$, I reduce a into the form of a Fraction whoſe Denominator ſhall be $a+b$, and ſo inſtead of a, I find $\frac{aa+ab}{a+b}$, from which ſubtracting $\frac{aa-ab}{a+b}$, there remains $\frac{2ab}{a+b}$

Multiplication of Algebraical Fractions.

XIX. When two Algebraic Fractions are given to be multiplied one by the other, multiply their Numerators one into the other, and take the Product for a new Numerator; likewiſe multiplying the Denominators one into the other, this Product ſhall be a new Denominator, and the new Fraction is the Product ſought.

As, for Example, to multiply $\frac{2a}{c}$ by $\frac{b}{3d}$, I multiply (as in vulgar Fractions) the Numerator $2a$ by the Numerator b, and the Product $2ab$ is a new Numerator; likewiſe I multiply the Denominators, $3d$ and c one into the other, and the Product $3dc$ ſhall be a new Denominator; ſo $\frac{2ab}{3dc}$ is the Product ſought.

In like manner, $\frac{aa-bb}{c}$ multiplied by $\frac{2ab}{b+c}$ gives the Product $\frac{2aaab-2abbb}{bc+cc}$

XX. When either or both the given Terms are mixed Quantities, reduce the mixt Quantity to the form of a Fraction (by the Rule in *Sect.* 5. of this *Chap.*) and then multiply as before: So to multiply $c+\frac{bb}{d}$ by $a+\frac{ad}{c-d}$, I firſt Reduce thoſe mixt Quantities to theſe Fractions, $\frac{cd+bb}{d}$ and $\frac{ac}{c-d}$, then multiplying the Numerator $cd+bb$ by the Numerator ac, the Product is $accd+acbb$ for a new Numerator; alſo multiplying the Denominators d and $c-d$ one by the other, the Product is $dc-dd$ for a new Denominator; and the Product ſought is $\frac{accd+acbb}{dc-dd}$.

XXI. When an Integer is to be Multiplied by a Fraction, expreſs the Integer Fraction-wiſe by giving it unity, (to wit, 1) for a Denominator, (according to *Sect.* 4. of this *Chap.*) and then multiply as in the preceding Examples.

As, to multiply a by $\frac{b}{c}$, that is, $\frac{a}{1}$ by $\frac{b}{c}$, the Product will be $\frac{ab}{c}$. Likewiſe to Multiply $aa-bb$ by $\frac{aa+bb}{cd+fg}$, I reduce $aa-bb$ to $\frac{aa-bb}{1}$, then multiplying the Numerator $aa+bb$ by the Numerator $aa-bb$, the Product $aaaa-bbbb$ ſhall be a New Numerator; Likewiſe the Denominator $cd+fg$ multiplied by the Denominator 1 gives $cd+fg$ for a New Denominator, and the New Fraction $\frac{aaaa-bbbb}{cd+fg}$ is the Product ſought.

XXII. But oftentimes there may be this uſeful Contraction in the Multiplication of Fractions, *viz.* When the Numerator of the one and the Denominator of the other may be ſeverally divided by ſome common Diviſor without a Remainder, take the Quotients inſtead of the ſaid Numerator and Denominator, and then multiply as in the preceding Examples.

As, for Example, to multiply $\frac{aa+2ab+bb}{cd-dd}$ by $\frac{dd}{a+b}$:

Foraſmuch as the Numerator of the firſt Fraction and the Denominator of the latter may be divided ſeverally by $a+b$ without a Remainder, I ſet the Quotients $a+b$ and 1 in the places of $aa+2ab+bb$ and $a+b$, and by that exchange theſe Fractions will ariſe, to wit;

$$\frac{a+b}{cd-dd} \text{ and } \frac{dd}{1}.$$

In like manner, becauſe $cd-dd$ the Denominator of the firſt of the two Fractions laſt above-written, and dd the Numerator of the latter Fraction, may be ſeverally divided by d without a Remainder, I ſet the Quotients $c-d$ and d in the Places of $cd-dd$, and dd, and ſo theſe new Fractions ariſe, to wit;

$$\frac{a+b}{c-d} \text{ and } \frac{d}{1},$$

Then I multiply (as before) the Numerators $a+b$ and d, one by the other, and the Product $da+db$ is a New Numerator: Alſo multiplying the Denominator $c-d$ by the Denominator 1, the Product $c-d$ is a new Denominator, and the new Fraction $\frac{da+db}{c-d}$ is the Product ſought; being equal to that which would be made by the mutual Multiplication of $\frac{aa+2ab+bb}{cd-dd}$ and $\frac{dd}{a+b}$ the Fractions firſt propoſed to be Multiplied.

So as alſo, If it be deſired to Multiply $a+\frac{bb}{a-b}$ by $a-2b+\frac{bb}{a}$, that is, $\frac{aa-ab+bb}{a-b}$ by $\frac{aa-2ab+bb}{a}$; Foraſmuch as the Numerator $aa-2ab+bb$ of the latter Fraction, and the Denominator $a-b$ of the former, being ſeverally divided by their common Diviſor $a-b$ will give the Quotients $a-b$ and 1; therefore I ſet theſe in the places of $aa-2ab+bb$ and $a-b$, whence theſe Fractions will ariſe, to wit,

$$\frac{aa-ab+bb}{1} \text{ and } \frac{a-b}{a}:$$

Which being multiplied one by the other will give $\frac{aaa-2aab+2abb-bbb}{a}$, or $aa-2ab+2bb-\frac{bbb}{a}$, the Product ſought.

Again, this Fraction $\frac{aac-aad-bbc+bbd}{aa+2ab+bb}$ multiplied by $\frac{aaa-abb}{cd-dd}$, will produce $\frac{aaaa-aaab-aabb+abbb}{ad+bd}$; For the Numerator of the firſt Fraction and the Denominator of the latter being ſeverally divided by their common Diviſor $c-d$ the Quotients will be $aa-bb$ and d; Alſo, the Denominator of the firſt Fraction and the Numerator of the ſecond being ſeverally divided by their common Diviſor $a+b$, the Quotients will be $a+b$ and $aa-ab$; then ſetting the two former Quotients in the places of the two firſt Dividends, and the two latter Quotients in the places of the two latter Dividends, theſe two Fractions will ariſe, to wit;

$$\frac{aa-bb}{a+b} \text{ and } \frac{aa-ab}{d}:$$

Laſtly, multiplying the Numerators $aa-bb$ and $aa-ab$ one into the other; as alſo the Denominators $a+b$ and d, (as in former Examples,) you will find the Product ſought, to wit;

$$\frac{aaaa-aaab-aabb+abbb}{ad+bd}.$$

XXIII. When a Fraction is to be multiplied by ſome Integer that happens to be the ſame with the Denominator of the Fraction, take the Numerator for the Product required. As, for Example, to multiply $\frac{aa+ab+bb}{a+d}$ by $a+d$; I write $aa+ab+bb$ for the Product of their multiplication.

Likewiſe, If $\frac{b}{c}$ be to be multiplied by the Denominator c; I write the Numerator b for the Product. The reaſon of this Contraction is Evident; for if $\frac{b}{c}$ be multiplied by c, or $\frac{c}{1}$, in the ordinary way, the Product will ſtand thus, $\frac{bc}{c}$, which, by caſting away the common Factor c out of the Numerator and Denominator, gives b for the Product; to wit, the Numerator of the given Fraction $\frac{b}{c}$.

Hence alſo, if an Algebraical Fraction be to be multiplied by ſome letter or letters that are found among others in every Member of the Denominator, that multiplication needs no other work but the caſting away ſuch letter or letters out of the Denominator: As to multiply $\frac{ab}{cd}$ by c, the Product is $\frac{ab}{d}$; where obſerve, that becauſe the multiplier c is found in the given Denominator cd, I ſtrike it quite out.

Likewiſe, to multiply $\frac{ab}{cd}$ by d, I write $\frac{ab}{c}$ for the Product: And to multiply $\frac{bbb-ccc}{3faa-3gaa}$ by $3aa$, I cancel $3aa$ in the Denominator, and write $\frac{bbb-ccc}{f-g}$ for the Product required.

Note. The taking of $\frac{2}{3}$ parts of the quantity a, imports the ſame thing with the multiplying of a by $\frac{2}{3}$, and the Product may be expreſt either thus, $\frac{2a}{3}$; or thus, $\frac{2}{3}a$.

Likewiſe $\frac{2}{5}$ of $b+c$, or the Product of $b+c$ multiplied by $\frac{2}{5}$, may be expreſt ei- either thus $\frac{2b+2c}{5}$, or thus, $\frac{2}{5}b+\frac{2}{5}c$. And ſo of others.

Diviſion in Algebraical Fractions.

XXIV. When the two given Fractions, to wit, the Dividend and Diviſor, have a common Denominator, caſt away the Denominator, and divide the Numerator of the Dividend by the Numerator of the Diviſor; ſo that which ariſes ſhall be the Quotient ſought. As, to divide $\frac{aab}{c}$ by $\frac{bb}{c}$; I caſt away the common Denominator c, and divide aab by bb, ſo the Quotient ſought is $\frac{aab}{bb}$, that is, $\frac{aa}{b}$.

In like manner, $\frac{aabb}{d}$ divided by $\frac{ab}{d}$ gives $\frac{aabb}{ab}$, that is, ab for the Quotient.

Again, If $\frac{aaa-abb}{c-d}$ be divided by $\frac{aa+2ab+bb}{c-d}$ there will ariſe $\frac{aaa-abb}{aa+2ab+bb}$, which abbreviated (by dividing the Numerator and Denominator ſeverally by their common Diviſor $a+b$) gives $\frac{aa-ab}{a+b}$ the Quotient ſought.

XXV. If the given Fractions have not a common Denominator, then (as in Diviſion of vulgar Fractions) multiply the Numerator of the Dividend by the Denominator of the Diviſor, and the Product ſhall be a new Numerator; alſo, multiply the Denominator of the Dividend by the Numerator of the Diviſor, and the Product ſhall be a new Denominator; ſo the new Fraction is the Quotient ſought.

As, for Example, to divide $\frac{ab}{c}$ by $\frac{dd}{a}$, I multiply ab by a, and the Product is aab for a new Numerator; alſo, multiplying c by dd, the Product is ddc for a new Denominator; ſo the Quotient ſought is $\frac{aab}{ddc}$.

$$\left.\frac{dd}{a}\right)\quad \frac{ab}{c}\quad \left(\frac{aab}{ddc}\right.$$

Likewiſe, If $\frac{aa-bb}{c+d}$ be divided by $\frac{c-d}{aa+bb}$, the Quotient will be $\frac{aaaa-bbbb}{cc-dd}$; For $aa-bb$ the Numerator of the Dividend being multiplied by $aa+bb$ the Denominator of the Diviſor, the Product $aaaa-bbbb$ is the new Numerator: and $c+d$ the Denominator of the Dividend being multiplied by $c-d$ the Numerator of the Diviſor produces $cc-dd$ for a new Denominator; whence the Quotient ſought is $\frac{aaaa-bbbb}{cc-dd}$.

XXVI. But oftentimes there may be this uſeful Contraction in the Diviſion of Fractions, *viz*, when either the two Numerators, or the two Denominators may be divided by ſome common Diviſor without a Remainder, ſet the Quotients ariſing out of ſuch Diviſion (or imagine them to be ſet) in the places of the ſaid Numerators or Denominators that were divided, and then divide as in the former Examples.

As, to divide $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$; Foraſmuch as the Numerators $aa-ab$ and $a-b$ may be reduced to more ſimple Terms, to wit, a and 1, (for $aa-ab$ and $a-b$ being ſeverally divided by their common Meaſure $a-b$ give a and 1. And, becauſe the Denominators cc and cd may likewiſe be reduced to more ſimple Terms c and d, (by dividing the ſaid cc and cd by their common Diviſor c,) therefore in the places of the two given Numerators $aa-ab$ and $a-b$ I ſet the two former Quotients a and 1, and in the places of the two given Denominators cc and cd I ſet the two latter Quotients c and d; ſo there will be $\frac{a}{c}$ and $\frac{1}{d}$ for a new Dividend and Diviſor; then (as before) I multiply a by d, and the Product is ad or da for a new Numerator; Alſo, c multiplied by 1 gives c for a new Denominator, and the new Fraction $\frac{da}{c}$ is the

$$\left.\frac{1}{d}\right)\quad \frac{a}{c}\quad \left(\frac{da}{c}\right.$$

the Quotient ſought; which is equal to that which would ariſe by dividing $\frac{aa-ab}{cc}$ by $\frac{a-b}{cd}$, to wit, the Fractions firſt propoſed.

Again, If it be deſired to divide $\frac{aaaa-bbbb}{aa-2ab+bb}$ by $\frac{aa+ab}{a-b}$; Foraſmuch as the Numerators $aaaa-bbbb$ and $aa+ab$ may be reduced to $aaa-aab+abb-bbb$ and a by their common Diviſor $a+b$; and the Denominators $aa-2ab+bb$ and $a-b$ may be reduced to $a-b$ and 1, by the common Diviſor $a-b$; therefore inſtead of multiplying $aaaa-bbbb$ by $a-b$, I multiply the ſaid $aaa-aab+abb-bbb$ by 1, and the Product is $aaa-aab+abb-bbb$ for a new Numerator; and inſtead of multiplying $aa-2ab+bb$ by $aa+ab$, I multiply $a-b$ by a; ſo the Product $aa-ab$ ſhall be a new Denominator, whence the Quotient ſought is $\frac{aaa-aab+abb-bbb}{aa-ab}$.

In like manner, If $\frac{aaaa-625}{aa-10a+25}$ be divided by $\frac{aa+5a}{a-5}$, the Quotient will be $\frac{aaa-5aa+25a-125}{aa-5a}$; For $aaaa-625$ and $aa+5a$ may be reduced to $aaa-5aa+25a-125$, and a by the common Diviſor $a+5$; Alſo, $aa-10a+25$ and $a-5$ may be reduced to $a-5$ and 1 by the common Diviſor $a-5$ and 1; whence inſtead of the Fractions given we may divide

$$\frac{aaa-5aa+25a-125}{a-5} \text{ by } \frac{a}{1},$$

and the Quotient ſought will be $\frac{aaa-5aa+25a-125}{aa-5a}$.

Again, to divide $aaa-2aab+abb$ by $\frac{aa-ab}{a+b}$, I ſet 1 for a Denominator under the Dividend $aaa-2abb+abb$, and it ſtands thus $\frac{aaa-2aab+abb}{1}$; then foraſmuch as the Numerators $aaa-2aab+abb$ and $aa-ab$ may be reduced to $a-b$ and 1, (by the common Diviſor $aa-ab$) therefore inſtead of the given Dividend and Diviſor we may take $\frac{a-b}{1}$ and $\frac{1}{a+b}$, whence the Quotient ſought will be found $aa-bb$.

So alſo, If $a+\frac{3abb}{a+4b}$ be to be divided by $a+b$, that is, $\frac{aaa+4aab+3abb}{a+4b}$ by $\frac{a+b}{1}$, the Quotient will be found $\frac{aa+3ab}{a+4b}$: And $\frac{xx+5x}{x-5}$ divided by $xx+5x$, gives the Quotient $\frac{1}{x-5}$: Laſtly, $\frac{xx+5x}{x-5}$ divided by $x+5$ gives the Quotient $\frac{x}{x-5}$.

CHAP. VII.

The Rule of Three in Quantities repreſented by Letters.

I. AS in Vulgar Arithmetic ſo here in Algebraical, if three Quantities be given to find out a fourth in a direct Proportion, that is, when the Nature of the Queſtion is ſuch; that as the firſt Term is in proportion to the ſecond, ſo is the third to the fourth ſought; then (reſpect being had to the preceding Rules of Algebraical Multiplication and Diviſion) multiply the ſecond and third Terms one into another, and divide the Product by the firſt Term; ſo the Quotient ſhall be the fourth Proportional ſought.

As for Example, If the Quantity a give b, what ſhall c give, in a direct Proportion? Or, to the ſame Effect, find out a Quantity which ſhall have the ſame proportion to c, as b has to a; here I multiply b by c, and then dividing the Product bc by a, the Quotient, $\frac{bc}{a}$ is the fourth Proportional ſought; as will appear by the Proof of the Rule of Three direct: For if the fourth Term $\frac{bc}{a}$ be multiplied by the firſt Term a, the Product will be $\frac{abc}{a}$, which (by *Sect.* 5. *Chap.* 5.) is equal to bc, to wit, the Product of the ſecond Term multiplied by the third.

$$a . b :: c . \frac{bc}{a}$$

The Proof,

$$\frac{abc}{a} = bc.$$

In like manner, If $a+b$ give d, what ſhall $c+d$ give, in a Direct proportion? Anſwer, $\frac{dc+dd}{a+b}$

Again, If 4 gives 3, what ſhall $8aa$ give? Anſw. $\frac{24aa}{4}$ that is, $6aa$.

Moreover, If $aaa-aab+abb-bbb$ give $aa+bb$, what ſhall $aa-bb$ give? Anſw. $a+b$: For the ſecond and third Term being multiplied one by the other will produce $aaaa-bbbb$, which divided by the firſt Term $aaa-aab+abb-bbb$ (according to the general Method of Diviſion in *Sect.* 9. *Chap.* 5.) gives $a+b$ the fourth Proportional ſought.

II. When any one of the three given Quantities is an Algebraic Fraction, ſet the other two if they be Integers, in the form of Fractions, by placing 1 as a Denominator under each Integer.

Alſo, when any one of the three given Quantities is compos'd of an Integer and a Fraction, let it be reduced into the Form of a Fraction, (by *Sect.* 5. *Chap.* 6.) then if the Proportion be Direct, multiply and divide as before.

As for Example, If $a+\frac{bb}{c}$ give cd, what ſhall $\frac{ab}{f}$ give in a direct proportion? Anſw. $\frac{abccd}{acf+bbf}$: For firſt, $a+\frac{bb}{c}$ being reduced to the Form of a Fraction will ſtand thus $\frac{ac+bb}{c}$; alſo cd ſet Fraction-wiſe is $\frac{cd}{1}$; then multiplying the third Term $\frac{ab}{f}$ by the ſecond Term $\frac{cd}{1}$, the Product is $\frac{abcd}{f}$, which divided by the firſt Term $\frac{ac+bb}{c}$ gives $\frac{abccd}{acf+bbf}$ for the fourth Proportional ſought.

In like manner, If $\frac{ab}{c}$ give d, then $\frac{bb}{d}$ will give $\frac{cdbb}{abd}$, that is, $\frac{cb}{a}$, (for $\frac{cdbb}{abd}$ being abbreviated according to *Sect.* 5. *Chap.* 5. gives $\frac{cb}{a}$.

Alſo, If $\frac{a+c}{d}$ give $\frac{aa}{bb}$; then $\frac{bb}{a-c}$ will give $\frac{daa}{aa-cc}$.

III. If after the three given Quantities are ordered or ſet in the Rule according to the uſual manner in Vulgar Arithmetic; the Proportion flows backwards, *viz.* if the Nature of the Queſtion be ſuch, that as the third Term is in proportion to the ſecond, ſo is the firſt to the fourth Term ſought; then (as in the Inverſe or backward Rule of Three in Vulgar Arithmetic) multiply the firſt and ſecond Terms one by the other, and divide the Product by the third, ſo the Quotient ſhall be the fourth Proportional ſought. But I ſhall not need to give Examples of this Rule, nor to make application of Algebraical Arithmetic to the Double Rule of Three, Rules of Fellowſhip and Alligation; ſince he that underſtands the manner of working thoſe Rules in Vulgar Arithmetic, as alſo the Rules of Algebraical Arithmetic before delivered, cannot miſs of performing the like work Algebraically when there is occaſion.

CHAP. VIII.

An Introduction to the Extraction of ROOTS out of Algebraical Quantities.

I. IT is not my deſign in this Chapter to treat of the Extraction of *Roots* in general, (that Doctrine being hereafter handled in the third and fourth Chapter of the ſecond Book) but chiefly to ſhew how to extract the Roots or ſides of Simple Powers expreſs'd by Letters, as alſo of Squares formed from Rational Binomial Roots, in order to the Explication of divers Equations in the following Chapters: For I would not willingly affright the Learner with tedious and intricate Operations until he has had a conſiderable Taſte of the practice of Algebra in the ſolving of Arithmetical Queſtions.

II. As in Vulgar Arithmetic, the Extraction of the Square root of a given Number imports nothing elſe but the finding out ſuch a Number that being multiplied by it ſelf will produce the given Number; ſo the Extracting of the Square root of the Quantity aa implies only the finding out ſuch a Quantity, which if it be multiplied by it ſelf will produce aa; and ſince a multiplied by a produces aa, therefore a is the Root or ſide of the Square aa.

Likewiſe the ſquare Root of $4bb$ is $2b$; for $2b$ multiplied by $2b$ produces $4bb$: And for the ſame Reaſon, the ſquare Root of $\frac{1}{4}aa$ (or $\frac{aa}{4}$) is $\frac{1}{2}a$; (or $\frac{a}{2}$.) Alſo, the ſquare Root of $bbaa$ is ba; and the ſquare Root of $aaaa$ is aa.

Moreover, Foraſmuch as aa, or the Square of the Root a, being multiplied by the Root a produces aaa, or the Cube of a, therefore the Cubic Root of aaa being extracted there will come forth again the Root a. In like manner, the Cubic Root of $8aaa$ is $2a$; for $2a$ multiplied cubically, (that is, firſt by it ſelf and then again by the Product) produces $8aaa$.

III. The like is to be underſtood in the Extraction of the Root of a compound Power; For, as the Binomial Root $a+b$, which may repreſent the Sum of the two parts into which ſome Number or Right-line is divided, being ſquared or multiplied by it ſelf, produces the Square $aa+2ab+bb$; So the ſquare Root of $aa+2ab+bb$ being extracted, there will ariſe the Root $a+b$. Here the Learner may obſerve, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa+2ab+bb$) which is made of $a+b$ the Sum of the Parts, is compoſed of (aa and bb) the Squares of the Parts, and of ($2ab$) the Double Product made by the Multiplication of the Parts (a and b) one into the other.

$$\begin{array}{ll} a+b. & \text{The Root.} \\ \underline{a+b} & \\ aa+ab & \\ \underline{+ab+bb} & \\ aa+2ab+bb. & \text{The Square.} \end{array}$$

So the Square of 8, or of $5+3$, is equal to $25+9+30$, that is, 64.

Again, As the Binomial, or (as ſome call it) the Reſidual Root $a-b$, or $b-a$ being multiplied by it ſelf produces the Square $aa-2ab+bb$; So the ſquare Root of $aa-2ab+bb$ being extracted, there will come forth the Root $a-b$, or $b-a$; (for either of theſe Roots will produce the ſame Square.) Here alſo the Learner may obſerve, That if a Number or Right-line be divided into any two parts, (a and b) the Square ($aa-2ab+bb$) which is made by the Multiplication of ($a-b$, or $b-a$) the Difference of the Parts into it ſelf, is equal to ($aa+bb$) the Sum of the Squares of the Parts, leſs by ($2ab$) the double Product of the Multiplication of the Parts one into the other: So the Square of $5-3$, that is, of 2, is equal to $25+9-30$, that is, 4.

$$\begin{array}{rl} a-b. & \text{The Root.} \\ a-b & \\ \hline aa-ab & \\ -ab+bb & \\ \hline aa-2ab+bb. & \text{The Square.} \end{array}$$

IV. From what has been ſaid in the laſt Section, this Theorem may be inferr'd, *viz.* If a compound Quantity conſiſts of three ſuch Members or ſimple Quantities, that two of them are Squares, each of them having the ſign + prefix'd to it, and the third is the double Product made by the mutual Multiplication of the Roots of thoſe ſimple Squares, the ſaid double Product alſo having the ſign + prefix'd to wit; that compound Quantity ſhall be a Square whoſe Root is the Sum of the two Roots of the ſaid two ſimple Squares: But if the ſaid double Product has the Sign — prefix'd to it, then the difference of the ſaid Roots ſhall be the Root of the ſaid compound Square.

Hence $aa+6a+9$ will be found a Square, whoſe Root is $a+3$; for it is evident that aa and 9 are Squares, whoſe Roots are a and 3; and $6a$ is the double Product of the Multiplication of thoſe Roots a and 3 one by the other.

Likewiſe, $9bb+6bc+cc$ is a Square, whoſe Root is $3b+c$; for $9bb$ and cc are Squares whoſe Roots are $3b$ and c, and $6bc$ is the double Product of the Multiplication of the Roots $3b$ and c one into the other. Alſo, $aaaa+baa+\frac{1}{4}bb$ will be found a Square, whoſe Root is $aa+\frac{1}{2}b$.

Moreover, (agreeable to the latter Caſe in the Theorem) This compound Quantity $aa-10a+25$ will be diſcovered to be a Square whoſe Root is $a-5$, or $5-a$. And $bbaa-2bca+cc$ is a quare whoſe Root is $ba-c$, or $c-ba$; For from either of theſe Roots the ſame Square $bbaa-2bca+cc$ will be produced by Algebraical Multiplication.

If the Learner be well vers'd in this Theorem, he may oftentimes diſcern at firſt ſight whether a compound Quantity that conſiſts of three Members or ſingle Quantities be a Square or not; and if a Square, what its Root is.

V. If a Quantity out of which a Root is to be extracted be ſuch, that the Root cannot any manner of way be exactly extracted; that Root is uſually deſign'd or repreſented by prefixing the radical ſign before the Quantity propoſed. So to extract the ſquare Root of the Quantity a, (whether it repreſents a plane Number or a Superficies) I write $\sqrt{}a$, or $\sqrt{}(2)a$, which ſignifies that the ſquare Root of a is extracted or to be extracted.

So alſo, $\sqrt{}:\overline{aa+bb}:$ or, $\sqrt{}(2):\overline{aa+bb}:$ denotes the ſquare Root of the Sum of the Squares aa and bb.

Likewiſe, to extract the Cubic Root of b, I write $\sqrt{}(3)b$; as alſo $\sqrt{}(3)\,aab$, to ſignifie the Cubic Root of aab; which kind of Roots are called Surd or Irrational Quantities. (As hereafter in *Chap.* 9. of the II. *Book* will be more fully declared.)

VI. When it is required to extract the Root of a Fraction, the Root of the Numerator and the Root of the Denominator ſhall give a new Fraction which is the Root ſought. As for Example, If the ſquare Root of $\frac{aa}{bb}$ be deſired; foraſmuch as the ſquare Root of aa is a, and the ſquare Root of bb is b; I write $\frac{a}{b}$ for the Root ſought.

In like manner, the ſquare Root of $\frac{aabb}{dd}$ is $\frac{ab}{d}$; (for the ſquare Root of $aabb$ is ab, and the Root of dd is d.)

Again, the ſquare Root of $\frac{aa+6a+9}{bb}$ is $\frac{a+3}{b}$; For (by the foregoing *Sect.* 4.) the ſquare Root of the Numerator $aa+ba+9$ is $a+3$; and the ſquare Root of the Denominator bb is b. Alſo, the ſquare Root of $\frac{9bb+6bc+cc}{\frac{1}{4}dd}$ is $\frac{3b+c}{\frac{1}{2}d}$; and the Cubic Root of $\frac{27ddd}{64}$ is $\frac{3d}{4}$, or $\frac{3}{4}d$.

VII. But if the Root ſought cannot be extracted out of the Numerator and Denominator as before, the Radical ſign is to be ſet before the given Fraction; as to extract the ſquare Root of $\frac{aa}{b}$ I write $\surd\frac{aa}{b}$; or becauſe the ſquare Root of the Numerator is a, the ſquare Root of $\frac{aa}{b}$ may be expreſs'd thus $\frac{a}{\surd b}$; likewiſe the ſquare Root of $\frac{aa+bb}{aabb}$ may be written either thus, $\surd\frac{aa+bb}{aabb}$; or thus, $\surd\frac{:\overline{aa+bb}:}{ab}$

CHAP. IX.

Which teaches how by any two of the three Members of a Square formed from a Binomial Root, to find out the third Member.

I. FRom *Sect.* 3. of the precedent 8. *Chap.* it is evident that every Square formed from a Binomial Root, that is, a Root of two Names or Parts, conſiſts of three Members or diſtinct Quantities, to wit, two Affirmative Squares, and the double of the Product made by the mutual Multiplication of the two Roots of thoſe ſquares; which double Product is ſometimes Affirmative, and ſometimes Negative: So each of theſe compound Squares $9aa+12a+4$, and $9aa-12a+4$, whoſe Roots are $3a+2$, and $3a-2$, (or $2-3a$) conſiſts of two Squares, to wit, $9aa$ and 4, together with $12a$, the double Product of $3a$ multiplied by 2; which $3a$ and 2 are the Roots of the ſaid Squares $9aa$ and 4: Now if any two of the three Members of a Square formed from a Binomial root be given, we may find out the third Member by one of theſe two following Rules.

II. When two Affirmative Squares are given as two of the three Members or Parts of a compound Square formed from a Binomial root to find out the third or mean Member; extract the Square root out of each of thoſe given Squares, then the double of the Product made by the Multiplication of thoſe Roots one into the other ſhall be the mean or middle Member ſought, which if it be annexed to the two given Squares either by $+$ or $-$, will make a compleat compound Square having a Binomial Root.

As for Example, If the Squares $9aa$ and 4 be given, firſt I extract their Roots which are $3a$ and 2, then multiplying theſe Roots one by the other the Product is $6a$, which doubled makes $12a$, the middle Member ſought; this joined by $+$ to the Sum of the given Squares $9aa$ and 4 makes the compound Square $9aa+4+12a$, or $9aa+12a+4$, whoſe Root is $3a+2$: But if the ſaid double Product $12a$ be joined to the Sum of the Squares by $-$, there will ariſe the compound ſquare $9aa+4-12a$, or $9aa-12a+4$; whoſe Root is $3a-2$, or, $2-3a$.

In like manner, If $4aa$ and $9bb$ be propos'd as two of the three Members of a compound Square that has a Binomial Root, the third Member will be found $12ab$; and the Square ſought will be either $4aa+12ab+9bb$, whoſe Root is $2a+3b$; or elſe $4aa-12ab+9bb$, whoſe Root is $2a-3b$, or $3b-2a$.

III. When the double Product and either of the two Affirmative Squares aforeſaid are given as two of the three Members of a compound Square having a Binomial Root, to find out the other Square or third Member; divide half the ſaid double Product by the Root of the given Square, and the Square of the Quotient ſhall be the third Member ſought, which added by $+$ to the two given Members will compleat the compound Square.

As for Example, If $9aa+12a$ be propoſed; the half of $12a$ is $6a$; this divided by $3a$ (the ſquare Root of $9aa$) gives 2 whoſe Square is 4, which added by $+$ to $9aa+12a$ makes $9aa+12a+4$, which is a compleat Compound Square, whoſe Root is $3a+2$.

In like manner, If $12a+4$ be given; the half of $12a$ is $6a$, which divided by 2, (the ſquare Root of 4) gives $3a$, whoſe Square is $9aa$, which added by $+$ to $12a+4$, makes the compound Square $12a+4+9aa$, that is, $9aa+12a+4$, whoſe Root is $3a+2$.

Again, If $aa-2ba$ be given; the half of $2ba$ is ba, which divided by a, (the ſquare Root of aa) gives the Quotient b, whoſe Square is bb; which added to $aa-2ba$ makes the Square $aa-2ba+bb$, whoſe Root, becauſe $-$ is prefix'd to $2ba$, ſhall be $a-b$, or, $b-a$; but if $+$ had been prefix'd to $2ba$, then the Root would have been $a+b$, or $b+a$.

Note: If the ſaid Affirmative Square given be expreſs'd by Letters, and has only 1 (to wit, Unity) prefix'd to it, then inſtead of the Rule above delivered in this *Sect.* 3. there may be this *Compendium*, *viz.* The Square of half that Quantity which in the double Product given is drawn into the Root of the given Square ſhall be the third Member ſought to compleat the compound Square: As in the laſt Example, where $aa-2ba$ was given, becauſe 1 is prefix'd (or muſt be imagined to be prefix'd) to aa; I take the half of $2b$ to wit, b, which multiplied by it ſelf gives bb, which added by $+$ to $aa-2ba$, will make (as before) the compleat Compound Square $aa-2ba+bb$. So alſo to make $aa+6da$ a compleat Square, I take the half of $6d$ which is $3d$, whoſe Square $9dd$ added by $+$ to $aa+6da$ makes the compound Square $aa+6da+9dd$, whoſe Root is $a+3d$. This will be further illuſtrated in the next *Section.*

IV. If a compound Quantity conſiſts of two ſuch Quantities that one of them is an Affirmative Square expreſs'd by Letters, before which 1 is prefix'd, (or ſuppos'd to be prefix'd) and the other is the Product made by the Multiplication of the Root of that Square by ſome Quantity, which is uſually called the Coefficient; that compound Quantity may be made a compleat Square thus, *viz.* Add by the Sign $+$ the Square of half the Coefficient to the compound Quantity given, ſo ſhall the Sum be a Square, whoſe Root, when $+$ is prefix'd to the ſaid Product, is the Sum of the Roots of the Square given and the Square added: But when $-$ is prefix'd to the ſaid Product, then the Root of the compound Square found ſhall be the difference of thoſe two Roots.

As for Example, If the compound Quantity $aa+ca$ be propoſed, I take the half of the Coefficient c, to wit, $\frac{1}{2}c$; then the Square of $\frac{1}{2}c$ is $\frac{1}{4}cc$, which added to $aa+ca$ makes $aa+ca+\frac{1}{4}cc$; which is a Square whoſe Root or Side is $a+\frac{1}{2}c$, to wit, the Sum of the Roots of the Squares aa and $\frac{1}{4}cc$; But if the ſaid $\frac{1}{4}cc$ be added to $aa-ca$, then there will ariſe the Square $aa-ca+\frac{1}{4}cc$, whoſe Root is $a-\frac{1}{2}c$, or $\frac{1}{2}c-a$.

In like manner, To make $aa+5ba$ a compleat Square, and to diſcover its Root; I take the half of $5b$, to wit, $\frac{5}{2}b$, the Square whereof is $\frac{25}{4}bb$, which added to the given compound Quantity $aa+5ba$ makes $aa+5ba+\frac{25}{4}bb$, which is a Square whoſe Root is $a+\frac{5}{2}b$, as will eaſily appear by multiplying the ſaid Root into it ſelf.

So alſo, To make $aa-12a$ a perfect Square, I add 36 (the Square of half the Coefficient 12) to $aa-12a$, and it makes the compound Square $aa-12a+36$, whoſe Root is $a-6$, or $6-a$.

Again, To find what Quantity muſt be added to $aaaa+aa$, or $aaaa+1aa$, to make a compleat Square; I take $\frac{1}{2}$, to wit, half the Coefficient 1 which is prefix'd to aa, (the ſquare Root of $aaaa$) and then the Square of the ſaid $\frac{1}{2}$ is $\frac{1}{4}$; this added to $aaaa+1aa$ makes the Square $aaaa+1aa+\frac{1}{4}$, or, $aaaa+aa+\frac{1}{4}$, whoſe Root is $aa+\frac{1}{2}$, to wit, the Sum of the Roots of the Squares $aaaa$ and $\frac{1}{4}$.

After

After the ſame manner, to make this Compound Quantity a compleat Square, ...	$aa+\frac{2b+3c}{d}a.$
I take the half of the Coefficient $\frac{2b+3c}{d}$, to wit, ...	$\frac{2b+3c}{2d},$
Then the Square of that half Coefficient is	$\frac{4bb+12bc+9cc}{4dd}$
Which Square added to the Compound Quantity propoſed, makes	$aa+\frac{2b+3c}{d}a+\frac{4bb+12bc+9cc}{4dd},$
Which laſt Compound Quantity is a Square, whoſe Root is	$a+\frac{2b+3c}{2d}.$

Likewiſe, If it be deſired to make this compound Quantity a compleat Square, to wit, $aaaaaa+baaa$, I add to it the Square of half the Coefficient b, to wit, $\frac{1}{4}bb$; ſo there will be $aaaaaa+baaa+\frac{1}{4}bb$ the Square deſired, whoſe Root is $aaa+\frac{1}{2}b$.

CHAP. X.

A Collection of eaſie Queſtions to exerciſe the Rules hitherto delivered.

I. THere are two Quantities, whereof the greater is a (or 3,) the leſſer is e (or 2,) What is their Sum? What is their difference? What is the Product of their Multiplication? What is the Quotient of the greater divided by the leſſer? What is the Quotient of the leſſer divided by the greater? What is the Sum of their Squares? What is the difference of their Squares? What is the Sum of the Sum and difference of the two Quantities firſt propoſed? What is the difference of their Sum and Difference? What is the Product made by the Multiplication of the Sum by the Difference? What is the Square of the Sum? What is the Square of the Difference? What is the Sum of the Squares of the Sum and Difference? What is the Difference between the Square of the Sum, and the Square of the Difference? What is the Square of the Product of the Multiplication of the ſaid two Quantities?

	Anſwers by Letters,	*by Numbers.*
1. The Sum of the two Quantities propoſed is	$a+e$	5
2. Their Difference, or the exceſs of the greater above the leſs, is	$a-e$	1
3. The Product of their Multiplication is	ae	6
4. The Quotient of the greater divided by the leſs is...	$\frac{a}{e}$	$\frac{3}{2}$
5. The Quotient of the leſſer divided by the greater is	$\frac{e}{a}$	$\frac{2}{3}$
6. The Sum of their Squares is	$aa+ee$	13
7. The Difference of their Squares is	$aa-ee$	5
8. The Sum of the Sum and Difference of the two Quantities firſt propoſed is	$2a$	6
9. The difference of their Sum and Difference is	$2e$	4
10. The Product of the Multiplication of the Sum by the Difference is	$aa-ee$	5
11. The Square of the Sum is	$aa+2ae+ee$	25
12. The Square of the Difference is	$aa-2ae+ee$	1

13. The Sum of the Squares of the Sum and Difference is	$2aa+2ee$	26
14. The difference between the Square of the Sum and the Square of the Difference is	$4ae$	24
15. The Square of the Product of the Multiplication of the two Quantities is	$aaee$	36

In like manner, If the greater of two Quantities be c, (or 4,) and the leſſer be $\frac{b-d}{c}$; (which we may ſuppoſe to repreſent $\frac{20-12}{4}$, that is, 2; by putting b for 20, and d for 12;) then

1. The Sum of thoſe two Quantities will be......	$c+\frac{b-d}{c}$	6
2. Their Difference is	$c-\frac{b-d}{c}$	2
3. The Product of their Multiplication is	$b-d$	8
4. The Quotient of the greater divided by the leſs is	$\frac{cc}{b-d}$	2
5. The Quotient of the leſſer divided by the greater is	$\frac{b-d}{cc}$	$\frac{1}{2}$
6. The Sum of their Squares is	$cc+\frac{bb-2bd+dd}{cc}$	20
7. The Difference of their Squares is	$cc-\frac{bb-2bd+dd}{cc}$	12
8. The Sum of the Sum and Difference of the two Quantities is	$2c$	8
9. The Difference between the Sum and Difference is	$\frac{2b-2d}{c}$	4
10. The Product of the Sum multiplied by the Difference is	$cc-\frac{bb-2bd+dd}{cc}$	12

II. There are two Quantities whoſe Sum is b, (or 20,) and the greater of them is put a, (or 12;) What is the Leſſer? What is their Difference? What is the Product of their Multiplication? What is the Sum of their Squares? What is the Difference of their Squares?

1. If from the Sum of two Quantities the greater be ſubtracted, the Remainder ſhall be the leſſer; therefore the leſſer Quantity ſought is	$b-a$	8
2. If from the greater Quantity a, the leſſer $b-a$ be ſubtracted, the Remainder or Difference will be..	$2a-b$	4
3. The Product of the Multiplication of the two Quantities is	$ba-aa$	96
4. The Sum of their Squares is	$2aa+bb-2ba$	208
5. The Difference of their Squares is	$2ba-bb$	80

---

1. But if the Sum of two Quantities be repreſented by	b	20
2. And for the leſſer of them there be put	e	8
3. The greater Quantity ſhall be	$b-e$	12
4. Their Difference ſhall be	$b-2e$	4
5. The Product of their Multiplication	$be-ee$	96
6. The Sum of their Squares	$2ee+bb-2be$	208
7. The Difference of their Squares	$bb-2be$	80

III. There are two Quantities whoſe Difference is *d*, (or 4,) and if for the Greater Quantity there be put *a*, (or 12;) What is the Leſſer? What is their Sum? What is the Product of their Multiplication? What is the Sum of their Squares? What is the Difference of their Squares?

1. By ſubtracting the Difference from the Greater quantity, the Leſſer will be	$a-d$	8
2. The Sum of the two Quantities is . . .	$2a-d$	20
3. The Product of their Multiplication is . .	$aa-da$	96
4. The Sum of their Squares is	$2aa+dd-2da$	208
5. The Difference of their Squares is	$2da-dd$	80

1. But if the Difference of two quantities be . . .	d	4
2. And for the Leſſer quantity you put	e	8
3. The Greater ſhall be the ſum of the Difference and the Leſſer, to wit, ...	$d+e$	12
4. The Sum of the two Quantities is	$d+2e$	20
5. The Product of their Multiplication is . .	$de+ee$	96
6. The Sum of their Squares is	$dd+2de+2ee$	208
7. The Difference of their Squares is	$dd+2de$	80

IV. There are two Quantities, whereof the Greater has ſuch Proportion to the Leſſer as *r* (3) to *s*, (2,) now if for the greater quantity there be put *a*, (15,) What is the Leſſer? What is their Sum? What is their Difference? What is the Product of their Multiplication? What is the Sum of their Squares? What is the Difference of their Squares?

1. Firſt, ſay by the Rule of Three, If *r* give *s*, what will *a* give? *Anſw.* $\frac{sa}{r}$, which is the Leſſer quantity ſought	$\frac{sa}{r}$	10
2. Then the Sum of the two quantities will be . .	$a+\frac{sa}{a}$	25
3. Their Difference is	$a-\frac{sa}{r}$	5
4. The Product of their Multiplication is . . .	$\frac{saa}{r}$	150
5. The Sum of their Squares is	$aa+\frac{ssaa}{rr}$	325
6. The Difference of their Squares is	$aa-\frac{ssaa}{rr}$	125

But if the Leſſer of two Quantities be *e* (10,) and has ſuch Proportion to the Greater as *s* (2,) to *r* (3;) Then

1. The Greater Quantity will by the Rule of Three be found	$\frac{re}{s}$	15
2. And the Sum of the two Quantities will be . .	$\frac{re}{s}+e$	25
3. Their Difference is	$\frac{re}{s}-e$	5
4. The Product of their Multiplication is . . .	$\frac{ree}{s}$	150
5. The Sum of their Squares is	$\frac{rree}{ss}+ee$	325
6. The Difference of their Squares is	$\frac{rree}{ss}-ee$	125

V. There are two Quantities, the Product of whoſe Multiplication is b (20;) and if for the Greater quantity there be put a (5,) What is the Leſſer? What is their Sum? What is their Difference? What is the Sum of their Squares? What is the Difference of their Squares?

1. The Product b divided by the Greater quantity a gives the Leſſer, to wit, ...	$\frac{b}{a}$	4
2. The Sum of the two Quantities is . . .	$a+\frac{b}{a}$	9
3. Their Difference is	$a-\frac{b}{a}$	1
4. Then the Sum of their Squares is	$aa+\frac{bb}{aa}$	41
5. The Difference of their Squares is	$aa-\frac{bb}{aa}$	9

But if the Product of the multiplication of two Quantities be b (20,) and for the Leſſer there be put e (4.)

1. The Greater quantity will be	$\frac{b}{e}$	5
2. The Sum of the two quantities is	$\frac{b}{e}+e$	9
3. The Difference is	$\frac{b}{e}-e$	1
4. The Sum of their Squares is	$\frac{bb}{ee}+ee$	41
5. The difference of their Squares is	$\frac{bb}{ee}-ee$	9

VI. The extraction of Roots may be exerciſed by theſe following Queſtions, reſpect being had to *Sect.* 28. *Chap.* 1. as alſo *Chap.* 8.

1. What is the Square Root of $144aa$? *Anſw.* $12a$.
2. What is the Square Root of $\frac{169}{16}aabb$? *Anſw.* $\frac{13}{4}ab$.
3. What is the Square Root of $9aa-6ab+bb$? *Anſw.* $3a-b$, or, $b-3a$.
4. What is the Square Root of $\frac{4aa+16ab+16bb}{9cc}$? *Anſw.* $\frac{2a+4b}{3c}$.
5. What is the Cubic Root of $125aaabbb$? *Anſw.* $5ab$.

6. If b be put for 65, and c for 8. what number is ſignified by $\sqrt{:\overline{b+\frac{1}{4}cc}:}-\frac{1}{2}c$? *Anſw.* 5.

7. The ſame things being put as in the laſt Queſtion, what number is ſignified by $\sqrt{:\overline{b+\frac{1}{4}cc}:}+\frac{1}{2}c$? *Anſw.* 13.

8. If d be put for 8, and f for 48, what number is ſignified by $\sqrt{:\overline{\sqrt{f+\frac{1}{4}dd}-\frac{1}{2}d}:}$ *Anſw.* 2.

9. But the ſame things being put as in the laſt Queſtion, this quantity $\sqrt{:\overline{\sqrt{f+\frac{1}{4}dd}+\frac{1}{2}d}:}$ ſignifies $\sqrt{12}$, or, 3. 464, &c. that is $3\,\frac{464}{1000}$, &c.

10. If g be put for 4, and b for 837, what Number is ſignified by $\sqrt{(3)}:\overline{\sqrt{b+\frac{1}{4}gg}-\frac{1}{2}g}:$ *Anſw.* 3.

11. But the ſame things being put as in the laſt Queſtion, this Quantity $\sqrt{(3)}:\overline{\sqrt{b+\frac{1}{4}gg}+\frac{1}{2}g}:$ ſignifies $\sqrt{(3)}\,31$, or, 3. 141, &c.

VII. The Rules of the ninth *Chap.* may be exerciſed by theſe following Queſtions.

1. What Quantity is that which if it be added to $aa+25$, will make the Sum a Square? *Anſw.* The Quantity to be added may be either $+10a$, or $-10a$;

and the Square ſought is either $aa+10a+25$, whoſe Root or ſide is $a+5$, or elſe the Square is $aa-10a+25$, whoſe Root is $a-5$, or $5-a$.

2. What Quantity is that which if it be added to $\frac{9}{16}aa+\frac{4}{9}bb$, will make the Sum a Square? *Anſw.* The Quantity to be added may be either $+ab$, or $-ab$; and the Square is either $\frac{9}{16}aa+ab+\frac{4}{9}bb$, whoſe Root is $\frac{3}{4}a+\frac{2}{3}b$: Or elſe the Square is $\frac{9}{16}aa-ab+\frac{4}{9}bb$, whoſe Root is $\frac{3}{4}a-\frac{2}{3}b$; or $\frac{2}{3}b-\frac{3}{4}a$.

3. What Quantity is that which if it be added to $aa+3a$ will make the Sum a Square? *Anſw.* The Quantity to be added is $\frac{9}{4}$; and the Square is $aa+3a+\frac{9}{4}$, whoſe Root is $a+\frac{3}{2}$.

4. What Quantity is that which together with $aaaa-2bbaa$ will make a perfect Square? *Anſw.* The Quantity to be added is $bbbb$; and the Square is $aaaa-2bbaa+bbbb$, whoſe Root is $aa-bb$, or $bb-aa$.

5. What Quantity is that which if it be added to $aa+\frac{bb}{c}a$ will make the Sum a Square? *Anſw.* The Quantity to be added is $\frac{bbbb}{4cc}$; and the Square is $aa+\frac{bb}{c}a+\frac{bbbb}{4cc}$, whoſe Root is $a+\frac{bb}{2c}$.

6. What Quantity is that which together with $aaaaaa-aaa$ will make a compleat Square? *Anſw.* The Quantity to be added is $\frac{1}{4}$; and the Square ſought is $aaaaaa-aaa+\frac{1}{4}$, whoſe Root is $aaa-\frac{1}{2}$, or $\frac{1}{2}-aaa$.

CHAP. XI.

Concerning an Equation, and the Reduction of Equations.

I. AN *Equation* in the Algebraical Art is a mutual Comparing of two Equal quantities or things of different Denominations: as, If the value of three Shillings be compared to thirty ſix pence of *Engliſh* Money, that compariſon imports an Equation, which may be Symbolically expreſt thus, $3\,s = 36\,d$, that is, three Shillings are equal to thirty ſix pence. Likewiſe, foraſmuch as nine Crowns are of equal value with the Sum of two Pounds and five Shillings of *Engliſh* Money; the comparing of theſe two Sums to one another is nothing elſe but an Equation which may be briefly expreſt thus, $9c=2l+5s$. In each of which Equations the Moneys compared are of different kinds; for Equations between equal things of one and the ſame name, as $2s=2s$, or $5=5$, and ſuch like, are fruitleſs.

After the ſame manner, this Equation $a=b+c$ may ſignifie that ſome Number or line repreſented by a is equal to two other Numbers or Lines b and c taken together as one; or, if the number or Line a be divided into two parts b and c, then alſo $a=b+c$; for the whole is equal to all its parts.

II. Every Equation conſiſts of two Parts, which are uſually ſeparated one from another by this Character $=$; ſo in the firſt Equation in the preceding *Sect.* $3s$ is the firſt Part, and $36d$ the latter; alſo in the ſecond Equation, $9c$ is the firſt Part, and $2l+5s$ is the latter; likewiſe in the laſt Equation of the ſame *Section*, a is the firſt Part, and $b+c$ the latter.

III. The ſingle Quantities or things, whereof each part of an Equation is compoſed, are called the Terms of an Equation; as in this Equation, $a=b+c$, the Terms are a, b and c.

IV. How Equations are found out, the Reſolution of Queſtions will hereafter ſhew; but when known quantities are intermingled with unknown in an Equation, the firſt Scope is to clear the Equation from all ſuperfluous quantities, and to ſeparate the known quantities from the unknown, that at length an Equation may remain in the

BIBL. UNIV. EDINBURGEN. EX

feweſt and ſimpleſt Terms, ſo diſpoſed, that the unknown quantity or quantities may poſſeſs one part of the Equation, and the known the other, this work is called *Reduction*, and how 'tis perform'd the Examples in the following *Sections* will make manifeſt.

Reduction by Addition.

V. Reduction by Addition is grounded upon this Axiom, (or common Notion) *viz.* If equal quantities, or one and the ſame quantity, be added to equal quantities, the wholes or totals ſhall be equal. As, for Examples;

If the letter *a* repreſent ſome number unknown, and it be granted or found out that	$a - 3 = 12$
Then by adding + 3 to each part of that Equation, this ariſes, to wit,	$a - 3 + 3 = 12 + 3$
That is, (becauſe — 3 and + 3 added together make 0,)	$a = 15$

In like manner, to reduce this Equation.	$3a - 4 = 6 - a$
I add + 4 to each part, and there ariſes	$3a - 4 + 4 = 6 - a + 4$
Which Equation contracted makes	$3a = 10 - a$
Then by adding + *a* to each part of the laſt Equation, this ariſes,	$3a + a = 10 - a + a$
That is, after each part is contracted,	$4a = 10$

Again, If this Equation be propos'd to be reduced	$aa - b = d + b$
By adding + *b* to each part, this Equation ariſes,	$aa - b + b = d + b + b$
Which laſt Equation, after due contraction gives	$aa = d + 2b$

So alſo, If	$a - b = 0$
By adding + *b* to each Part, there ariſes	$a = b$

Likewiſe, If	$b - a = 0$
By adding *a* to each part there ariſes	$b = a$

Moreover, If	$aa - bb - cc = dd$
Then by adding *bb* + *cc* to each part of this Equation comes forth,	$aa = dd + bb + cc$

Laſtly, If	$aa - bb = cc - da$
By adding + *bb* to each part, this Equation ariſes,	$aa = cc - da + bb$
And by adding + *da* to each part of the laſt Equation, this ariſes, to wit,	$aa + da = cc + bb$

From the premiſes it is evident, That if in any Equation any Quantity which has the ſign — prefixed to it, be transfer'd to the other part of the Equation with the ſign +, that work effects the ſame thing as the adding of that Quantity to each part of the Equation, and is called *Tranſpoſition*.

Reduction by Subtraction.

V. If from equal Quantities you take away equal Quantities, or one and the ſame Quantity, the Quantities remaining will be equal; therefore,

If it be taken for granted that . . . . $a+3=12$
Then by ſubtracting $+3$ from each part, there ariſes . . . . $a=9$

In like manner, If . . . . $b+a=4b$
I Subtract $+b$ from each Part, and there ariſes . . . . $a=3b$

Again, If . . . . $bb+2aa=aa+cc$
Firſt, I ſubtract bb from each part, and there remains . . . . $2aa=aa+cc-bb$
Then aa Subtracted from each part of the laſt Equation, leaves this, to wit, ... $aa=cc-bb$

So alſo, If . . . . $aa+b+c=2ca+df$
By ſubrracting $+b+c$ from each part, rhere ariſes . . . . $aa=2ca+df-b-c$
And by ſubtracting $2ca$ from each part of the laſt Equation, this ariſes, to wit, ... $aa-2ca=df-b-c$

Hence it is evident, That if in any Equation any Quantity which has the ſign $+$ prefixed to it be transferr'd to the other part of the Equation with the ſign $-$, that work Effects the ſame thing as the ſubtracting of that Quantity from each part of the Equation, and is alſo called *Tranſpoſition.*

Reduction by Multiplication.

VII. If equal Quantities be multiplied by equal Quantities, or by one and the ſame Quantity, the Products ſhall be equal: Hence Equations expreſt by Algebraical Fractions are reduced to other Equations conſiſting altogether of Integers.

As, for Example, If . . . . $\frac{a}{5}=6$
Then by multiplying each part by 5, this Equation is produced . . . . $a=30$

Again, to reduce this Equation to another in Integers, *viz.* . . . . $a=\frac{dd}{a-b}$
I multiply each part by $a-b$ and there comes forth . . . . $aa-ab=dd$

Likewiſe, to reduce this Equation to another in Integers, ... $\frac{3aa}{c}=\frac{dd}{b}$
Firſt, I multiply each part by the Denominator b, and there will be produced . . . . . $\frac{3aab}{c}=dd$
Then Multiplying each part of the laſt Equation by the Denominator c, I find this Equation . . . . $3aab=cdd$

Hence it is manifeſt, That an Equation whereof each part is a Fraction, may be reduced to another Equation in Integers, by multiplying croſs-wiſe, as in the Reduction of Fractions to a common Denominator, and then omitting the common Denominator, a new Equation may be inſtituted between the new Numerators only.

When either part of an Equation is compoſ'd of Integers and Fractions, firſt reduce that part into a Fraction, (after the manner of the latter Example in *Sect.* 16. *Chap.* 6.) and then multiply as in the preceding Examples: as,

If this Equation be propoſed, $\frac{aa}{b}+c+d=bc+\frac{dd}{a}$

Firſt, I reduce that Equation to this, $\frac{aa+bc+bd}{b}=\frac{bca+dd}{a}$

Which laſt Equation reduced by Multiplication as in the preceding Examples, gives $aaa+abc+abd=bbca+bdd$

But here is to be noted, that in reducing Equations which conſiſt of Fractions into other Equations in Integers, the Operation may oftentimes be facilitated by the ſame compendium that has before been ſhewn in the Diviſion of Fractions (in *Sect.* 26. *Chap.* 6.) *viz.* When either the Numerators or Denominators can be reduced to more ſimple Terms by ſome common Diviſor, ſet the Quotients in the Places of thoſe Numerators or Denominators; and then reduce theſe new Fractions into an Equation in Integers, by multiplying croſs-wiſe as before: As for Example,

To reduce this Equation to another in Integers, $\frac{aaa}{aa-bb}=\frac{ba-bb}{a+b}$

Firſt, after the Denominators $aa-bb$ and $a+b$, are reduced to $a-b$ and 1, by the common Diviſor $a+b$, this New Equation ariſes, $\frac{aaa}{a-b}=\frac{ba-bb}{1}$

Whence by multiplying croſs-wiſe, (as in the preceding Examples) this Equation in Integers is produced, $aaa=baa-2bba+bbb$

Again, to reduce this Equation to another in Integers, $\frac{bba-cca}{a+b}=\frac{bbb-bcc}{a}$

Firſt, the Numerators reduced to a and b by the common Diviſor, $bb-cc$ will give $\frac{a}{a+b}=\frac{b}{a}$

Whence by Multiplying croſs-wiſe, this Equation is produced $aa=ba+bb$

In like manner, to reduce this Equation, $\frac{baa-caa}{cc-ca}=\frac{bb-bc}{c}$

Firſt, I reduce the Numerators to aa and b, by the common Diviſor $b-c$; alſo the Denominators to $c-a$ and 1, by the common Diviſor c; which new Numerators and Denominators conſtitute this Equation, $\frac{aa}{c-a}=\frac{b}{1}$

Whence by multiplying croſs-wiſe, this Equation is produced $aa=bc-ba$

So alſo to reduce this Equation $\frac{ba^3-ca^3}{aa-ba+bb}=bc-cc$

Firſt, I ſet 1 for a Denominator under the Integer $bc-cc$, ſo the Equation propoſed will ſtand thus, $\frac{ba^3-ca^3}{aa-ba+bb}=\frac{bc-cc}{1}$

Then, after the Numerators $ba^3 - ca^3$ and $bc - cc$ are reduced to a^3 and c,) by the common Diviſor $b - c$, this Equation ariſes . . . . $\frac{a^3}{aa - ba + bb} = \frac{c}{1}$

Which laſt Equation, by multiplying croſs-wiſe, gives this in Integers, ... $aaa = caa - cba + cbb$

When one part of an Equation is a Surd quantity, (that is, ſuch which has a Radical ſign prefixt to it, as, $\surd$, or $\surd(3)$, *&c.*) and the other part is a rational Quantity; that Equation may be reduced to another which ſhall be free from any Surd quantity, by caſting away the Radical ſign, and multiplying the rational part of the given Equation either quadraticaly or cubicaly, *&c.* according to the import of the Radical ſign; as,

If there be propoſed . . . . $\surd a = 6$

Foraſmuch as the Squares of equal Roots or Sides are alſo equal, therefore by ſquaring each part of that Equation, this is produced, to wit, ... $a = 36$

Likewiſe, If . . . . $\surd a = bc$

By multiplying each part into it ſelf, this Equation is produced, ... $a = bbcc$

Again, If . . . . $\surd a = \surd 5$

By ſquaring each part, there comes forth . . . . . $a = 5$

And, If . . . . $\surd a = \surd{:}\overline{bcc - b}{:}$

By ſquaring each part, which is done by caſting away $\surd$, there will ariſe . . . . $a = bcc - b$

So alſo if this Equation be propoſed, ... $\surd ca = b - d$

By multiplying each part into it ſelf, this Equation is produced, ... $ca = bb - 2bd + dd$

And, If . . . . $\surd(3)a = 8.$

By multiplying each part into it ſelf cubically, there ariſes . . . . $a = 512.$

Alſo, If . . . . $\surd(3)a = \surd(3){:}\overline{b + c}{:}$

By caſting away $\surd(3)$ from each part it gives . . . . $a = b + c.$

Reduction by Diviſion.

VIII. If equal Quantities be divided by equal Quantities, or by one and the ſame Quantity, there will come forth equal Quotients. Hence Equations are reduced to others of lower Degrees: As, for example;

If it be granted or found out that . . . . $aa = 5a$

Then by dividing each part by a, you will find . . $a = 5$

Again, If . . . . $aaa + baa = bba$

By dividing each part by a, this Equation ariſes . . . . $aa + ba = bb.$

Alſo, If . . . . $5a = 15$

By dividing each part by 5, there ariſes . . . . . $a = 3$

Likewiſe, If . . . . $ba = bc$

By dividing each part by b, this Equation ariſes, ... $a = c$

Again, If . . . . $ba - ca = cc$

By dividing each part by $b - c$, there ariſes . . . $a = \frac{cc}{b - c}$

Alſo, If . . . . $baa + caa = bd + cd$

By dividing each part by $b + c$, there ariſes . . . . $aa = d$

Moreover, If . . . . $3aa+4a=39$
By dividing each part by 3, there ariſes . . . $aa+\frac{4}{3}a=13$
Likewiſe, If . . . . $caa-ba=cdd$
By dividing each part by c, there ariſes . . . $aa-\frac{b}{c}a=dd$

Reduction by Extraction of ROOTS.

IX. Foraſmuch as the Sides or Roots of equal Squares and Cubes, &c. are alſo equal between themſelves; therefore,

If there be propoſed . . . . $aa=36$
By extracting the Square Root of each part, there ariſes . . . . $a=6$
In like manner, If . . . . $aa=bb+2bc+cc$
By extracting the Square Root of each part, there comes forth . . . . $a=b+c$
Again, If . . . . $aa=29$
By extracting the Square Root of each part, there will ariſe . . . . $a=\surd 29$
Likewiſe, If . . . . $aa=bb-dd$
Then, by extracting the Square Root out of each part, there ariſes . . . . . . $a=\surd:\overline{bb-dd:}$
Again, If . . . . $aaa=27$
Then, the Cubic Root being extracted out of each part there comes forth . . . $a=3$
Alſo, If . . . . $aaa=12$
By extracting the Cubic Root out of each part, this Equation will ariſe . . . $a=\surd(3)12.$
Likewiſe, If . . . . $aaa=bbc+cdd$
Then, the Cubic Root extracted out of each part, gives . . . . $a=\surd(3):\overline{bbc+cdd:}$

X. By the help of ſome of the foregoing Reductions, I ſhall here ſhew (after the manner of *Fran van Scooten* in his *Principia Matheſ. Univerſal.*) the certainty of the Rule before given concerning + and — in the Algebraical Multiplication of Compound Quantities: *viz.* That + multiplied by —, or — by + makes —; alſo, That — multiplied by — makes +.

Firſt, let $a-b$ be to be multiplied by c, then the Product according to Algebraical Multiplication is $ac-bc$: now it muſt be proved that $-b$ multiplied by $+c$ makes $-bc$; to which end, let f be put equal to $a-b$, and then if it be proved that $ac-bc=fc$, it is evident that $ac-bc$ is the true Product ſought; and conſequently, $-b$ multiplied by $+c$ makes $-bc$: But that $ac-bc=fc$ may be proved thus,

Foraſmuch as by ſuppoſition, ... $a-b=f$
Therefore by adding b to each part, it makes . . . $a=f+b$
And by multiplying each part of the laſt Equation by c, there will be produced . . . . $ac=fc+bc$
Wherefore, by ſubtracting bc from each part of the laſt Equation there remains . . . $ac-bc=fc$
Which was to be proved.

After the ſame manner it may be proved that — multiplied by — makes +: For, If $a-b$ be to be multiplied by $c-d$, and there be put (as before) $f=a-b$, it may be ſhewn that $ac-bc-ad+bd$ is equal to $\overline{a-b}\times\overline{c-d}$ the Product ſought; and therefore $-b$ multiplied by $-d$ produces $+bd$. For, By ſuppoſition . . . . $f = a-b$

Therefore, by multiplying each part into $c-d$ . . . $f\times\overline{c-d} = \overline{a-b}\times\overline{c-d}$

That is, ... $fc-fd = \overline{a-b}\times\overline{c-d}$

But it has been proved in the former Example, that . . . . $ac-bc = fc$

Therefore inſtead of *fc* in the third Equation of this latter Example, taking $ac-bc$ (equal to *fc*) there ariſes . . . . . $ac-bc-fd = \overline{a-b}\times\overline{c-d}$

Again, If each part of the firſt Equation be multiplied by *d*, this will be produced, ... $fd = ad-bd$

Wherefore, If from $ac-bc$ in the fifth Equation there be ſubtracted $ad-bd$ inſtead of *fd* equal to $ad-bd$, there will remain according to the Rule of Algebraical Subtraction . . . . . . $ac-bc-ad+bd = \overline{a-b}\times\overline{c-d}$

Which was to be proved.

CHAP. XII.

Which ſhews in what Order the Reductions in the foregoing Chap. 11. *are to be uſed to reſolve Equations, or at leaſt to prepare them for Reſolution.*

I. BY the help of the precedent Reductions, either the value of the unknown Root or Quantity ſought in an Equation will be found equal to ſome known Quantity or Quantities, and conſequently the Quantity ſought is then known alſo; or elſe a new Equation will be diſcovered, from whence the ſame Quantity ſought may be made known by ſome other Rule or Rules hereafter delivered: But in the uſe of thoſe Reductions, the work may oftentimes be facilitated by an orderly proceſs, which is the Scope of the five following *Sections*; where I aſſume the Vowel *a* to ſtand for the unknown Root or Quantity ſought, and Conſonants for known Quantities.

II. If in any Equation the Quantity ſought, or any Power or Degree of it be found in a Fraction, reduce that Equation to another that may be expreſs'd altogether by Integers, (by *Sect.* 7. *Chap.* 11.) As for Example;

If this Equation be propoſed, ... $\frac{b-a}{c} = d+f-g$

By multiplying each part thereof by the Denominator *c*, this Equation ariſes in Integers, ... $b-a = cd+cf-cg$

After the ſame manner, this Equation multiplied by 4, ... $\frac{aa}{4}+6 = 15.$

Will be reduced into . . . . $aa+24 = 60.$

Likewiſe this Equation . . . . $\frac{aa+bb}{d}+b+c = a-c$

Will be reduced to . . . . $aa+bb+db+dc = da-dc.$

III. When Quantities given or known be intermingled with thoſe that are ſought in an Equation, let Quantities be transfer'd from one part of the Equation to the other under a contrary Sign, (according to *Sect.* 5. and 6. of *Chap.* 11.) until at length the unknown Quantity may make one part of an Equation, and all the known Quantities the other: As for Example;

If there be propoſed . . . . $2a-26=8$

By tranſpoſition of -26 to the other part of the Equation, under the contrary ſign $+$, there will ariſe . . . . $2a=8+26=34$.

In like manner, If . . . . $aa+24=60$

By tranſpoſition of $+24$, under the contrary ſign $-$ it gives . . . . $aa=60-24$

That is, ... $aa=36$

Again, If . . . . $6a-4=20-a$

Firſt, by tranſpoſition of -4, this Equation ariſes, ... $6a=20-a+4$

Then by tranſpoſition of $-a$, I find . . . . $6a+a=20+4$

Which laſt Equation being contracted by Addition, gives . . . . $7a=24$

Likewiſe, If . . . . $b-a=cd-cf$

After due Tranſpoſition, this Equation will ariſe, ... $b+cf-cd=a$

Or, ... $a=b+cf-cd$

IV. When ſome Power or Degree of the Quantity ſought happens to be multiplied into every Term or Member of an Equation, divide every Term by that Degree, ſo will that Degree or Power quite vaniſh, and conſequently the Equation will be depreſſed, that is, reduced to lower Degrees or more ſimple Terms: As for Example,

If there be propoſed . . . . $aa+3a=20a$

Foraſmuch as a is drawn into every Term of that Equation, I divide every Term by a, and there ariſes . . . . $a+3=20$

Whence by equal ſubtraction of 3 I find . . . . . $a=17$

In like manner, If . . . . $aaa=3aa$

By caſting away aa, that is, by dividing each part by aa, there will ariſe . . . $a=3$

Again, If . . . . $aaaa+baaa=ddaa$

By expunging aa out of every Term, there ariſes . . . . $aa+ba=dd$

V. When ſome known Quantity is multiplied into the higheſt Power or Degree of the Quantity unknown or ſought in an Equation; divide each part of the Equation by that known Quantity, to the end the ſaid higheſt unknown Power may have no Co-efficient or Fellow-multiplier but 1, (or Unity; As for Example,

If there be propoſed . . . . $5a=60$

Becauſe the unknown Quantity a is multiplied by 5, I divide each part of the Equation by 5, and there ariſes . . . . . $a=12$

Again, If . . . . $ca=cc+dd$

Becauſe c is drawn into a the Root ſought, I divide every Term of the Equation by c, and there ariſes . . . . $a=c+\frac{dd}{c}$

Likewiſe, If . . . . $2ba+3ca = 2ddb+3cdd$
Becauſe $2b+3c$ is drawn into the unknown Root a, I divide each part by $2b+3c$, and there ariſes . . . . . . $a = dd$

So alſo, If . . . . $4aa = 60$
By dividing each part by 4 which is drawn into aa, there ariſes . . . . $aa = 15$

Again, If . . . . $3aa-5a = 24$
Becauſe 3 is drawn into aa which is the higheſt unknown Power in the Equation, I divide every Term by 3, and there ariſes . . . . $aa-\frac{5}{3}a = 8$

Likewiſe, If . . . . $2ccaa-4dda = 5bbcc$
Becauſe $2cc$ is drawn into aa which is the higheſt unknown Power in the Equation, I divide every Term by $2cc$, and there ariſes . . . . . . $aa-\frac{2dd}{cc}a = \frac{5}{2}bb.$

Again, If . . . . $2bbaa+3cdaa-dda = ccdd$
Becauſe $2bb+3cd$ is drawn into aa the higheſt unknown Degree in the Equation, I divide each part by $2bb+3cd$, and there ariſes . . . . $aa-\frac{dd}{2bb+3cd}a = \frac{ccdd}{2bb+3cd}$

Alſo, If . . . . $3aaa+24aa-6a = 1200$
Becauſe 3 is drawn into aaa the higheſt unknown Power in the Equation, I divide each part by 3, and there ariſes . $aaa+8aa-2a = 400$

VI. If there be a ſurd Quantity in an Equation, that is, if a Radical ſign as $\surd$; or $\surd(3)$ be prefixed before ſome Quantity; firſt by Tranſpoſition (according to *Sect.* 5. or 6. of *Chap.* 11.) make the ſurd Quantity ſole poſſeſſor of one part of an Equation, then caſt away the Radical ſign, and exalt the other part of the Equation to the ſame Degree or Power which is denoted by the Radical ſign, by multiplying Quadratically or Cubically, *&c.* ſo at length an Equation will be found expreſs'd altogether by rational Quantities: As for Example;

If this Equation be propoſed . . . . $\surd a = 3$
By ſquaring each part, there will be produced . . . $a = 9$

In like manner, If . . . . $\surd ba = 3bc$
By multiplying each part into it ſelf quadraticaly, there comes forth . . . . $ba = 9bbcc$
Then dividing each part of the laſt Equation by b, there ariſes . . . . . . $a = 9bcc$

Again, If . . . . $b+\surd ba = c$
Firſt by tranſpoſition of b there ariſes . . . . . . $\surd ba = c-b$
Then by ſquaring each part of the laſt Equation, there will be produced . . . . . . $ba = cc-2cb+bb.$
Whence, by dividing each part by b, there ariſes . . . . $a = \frac{cc}{b}-2c+b$

Likewiſe, If . . . . $-d+\surd:\overline{ba+da}=b$

Firſt by tranſpoſition of $-d$, this Equation ariſes, ... $\surd:\overline{ba+da}=b+d$

Then by ſquaring each part, there will be produced . . . . $ba+da=bb+2bd+dd$

Laſtly, by dividing each part of the laſt Equation by $b+d$, there ariſes . . . $a=b+d$

Again, If . . . . $\surd(3)9a=3$

By multiplying each part Cubically, there will be produced . . . . $9a=27$

And, by dividing each part of the laſt Equation by 9 there ariſes . . . . . . $a=3$

Likewiſe, If . . . . $\surd(3):\overline{ba-ca}:+c=b$

Firſt, by tranſpoſition of $+c$ this Equation ariſes, ... $\surd(3):\overline{ba-ca}:=b-c$

Then multiplying each part of the laſt Equation cubically, this Equation will be produced, to wit,

$$ba-ca=bbb-3bbc+3bcc-ccc:$$

Whence, by dividing each part by $b-c$, the value of a will be diſcovered, *viz.*

$$a=bb-2bc+cc.$$

VII. When after the uſing of all, or any of the foregoing Rules of this Chapter an Equation ariſes between a perfect Square, Cube or other higher Power of the Quantity ſought, and ſome known Quantity; then extract ſuch a Root out of each part of the ſaid Equation as the Index of the ſaid unknown Power denotes, ſo will the value of the unknown Root or Quantity ſought be made known: As, for Example;

If this Equation be propoſed, to wit, $\frac{6aa}{5}+8=128$

Firſt by ſubtracting 8 from each part, this Equation ariſes, ... $\frac{6aa}{5}=120$

Then each part of the laſt Equation being multiplied by 5, gives . . . . $6aa=600$

And by dividing each part of the laſt Equation by 6, this ariſes, ... $aa=100$

Laſtly, the ſquare Root of each part of the laſt Equation being extracted, the value of a will be diſcovered, to wit, ... $a=10$

Again, If . . . . $\frac{3aaaa}{4}-8a=154a$

Then by tranſpoſition of $-8a$ there ariſes $\frac{3aaaa}{4}=162a$

And by multiplying each part of the laſt Equation by 4, this will be produced, $3aaaa=648a$

And by dividing each part of the laſt Equation by a this ariſes, to wit, ... $3aaa=648$

Likewiſe each part of the laſt Equation divided by 3 gives . . . . $aaa=216$

Laſtly, by extracting the Cubic Root out of each part of the laſt Equation, the value of a will be diſcovered, to wit, $a=6$

Likewiſe, If . . . . $aa+2ba+bb = cc$

The ſquare Root extracted out of each part, gives . . . . $a+b = c$

And then by tranſpoſition of b, the value of a is diſcovered, to wit, ... $a = c-b$

CHAP. XIII.

Which ſhews how to convert Analogies into Equations, and Equations into Analogies.

I. IF four right-lines or numbers be Proportionals, the Product made by the Multiplication of the two Extreams is equal to the Product of the two means. And if three right-lines or numbers be Proportionals, the Product of the Extreams is equal to the Square of the mean, (by *Prop.* 16. and 17. of 6. *Elem.* and by 19 and 20 of 7. *Elem. Euclid*) Hence Analogies may be converted into Equations, as in the following Examples; where for the greater evidence let a repreſent 2; b, 6; c, 12; and d, 3; Then

1. Let there be four Proportionals, ſuppoſe theſe, ... $d\ .\ b :: d-a\ .\ a$
$3\ .\ 6 :: 1\ .\ 2$

Then by the Theorem above expreſs'd, this Equation will follow, ... $da = bd-ba$

Now to find the value of a in that Equation, firſt by tranſpoſition of $-ba$ this Equation ariſes, ... $da+ba = bd$

Then each part divided by $d+b$ gives . . . . $a = \frac{bd}{d+b}$

---

2. If there be three continual proportionals, ſuppoſe theſe, ... $4a\ ;\ c\ ,\ 9a$
$8\ ;\ 12\ ,\ 18$

That is, If . . . . $4a\ .\ c :: c\ .\ 9a$

Then, by the latter part of the ſaid Theorem, this Equation will follow, ... $36aa = cc$

Now to find the value of a in that Equation, extract the ſquare Root out of each part, and there ariſes . . . . $6a = c$

Laſtly, each part of the laſt Equation divided by 6 gives . . . . $a = \frac{c}{6}$, or $\frac{1}{6}c$.

II. If the Product of the multiplication of two Quantities be found equal to the Product of two other Quantities, that Equation may be reſolved into Proportionals; for as either of the Factors in either of the two equal Products is to a Factor of the ſame kind in the other Product, ſo is the remaining Factor in this latter Product to the other Factor in the former. Hence Equations may oftentimes be reſolved into Proportionals; as,

If there be propoſed . . . . $3ba = cd$

From that Equation this Analogy may be infer'd, *viz.* As . . . . $3b\ .\ c :: d\ .\ a$

Again, If . . . . $bd = da+ba$

That Equation may be reſolved into theſe Porportionals, *viz.* As . . . . $d+b\ .\ b :: d\ .\ a$

Likewiſe, If . . . . $6da = bb$

Then it ſhall be, As . . . . $6d\ .\ b :: b\ .\ a$

III. When there happens to be an Equation between an Algebraical Fraction and an Integer, and the Numerator of the Fraction can be reſolved into two ſuch Quantities that being mutually multiplied will produce the ſaid Numerator, then that Equation may be reſolved into Proportionals in this manner, *viz.* Let the Denominator of the Fraction, and the Integer to which the Fraction is equal, be made the extream Terms of an Analogy; and let the two Quantities which being mutually multiplied will conſtitute the Numerator be made the mean Terms; but with this caution in Geometrical Queſtions, that the firſt and ſecond Terms be of one and the ſame kind, that is, either both Lines, or both Planes, or both Solids. As for Example;

If this Equation be propoſed, ... $\frac{cd}{3b} = a$

It may be reſolved into theſe Proportionals, . $3b\ .\ c :: d\ .\ a$

But that they are Proportionals, I prove thus;

First, It is evident that theſe are Proportionals, (becauſe the Product of the extreams is equal to the Product of the means) . . . . $3b\ .\ c :: d\ .\ \frac{cd}{3b}$

And by the Equation propoſed, ... $a = \frac{cd}{3b}$

Therefore . . . . $3b\ .\ c :: d\ .\ a\ .\ (\frac{cd}{3b})$

Again, If . . . . $\frac{bb}{b+c} = a$

That Equation may be reſolved into theſe Proportionals, ... $.\ b+c\ .\ b :: b\ .\ a$

Likewiſe this Equation . . . . $\frac{cc-bb}{5b+2c} = a$

may be reſolved into this Analogy, . . $5b+2c\ .\ c+b :: c-b\ .\ a$

And this Equation . . . . $\frac{bb+2bc+cc}{54d} = a$

may be converted into theſe Proportionals, $54d\ .\ b+c :: b+c\ .\ a$

Alſo, this Equation . . . . $\frac{bbc}{36d} = aa$

may be reſolved into theſe Proportionals, . . . $36d\ .\ b :: bc\ .\ aa$

Or into theſe, ... $36d\ .\ c :: bb\ .\ aa$

But this Equation . . . . $\frac{b}{c} = a$

cannot be reſolved into Proportionals any otherwiſe than thus, ... $c\ .\ \sqrt{b} :: \sqrt{b}\ .\ a$

Nor can this Equation . . . . $\frac{bb+cd}{g} = a$

be converted into Proportionals, unleſs thus, $g\ .\ \sqrt{:\overline{bb+cd}:} :: \sqrt{:\overline{bb+cd}:}\ .\ a$

CHAP.

CHAP. XIV.

Various Arithmetical Queſtions Algebraically reſolved; whereby moſt of the Rules hitherto delivered are exercis'd, in the Invention and Reſolution of pure or ſimple Equations.

I. Equations may be divided into two kinds, *viz.* $\left\{\begin{array}{l}\text{1. Pure or Simple,}\\ \text{2. Adfected or Compounded.}\end{array}\right.$

II. A pure or ſimple Equation is of two kinds, *viz.* Firſt, when the Quantity ſought is expreſs'd by a ſimple Root only, as a; as in this Equation, $6a=12$: Secondly, when the Quantity ſought is expreſs'd by a ſimple Power only, as aa, or aaa, &c. as in this Equation, $3aaa=24$; likewiſe in this, $2aaaa=32$, and ſuch like.

III. An adfected or compounded Equation is that, wherein there are two or more different Degrees or Powers of the Quantity ſought; as in this Equation, $aa+6a=27$, where aa and a expreſs two different Degrees or Powers of the Quantity ſought, the one ſignifying a Square, and the other its Root or Side: alſo in this Equation, $aaa+6aa-2a=28$, there are three unlike Powers or Degrees of the Quantity ſought, to wit, aaa, aa, and a.

IV. The Invention and Reſolution of pure or ſimple Equations is copiouſly illuſtrated by Arithmetical Queſtions in this Chapter, as alſo in the ſecond and third Books of my *Algebraical Elements*; and the Reſolution of Adfected or Compound Equations in Numbers is handled in the 15, 16, and 17. Chapters of this Book, as alſo in the 10, and 11. Chapters of the Second Book. But how Algebraical Operations are applicable to the ſolving of Geometrical Problems, I ſhall ſhew in my fourth Book of *Algebraical Elements.*

V. When an Arithmetical Queſtion is propoſed, the number ſought muſt firſt of all be aſſumed or ſuppoſed to be known; and you may repreſent it by the Letter a, or any other Vowel: You may likewiſe repreſent the given Numbers by Conſonants, as, b, c, d, &c. *Renates des Cartes* puts for given Quantities the former Letters of the Alphabet, as, a, b, c, d, &c. but for Quantities ſought the latter Letters, z, y, x, &c. Then with the Letters repreſenting the Numbers given and ſought, an orderly proceſs muſt be made, by adding, ſubtracting, multiplying or dividing, *&c.* according to the Import of the Queſtion, until at length an Equation be found out between the Number ſought or ſome Power or Powers of it, and ſome Number or Numbers given: Laſtly, when the Equation ſo found out is a pure or ſimple Equation, the Number ſought may be diſcovered by ſome of the Reductions in the foregoing 12, and 13. Chapters; but when the Equation is Adfected or Compounded, the Reſolution thereof belongs either to the 15. Chapter of this firſt Book, or the 10, and 11. Chapters of the ſecond Book

VI. In the Reſolution of every Queſtion, I proceed from the Beginning to the End by ſteps numbred in the Margin, by 1, 2, 3, 4, 5, *&c.* And becauſe *Numeral Algebra* is more eaſie for Learners than the *Literal*, (though not ſo uſeful for the Reaſons before given in *Sect.* 8. *Chap.* 1.) I have in many Queſtions expreſs'd the Operation belonging to every ſtep in both kinds ~~of~~ *Algebra*, that the one may explain the other: So in the ſecond ſtep of the Reſolution of the following firſt Queſtion, the leſſer Number ſought is expreſs'd by *Numeral Algebra* thus, $26-a$; but by *Literal Algebra* thus, $b-a$. Alſo, in the fourth ſtep, the Equation by numeral Algebra is $2a-26=8$; but by literal Algebra it is $2a-b=c$.

VII. When an Equation is found out in any of the following Queſtions, I take it for granted that the Reader knows how to reduce it, if need be, according to the Rules in the foregoing 11, 12, and 13. Chapters, that I may avoid tedious repetitions of what has been already explain'd. Theſe things premiſed, I proceed to the Queſtions themſelves.

QUEST.

QUEST. 1.

There are two Numbers whoſe Sum is 26, (or b,) and their difference, (to wit, the exceſs of the greater above the leſſer) is 8, (or c;) What are the Numbers?

RESOLUTION:	*Numeral,*	*Literal.*
1. For the greater Number put	a	a
2. Then ſubtracting that Number a from the given Sum, the Remainder will be the leſſer Number, to wit, ...	$26-a$	$b-a$
3. And by ſubtracting the leſſer number from the greater, the Remainder will be their difference, to wit, ...	$2a-26$	$2a-b$
4. Which difference found out in the laſt ſtep muſt be equal to the given difference 8, (or c) whence this Equation ariſes, ...	$2a-26=8$	$2a-b=c$
5. From which Equation, after it is duly reduced according to *Sect.* 3. and 5. of *Chap.* 12. the greater number ſought will be diſcovered, to wit, ...	$a=17$	$a=\frac{1}{2}b+\frac{1}{2}c$
6. And conſequently from the fifth and ſecond ſteps the leſſer Number is alſo diſcovered, to wit, ...	9, that is,	$\frac{1}{2}b-\frac{1}{2}c$.

So the Numbers ſought are found 17 and 9, whoſe Sum is 26, and their difference is 8, as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be expreſs'd by words, they will give this

THEOREM.

Half the difference of any two Numbers added to half their Sum, gives the greater Number: But half the difference of any two Numbers ſubtracted from half their Sum, leaves the leſſer Number.

Therefore the Sum and difference of any two Numbers being given ſeverally, the Numbers themſelves are alſo given by the ſaid Theorem; but it preſuppoſes that the Number given for the Difference muſt be leſs than the Number given for the Sum.

Note here once for all, That the Numbers given in a Queſtion cannot always be choſen at pleaſure, but ſometimes, they muſt be ſubject to one or more Determinations, which for the moſt part (though not always) are diſcoverable by the Theorem or Canon that reſults from the Reſolution. But how Limits or Determinations are diſcovered, I ſhall have occaſion to ſhew hereafter in my ſecond, third, and fourth Books of *Algebraical Elements.*

QUEST. 2.

There are two Numbers whoſe Sum is 40, (or b,) and the greater Number has ſuch proportion to the leſſer as 3 to 2, or, as r to s;) What are the Numbers?

1. For the greater Number ſought put . . .	a	a
2. Then to find the leſſer Number, ſay by the Rule of Three, If 3 . 2 : : a . $\frac{2a}{3}$ Or, r . s : : a . $\frac{sa}{r}$ whence the leſſer Number is	$\frac{2a}{3}$	$\frac{sa}{r}$

3. There-

3. Therefore the Sum of the two Numbers ſought is	$\frac{5a}{3}$	$a+\frac{sa}{r}$
4. Which Sum found out in the laſt ſtep muſt be equal to the given Sum 40, (or *b*,) whence this Equation	$\frac{5a}{3}=40$	$a+\frac{sa}{r}=b.$
5. Which Equation, after due Reduction according to *Sect.* 2. and 5. of *Chap.* 12. gives the greater Number	$a=24$	$a=\frac{rb}{r+s}$
6. And from the fifth, firſt, and ſecond ſteps, the leſſer Number is alſo diſcovered, to wit,	16, or	$\frac{sb}{r+s}$

So the Numbers ſought are found 24 and 16, which will ſatisfie the Conditions in the Queſtion; for their Sum is 40; and the greater has ſuch proportion to the leſs as 3 to 2, as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be reſolved into Proportionals, according to *Sect.* 3. *Chap.* 13. there will ariſe this

THEOREM.

As the Sum of both the Terms which expreſs the Reaſon (or Proportion) of two Numbers, is to the Sum of the ſame two Numbers; ſo is the greater Term to the greater Number; and ſo is the leſſer Term to the leſſer Number.

Therefore the Sum of two Numbers being given, as alſo their Reaſon, or Proportion; the Numbers ſhall alſo be given ſeverally by the ſaid Theorem.

QUEST. 3.

There are two Numbers whoſe difference is 8, (or *d*,) and the greater Number has ſuch proportion to the leſſer as 3 to 2, (or as *r* to *s*;) what are the Numbers?

1. For the greater Number put	a	a
2. Then to find the leſſer Number ſay by the Rule of Three, If 3 . 2 :: a . $\frac{2a}{3}$ Or if r . s :: a . $\frac{sa}{r}$ whence the leſſer Number is	$\frac{2a}{3}$	$\frac{sa}{r}$
3. Therefore by ſubtracting the leſſer Number from the greater, the Remainder ſhall be their difference, to wit, ...	$\frac{a}{3}$	$a-\frac{sa}{r}$
4. Which difference muſt be equal to the given difference 8 (or *d*,) hence this Equation ariſes	$\frac{a}{3}=8$	$a-\frac{sa}{r}=d.$
5. Which Equation, after due Reduction, diſcovers the greater Number ſought, to wit,	$a=24$	$a=\frac{rd}{r-s}$
6. And from the fifth, firſt, and ſecond ſteps the leſſer number will be alſo made known, to wit,	$=16$	$=\frac{sd}{r-s}$

So the Numbers ſought are found 24 and 16, which will ſolve the Queſtion; for their difference is 8, and they are in the proportion of 3 to 2, as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be converted into Proportionals (according to *Sect.* 3. *Chap.* 13.) there will ariſe this

THEOREM.

As the difference of the two Terms which expreſs the Reaſon or Proportion of two Numbers is to the difference of the ſame two Numbers, ſo is the greater Term to the greater Number; and ſo is the leſſer Term to the leſſer Number.

Therefore the Difference and Reaſon of two Numbers being ſeverally given, the Numbers themſelves ſhall be alſo given by the ſaid Theorem.

QUEST.

QUEST. 4.

There are two Numbers whoſe Sum is 7, (or b,) and the difference of their Squares is 21, (or d;) what are the Numbers?

1. For the greater Number ſought put	a	a
2. Then ſubtracting the greater Number from the given Sum, the Remainder is the leſſer Number, to wit, ...	$7-a$	$b-a$
3. Therefore from the firſt ſtep the Square of the greater Number is	aa	aa
4. And from the ſecond ſtep the Square of the leſſer Number is	$aa-14a+49$	$aa-2ba+bb$
5. Therefore the difference of the Squares of the two numbers ſought ſhall be . . .	$14a-49$	$2ba-bb$
6. Which difference muſt be equal to the given difference 21 (or d,) whence this Equation ariſes	$14a-49=21$	$2ba-bb=d$
7. Which Equation, after due Reduction according to *Sect.* 3, and 5. of *Chap.* 12. diſcovers the greater number ſought, to wit,	$a=5$	$a=\frac{bb+d}{2b}$
8. And from the ſeventh and ſecond ſteps, the leſſer number will be alſo made known, to wit, ...	$=2$	$=\frac{bb-d}{2b}$

So the Numbers ſought are found 5 and 2, which will ſolve the Queſtion; for their Sum is 7, and the difference of their Squares is 21, (to wit, 25—4;) as was preſcribed.

Moreover, If the two laſt ſteps of the literal Reſolution be expreſs'd by words, they will give this

THEOREM.

If to the Square of the Sum of any two numbers the difference of their Squares be added, and the Sum of that addition be divided by the double Sum of the two Numbers, the Quotient will be the greater Number: But if from the Square of the Sum of two Numbers the difference of their Squares be ſubtracted, and the Remainder be divided by the double Sum of the two Numbers, the Quotient will give the leſſer Number.

Therefore the Sum of two numbers being given, as alſo the difference of their Squares, the numbers themſelves ſhall be given ſeverally; but it preſuppoſes the ſquare of the given Sum to exceed the given difference.

QUEST. 5.

There are two numbers whoſe difference is 3, (or c,) and the difference of their Squares is 21, (or d;) what are the Numbers?

1. For the leſſer number ſought put	a	a
2. To which adding the given difference 3, (or c,) the Sum will make the greater number, to wit, ...	$a+3$	$a+c$
3. Therefore the ſquare of the greater number is	$aa+6a+9$	$aa+2ca+cc$
4. And the ſquare of the leſſer number is	aa	aa
5. Therefore the difference of thoſe Squares is	$6a+9$	$2ca+cc$
6. Which difference muſt be equal to the given difference of the ſquares; whence this Equation ariſes, to wit, ...	$6a+9=21$	$2ca+cc=d$
7. Which Equation, after due Reduction (according to *Sect.* 3, and 5. of *Chap.* 12.) diſcovers the leſſer number, to wit, . . .	$a=2$	$a=\frac{d-cc}{2c}$
8. And from the ſeventh and ſecond Equations, the greater number will be found . . .	$=5$	$=\frac{d+cc}{2c}$

So the Numbers ſought are 5 and 2, which will ſolve the Queſtion; for their difference is 3, and the difference of their Squares is 21; as was preſcribed.

Moreover, the two laſt ſteps of the literal Reſolution afford this

THEOREM.

If to the difference of the Squares of any two Numbers the Square of their difference be added, and the Sum of that Addition be divided by the double of the difference of thoſe two Numbers, the Quotient will give the greater Number: But if from the difference of the Squares of two Numbers the Square of their difference be ſubtracted, and the Remainder be divided by the double of the difference of thoſe two Numbers, the Quotient ſhall be the leſſer Number.

Therefore the difference of any two Numbers being given, as alſo the difference of their Squares, the Numbers themſelves ſhall alſo be given ſeverally by this Theorem; but it preſuppoſes the given difference of the Squares of the two Numbers to exceed the Square of the given difference of the ſame two Numbers.

QUEST. 6.

A certain Perſon being asked what was the Age of every one of his four Sons, anſwered; the eldeſt was four Years (or b) elder than the ſecond, the ſecond was four Years elder than the third, the third was four Years elder than the fourth or youngeſt; and the double of the youngeſt Sons Age was equal to the Age of the eldeſt; what was the Age of each Son?

1. For the Age of the eldeſt Son put	a	a
2. Then from the Age of the eldeſt Son ſubtracting 4 (or b) there will remain the ſecond Sons Age, to wit, ...	$a-4$	$a-b$
3. Likewiſe from the ſecond Son's Age ſubtracting 4 (or b) the Remainder will be the third Son's Age, to wit, ...	$a-8$	$a-2b$
4. Again, from the third Son's Age ſubtracting 4 (or b) there will remain the fourth or youngeſt Son's Age, to wit, ...	$a-12$	$a-3b$
5. But according to the Queſtion, the double of the Age in the fourth ſtep muſt be equal to the Age in the firſt ſtep, whence this Equation will ariſe, ...	$2a-24=a$	$2a-6b=a$
6. Which Equation duly reduced diſcovers the Age of the eldeſt Son, to wit, ...	$a=24$	$a=6b$

Wherefore the Ages of the four Sons were 24, 20, 16, and 12; for the firſt exceeds the ſecond by 4, which is alſo the exceſs of the ſecond above the third, the third above the fourth, and the double of the fourth is equal to the firſt, as was preſcribed in the Queſtion.

Moreover the laſt ſtep of the literal Reſolution ſhews, that if inſtead of 4, any other Number be given for the common difference of the four Sons Ages, then ſix times that common difference will give the eldeſt Sons Age, which ſhall be equal to the double of the Age of the youngeſt.

QUEST. 7.

A Merchant began to Trade with a certain Number of Pounds: By his firſt Voyage he doubled that Stock; by his ſecond he loſt 1200 Pounds (or b) by his third he doubled his remaining Stock; by his fourth he loſt again 1200 Pounds, and then had no money left. The Queſtion is, to find how many Pounds the Merchant began to Trade with?

1. For

1. For the number of Pounds which the Merchant began to trade with put	a	a
2. Then the double of that number gives the number of Pounds he had at the end of his firſt Voyage, to wit, ...	$2a$	$2a$
3. From which laſt number ſubtracting 1200 (or *b*,) the Remainder ſhews the number of Pounds that remained to the Merchant at the end of his ſecond Voyage, to wit, . .	$2a-1200$	$2a-b$
4. Which remaining number being doubled gives the number of Pounds which the Merchant had at the end of his third Voyage, to wit, ...	$4a-2400$	$4a-2b$
5. From which laſt number ſubtracting again 1200 (or *b*) Pounds loſt by the fourth Voyage, the Remainder muſt be equal to nothing; hence this Equation, ...	$4a-3600=0$	$4a-3b=0$
6. Which Equation, after due Reduction, gives	$a=900$	$a=\frac{3}{4}b.$

Whence it is found that the Merchant began to trade with 900 Pounds; which number will ſatisfie the Conditions in the Queſtion.

Moreover the laſt ſtep of the literal Reſolution ſhews, that if inſtead of 1200 any other number were given, the Merchants ſtock at firſt would be three Quarters of that given number.

QUEST. 8.

A Gentleman hired a Servant for a Year, for 120 Shillings (or *c*,) together with a livery Cloak valued at a certain number of Shillings: But when $\frac{7}{12}$ (or *d*) parts of the Year were expired, the Maſter falling at variance with his Servant puts him away, and gives him the Cloak with 50 Shillings, (or *f*;) and ſo the Servant received full ſatisfaction for the time of his ſervice. The Queſtion is, to find how many Shillings the Cloak was valued at?

1. For the number of Shillings which the Cloak was valued at put	a	a
2. Then to find what part of the value of the Cloak was due to the Servant when $\frac{7}{12}$ (or *d*) parts of the Year were expired, ſay by the Rule of Three, If $1 : a :: \frac{7}{12} . (\frac{7a}{12}$ Or, if $1 . a :: d . (da$ whence the deſired part of the value of the Cloak is found	$\frac{7a}{12}$	da
3. Find likewiſe what part of the 120 (or *c*) Shillings was due to the Servant when $\frac{7}{12}$ (or *d*) parts of the Year were expired, and ſay, If $1 . 120 :: \frac{7}{12} . (70$ Or, $1 . c :: d . (cd$ whence the part deſired is found	70	cd

4. Now foraſmuch as the Cloak together with the 50 Shillings the Servant rece ought to be equal to the part of the Cloak, together with the part of the Shillings that was due to him at the time he left his ſervice; therefore from the premiſes there ariſes this Equation:

$$a+50=\frac{7a}{12}+70; \qquad \text{Or,} \qquad a+f=da+cd.$$

5. Which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12: will give the deſired value of the Cloak, to wit,

$$a = 48 = \frac{cd \backsim f}{1 \backsim d}:$$

Whence it is evident that the Cloak was valued at 48 Shillings; and the laſt Equation diſcovers this

CANON.

Multiply the Money which the Servant was to receive beſides the Cloak for a Years Wages, by the time he ſerved; then divide the difference between that Product and the Money he received when he left his ſervice by the difference between 1 (or unity) and the ſame time he ſerved; ſo the Quotient gives the value of the Cloak.

By which Canon the value of the Cloak will be found to be 48 *s.* as above.

The Proof.

$$48 + 50 = 98.$$

$$\tfrac{7}{12} \text{ of } 48, + \tfrac{7}{12} \text{ of } 120 = 98.$$

QUEST. 9.

A certain Man finding divers poor Perſons at his Door, gave every one of them three pence (or *b*,) and had ſix pence (or *c*) left; but if he would have given them four pence (or *f*) a piece, he ſhould have wanted two pence (or *g*.) How many poor Perſons were there?

1. For the number of poor Perſons put . . . . a
2. Then foraſmuch as that number multiplied by 3 (or *b*) and the Product increaſed with 6 (or *c*) makes the whole number of pence that the giver had: And, becauſe if the ſame number of poor Perſons be multiplyed by 4 (or *f*,) the Product leſs by 2 (or *g*) muſt alſo make the ſame number of pence: hence this Equation;

$$3a + 6 = 4a - 2:$$

$$\text{Or,}\quad ba + c = fa - g.$$

3. Which Equation after due Reduction according to *Sect.* 3, and 5. of *Chap.* 12. diſcovers the number of poor Perſons to be 8: *viz.*

$$8 = \frac{c+g}{f-b} = a.$$

QUEST. 10.

One being asked what a Clock it was, anſwer'd, That the time then paſt from Noon was equal to $\frac{33}{40}$ (or, *b*) parts of the time remaining until midnight: What was the preſent Hour? ſuppoſing the time between Noon and Midnight to be divided into 12 (or *c*) equal Hours.

1. For the Hour ſought after noon put . . .	a	a
2. Which ſubtracted from 12 (or *c*) leaves the time remaining until midnight, to wit,	$12-a$	$c-a$
3. Then $\frac{33}{40}$ (or *b*) parts of the ſaid remaining time will be	$\frac{396}{40}-\frac{33}{40}a$	$bc-ba$
4. Therefore from the firſt and third ſteps (according to the Queſtion) this Equation ariſes, to wit, ...	$a=\frac{396}{40}-\frac{33}{40}a$	$a=bc-ba$
5. Which Equation after due Reduction according to *Sect* 2, 3, and 5. of *Chap.* 12. gives the Hour ſought, to wit, ...	$a=5\frac{31}{73}$	$a=\frac{bc}{b+1}$

So the time ſought was $5\frac{31}{73}$ Hours after noon, and conſequently the remaining time until midnight was $\frac{480}{73}$ Hours, whereof $\frac{33}{40}$ is equal to the ſaid $5\frac{31}{73}$; as was preſcribed in the Queſtion.

QUEST.

QUEST. 11.

A General of an Army having ſet his Soldiers in a Square Battel, there happened to be 500 (or *b*) Soldiers to ſpare; but to increaſe the Square ſo as that its ſide might conſiſt of 1 (or *c*) Soldier more than the ſide of the former Square, there would be 29 (or *d*) Soldiers wanting. The Queſtion is, to find how many Soldiers the General had in his Army.

1. For the Number of Soldiers that made the ſide of the firſt Square, put	a	a
2. Then that ſide multiplied by it ſelf gives the Number of Soldiers in the firſt ſquare Battel, to wit,	aa	aa
3. Therefore the number of Soldiers in the whole Army was	$aa+500$	$aa+b$
4. Then to the end the ſide of another Square may exceed the ſide of the former by 1 (or *c*,) let it be	$a+1$	$a+c$
5. Which latter ſide multiplied by it ſelf gives the Number of Soldiers in the latter ſquare Battel, to wit,	$aa+2a+1$	$aa+2ca+cc$

6. But the number of Soldiers in the laſt ſtep exceeded the number of Soldiers in the Generals Army by 29 (or *d*;) therefore ſubtracting 29 (or *d*) from the number in the laſt ſtep, the Remainder muſt be equal to the number in the third ſtep: hence this Equation ariſes, to wit,

$$aa+2a+1-29 = aa+500,$$

$$\text{Or, } aa+2ca+cc-d = aa+b.$$

7. Which Equation after due Reduction (according to *Sect.* 3, and 5. of *Chap.* 12.) makes known the ſide of the firſt Square, *viz.*

$$a = 264 = \frac{b+d}{2c} - \tfrac{1}{2}c.$$

8. Laſtly, If the ſide or number found out in the laſt ſtep be multiplied by it ſelf, and the Product be increaſed with 500 (or *b*,) there will come forth the number of Soldiers that were in the Generals Army, to wit,

$$70196 = \frac{bb+2bd+dd}{4cc} + \tfrac{1}{4}cc + \tfrac{1}{2}b - \tfrac{1}{2}d.$$

Whence it is manifeſt that the General had 70196 Soldiers in his Army: Alſo, the ſide of the firſt ſquare Battel conſiſted of 264 Soldiers; and the ſide of the latter 265; this multiplied by it ſelf produces 70225, which exceeds the ſaid 70196 by 29: Moreover, the ſaid 70196 exceeds the Square of 264 by 500; as the Queſtion requires.

QUEST. 12.

Two Perſons, *A* and *B*, diſcourſe of their Money in this manner, *viz.* *A* ſaith, if *B* would give him a Crown (or *c*,) then *A* ſhould have as many Crowns as *B* had left; but *B* ſaith, if *A* would give him a Crown, then *B* ſhould have twice as many Crowns as *A* had left. How many Crowns had each Perſon?

1. For the number of Crowns which *A* had, put	a
2. Then, according to the Queſtion, if that number be increaſed with 1 Crown (or *c*,) the Sum will be the number of Crowns that remained to *B* after he had given 1 Crown to *A*, to wit,	$a+c$
3. And conſequently, by adding 1 Crown (or *c*) to the ſaid number of Crowns that remained to *B* after he had given 1 Crown to *A*, the Sum will be the number of Crowns which *B* had at firſt, to wit,	$a+2c$

4. Again, according to the Queſtion, if 1 Crown (or c) be added to the ſaid $a+2c$ in the laſt ſtep, and ſubtracted from a in the firſt ſtep, the Sum muſt be equal to the double of the Remainder; hence this Equation, ... $a+3c=2a-2c$

5. Which Equation, after due Reduction, diſcovers the number of Crowns that *A* had at firſt, to wit, ... $a=5c$

6. And from the fifth and third ſteps, the number of Crowns which *B* had at firſt will alſo be made known, to wit, ... $a+2c=7c$

So it is found that *A* had 5 Crowns, and *B* 7 Crowns, as will be evident by

The Proof.

$$5+1=7-1=6$$
$$7+1=4+4=8$$

QUEST. 13.

A Vintner having two ſorts of *French* Wines, to wit, one ſort worth 10*d.* (or b) the Quart, and the other 6 *d* (or c) *per* Quart, would have a mixed Quantity of both ſorts to conſiſt of 100 Quarts (or m) that might be worth 7 *d.* (or f) *per* Quart. The Queſtion is, to find what Quantity of each ſort of Wine muſt be taken to make that mixture?

1. For the number of Quarts that muſt be taken of the better ſort of Wine to make the mixture, put	a	a
2. Which number ſubtracted from 100 (or m) leaves the number of Quarts of the worſer ſort of wine in the mixture, to wit,	$100-a$	$m-a$
3. Then find the worth of the better ſort of Wine in the mixture at 10 *d.* (or b) *per* Quart, and ſay by the Rule of Three, If 1 . 10 :: a . ($10a$, Or, if 1 . b :: a . (ba. So the Quantity of the better ſort of Wine in the mixture is found worth	$10a$	ba
4. Find likewiſe the worth of the worſer ſort of Wine in the mixture at 6 *d.* (or c) *per* Quart, and ſay, If 1 . 6 :: $100-a$. ($600-6a$, Or, 1 . c :: $m-a$. ($cm-ca$. So the Quantity of the worſer ſort of Wine in the mixture is found worth . . .	$600-6a$	$cm-ca$
5. Therefore the Sum of the values of both the Quantities mentioned in the two laſt ſteps is	$4a+600$	$ba+cm-ca$

6. Which Sum muſt be equal to the Product made by the Multiplication of 100 (or m) the total mixed Quantity, by 7 (or f) the preſcribed mean price; hence this Equation ariſes, to wit,

$$4a+600=700,$$
Or, ... $ba+cm-ca=fm.$

7. Which Equation, after due Reduction, diſcovers the value of a, to wit, the number of Quarts that muſt be taken of the better ſort of Wine to make the mixture, *viz.*

$$a=25=\frac{fm-cm}{b-c}.$$

8. And from the ſeventh and ſecond ſteps the number of Quarts that ought to be taken of the worſer ſort of Wine to make the mixture will alſo be made known, *viz.*

$$75=\frac{bm-fm}{b-c}.$$

9. From the two laſt ſteps it is evident, That 25 Quarts of the better ſort of Wine, and 75 Quarts of the worſer ſort, muſt be taken to make the preſcribed mixture; for thoſe

quantities at their reſpective prices will be worth in the whole 700 pence, which is alſo the juſt worth of 100 quarts at 7 pence *per* quart.

Moreover, If the latter parts of the two laſt Equations be reſolved into Proportionals, (according to *Sect.*3. *Chap.*13.) and be expreſs'd by words, they will give this following

THEOREM.

As the difference between the given prices of two ſorts of Wines or other things whereof a mixture is deſired, is to the total Quantity required to be in the mixture; So is the exceſs by which ſome mean price preſcribed for the total Quantity mixed exceeds the leſſer of the two given prices, to the Quantity to be taken of the better ſort of Wine: And ſo is the exceſs of the greater of the two given prices above the mean price, to the Quantity that is to be taken of the worſer ſort of Wine.

This Theorem contains the ſubſtance of the Rule of Alligation-alternate in Vulgar Arithmetic. But how Queſtions of this nature, when three or more things are to be mixed, may be ſolved more generally than by that Rule, I ſhall hereafter ſhew in *Chap.* 13. of my ſecond Book of *Algebraical Elements.*

QUEST. 14.

A Ciſtern in a certain Conduit is ſupplied with Water by two Pipes, of ſuch capacities, that by both their Cocks *A* and *B* ſet open at once the Ciſtern will be filled in 12 (or *b*) Hours; but by the Cock *A* alone in 20 (or *c*) Hours: The Queſtion is, to find in what time the Ciſtern will be filled by the Cock *B* alone?

1. Suppoſe the time ſought to be	a	a
2. Then find what part of the Ciſtern will be filled by the Cock *B* alone in 12 (or *b*) Hours, and ſay by the Rule of Three, If $a \;.\; 1 :: 12 \;.\; (\frac{12}{a},$ Or, if $a \;.\; 1 :: b \;.\; (\frac{b}{a};$ whence the ſaid part is found	$\frac{12}{a}$	$\frac{b}{a}$
3. Find likewiſe what part of the Ciſtern will be filled by the Cock *A* alone in 12 (or *b*) Hours, and ſay, If $20 \;.\; 1 :: 12 \;.\; (\frac{3}{5},$ Or, if $c \;.\; 1 :: b \;.\; (\frac{b}{c};$ whence the ſaid part is found	$\frac{3}{5}$	$\frac{b}{c}$
4. But thoſe parts found out in the ſecond and third ſteps muſt be equal to the whole Ciſtern, to wit, 1; hence this Equation ariſes,	$\frac{12}{a}+\frac{3}{5}=1.$	$\frac{b}{a}+\frac{b}{c}=1.$
5. Which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. diſcovers the value of *a*, to wit, the time ſought, *viz.*	$a=30$	$a=\frac{bc}{c-b}$

Whence it appears, that by the Cock *B* ſet open alone the Ciſtern would be filled in 30 Hours: And, if the laſt Equation of the literal Reſolution be reſolved into Proportionals according to *Sect.* 3. *Chap.* 13. there will ariſe this following

CANON.

As the difference of the two numbers of ſpaces of Time given in the Queſtion is to either of them, ſo is the other to the Time ſought, *viz.*

$$\text{As } 8\ (20-12)\ .\ 12 :: 20\ .\ 30,$$

$$\text{Or, as} \ldots : c-b\ .\ b :: c\ .\ \frac{bc}{c-b}.$$

The Proof may be made by ſolving this Queſtion, *viz.*

If a Ciſtern will be filled with Water by a Cock *A* in 20 hours, and by another Cock *B* in 30 hours; in what time will the Ciſtern be filled by both Cocks ſet open at once? *Anſw.* 12 hours.

Firſt find what part or parts of the Ciſtern will be filled by each Cock in one and the ſame time; then it ſhall be, As the Sum of thoſe parts is to that common time, ſo is the whole Ciſtern (to wit, 1,) to the time wherein the whole Ciſtern will be filled by both Cocks ſet open at once; *viz.*

	ho.		*Ciſt.*		*ho.*		
Firſt, If	30	.	1	::	20	.	($\frac{2}{3}$ Ciſtern.
						add	1 Ciſtern.
						Sum,	$1\frac{2}{3}$ Ciſt.

So it is found that $1\frac{2}{3}$ Ciſtern will be filled in 20 hours by both Cocks *A* and *B* ſet open at once; then ſay again by the Rule of Three,

Ciſt.		*ho.*		*Ciſt.*		
$1\frac{2}{3}$	.	20	::	1	.	(12 hours.

If the Operation of this latter Queſtion be formed Algebraically by Letters, it will afford this

CANON.

As the Sum of the two given numbers expreſſing ſpaces of time in the latter Queſtion, is to either of them; So is the other to the time ſought.

QUEST. 15.

A Shepherd in the time of War driving a Flock of Sheep, fell into the hands of three Companies of plundering Soldiers, who compell'd him to deliver the half of his flock with half a Sheep over and above to the firſt Company; alſo half of his remaining flock with half a Sheep to the ſecond Company; likewiſe the half of the reſt of the flock with half a Sheep to the third Company: All which Diviſions the Shepherd exactly perform'd without killing a Sheep, and then there remained only 20 (or *b*) Sheep for himſelf. The queſtion is, to find How many Sheep the Shepherd had in his flock at firſt?

1. Let the Number of Sheep which the Shepherd had in his Flock at firſt be repreſented by	a
2. Then the half of that number is $\frac{1}{2}a$, to which adding $\frac{1}{2}$, (that is, half a Sheep.) the ſum will be the Number of Sheep delivered to the firſt Company of Soldiers, to wit, ...	$\frac{1}{2}a+\frac{1}{2}$
3. And by ſubtracting the ſaid $\frac{1}{2}a+\frac{1}{2}$ from a, the remainder will be the number of Sheep that were left to the Shepherd after he had ſatisfied the firſt Company of Soldiers, to wit, ...	$\frac{1}{2}a-\frac{1}{2}$
4. Then the half of that remaining Flock is $\frac{1}{4}a-\frac{1}{4}$, to which adding $\frac{1}{2}$, (that is, $\frac{1}{2}$ Sheep,) the ſum will be the Number of Sheep delivered to the ſecond Company of Soldiers, to wit, . . .	$\frac{1}{4}a+\frac{1}{4}$
5. Which $\frac{1}{4}a+\frac{1}{4}$ being ſubtracted from $\frac{1}{2}a-\frac{1}{2}$ in the third ſtep, the remainder will be the number of Sheep that were left to the Shepherd after he had ſatisfied the ſecond Company of Soldiers, to wit, ...	$\frac{1}{4}a-\frac{3}{4}$
6. Then the half of the remaining flock in the laſt ſtep is $\frac{1}{8}a-\frac{3}{8}$, to which adding $\frac{1}{2}$, (to wit, $\frac{1}{2}$ Sheep) the Sum will be the number of Sheep delivered to the third Company, to wit . . .	$\frac{1}{8}a+\frac{1}{8}$
7. Which $\frac{1}{8}a+\frac{1}{8}$ being ſubtracted from $\frac{1}{4}a-\frac{3}{4}$ in the fifth ſtep, the remainder will be the number of Sheep that were left to the Shepherd after he had ſatisfied all the three Companies, to wit,	$\frac{1}{8}a-\frac{7}{8}$
8. But the remainder in the laſt ſtep muſt be equal to 20 (or b) the number given in the Queſtion; hence this Equation, . . .	$\frac{1}{8}a-\frac{7}{8}=b=20.$
9. Which Equation, after due Reduction, diſcovers the Number ſought, to wit, ...	$a=8b+7=167$

So it appears that the Shepherd had 167 Sheep in his Flock at firſt.

The Proof.

1. The half of 167 is 83$\frac{1}{2}$, to which adding $\frac{1}{2}$, the ſum is 84, which was the number of Sheep delivered to the firſt Company of Soldiers; and then there remained 83 Sheep to the Shepherd.

2. Again, the half of 83 is 41$\frac{1}{2}$, which increaſed with $\frac{1}{2}$ makes 42, the number of Sheep delivered to the ſecond Company; and then there remained 41 Sheep to the Shepherd.

3. Laſtly, the half of 31 is 40$\frac{1}{2}$, which increaſed with $\frac{1}{2}$ makes 21, which was the number of Sheep delivered to the third Company; and ſo there remained 20 Sheep to the Shepherd, as the Queſtion declares.

Moreover, the Equation in the laſt ſtep of the Reſolution ſhews, That if any whole number inſtead of 20 be preſcribed in the Queſtion, that number multiplied by 8, and the Product increaſed with 7 will give a number capable of the like Diviſion as 167 that anſwered the Queſtion: So if there had been but one Sheep left for the Shepherd, then his Flock at firſt was 15 Sheep; if 2 had been left, his Flock at firſt was 23; if 3 Sheep had been left, then he had 31 when he firſt met with the Soldiers; and ſo by a continual addition of 8, all the odd Numbers capable of that Diviſion the Queſtion requires may be orderly found out. But to have nothing left after ſuch Diviſion is made, the Number firſt to be divided is 7.

It is alſo Evident, that by continuing the Reſolution an odd Number may be found out, that ſhall be capable of being divided according to the import of the Queſtion, as many times as ſhall be deſired.

QUEST. 16.

Two Merchants, *A* and *B*, were Co-partners in Traffic: the ſum of their Stocks was 300 *l.* (or b;) the Stock of *A* continued in Company 9 (or c) Months, and the Stock of *B* 11 (or d) Months; they gained a certain ſum of Money which they divided equally. The Queſtion is, to find what each Merchants Stock was at firſt?

1. For the Stock of *A* when he entred Partnerſhip, put	a	a
2. Then ſubtracting that ſtock from the Joynt ſtock 300 *l.* (or b) the Remainder will be the Stock of *B*, to wit, ...	$300 - a$	$b - a$
3. The firſt ſtock multiplied by the time it continued in Company produces . . .	$9a$	ca
4. And the other ſtock multiplied by its time produces.	$3300 - 11a$	$db - da$

5. Now foraſmuch as the Merchants divided the gain equally, therefore the Products in the third and fourth ſteps muſt be equal to one another, (according to the nature of the Rule of Fellowſhip with Time.) Hence this Equation ariſes;

$$9a = 3300 - 11a,$$

$$\text{Or, . . . } ca = db - da$$

6. Which Equation, after due Reduction, according to *Sect* 3, and 5. of *Chap.* 12. will diſcover the Stock which *A* put in, *viz.*

$$a = 165 = \frac{db}{c+d}.$$

7. And from the 6, and 2. ſteps the ſtock which *B* put in will alſo be made known, to wit,

$$135 = \frac{cb}{c+d}.$$

So it is found that the ſtock of *A* was 165 *l.* and that of *B*, 135 *l.* For, $165 \times 9 = 135 \times 11$.

Moreover, If the latter parts of the two Equations in the ſixth and ſeventh ſteps be reſolved into Proportionals, according to *Sect.* 3. *Chap.* 13. there will ariſe this

CANON.

As the ſum of both ſpaces of time given in the Queſtion, is to the given ſum of the two particular ſtocks ſought; ſo is the greater time to the particular ſtock belonging to the leſſer time: and ſo is the leſſer time to the ſtock belonging to the greater time.

QUEST.

QUEST. 17.

A certain Man being asked how many Years old he was, anſwered, If $\frac{1}{20}$ (or b) part of the Number of Years he had lived, were multiplied by $\frac{5}{8}$ (or c) parts of the ſame number, the Product would give his Age. What was his Age?

1. For the Number of the Years ſought put . .	a	a
2. Then according to the Queſtion, multiplying $\frac{1}{20}a$ by $\frac{5}{8}a$ (or ba by ca) the Product will be	$\frac{1}{32}aa$	$bcaa$
3. Which Product muſt be equal to the number of Years ſought, *viz.*	$\frac{1}{32}aa = a$	$bcaa = a$
4. Then, by reducing that Equation according to *Sect.* 4, and 5. of *Chap.* 12. the number of years ſought will be diſcovered, *viz.* .	$a = 32$	$a = \frac{1}{bc}$

Whence it is manifeſt that the Reſpondent was 32 Years of Age; for if $1\frac{3}{5}$, that is, $\frac{1}{20}$ of 32, be multiplied by 20, that is, $\frac{5}{8}$ of 32; the Product will be 32, to wit, the Number of Years ſought. It is alſo evident by the laſt Equation in the literal Reſolution, that if 1 (to wit Unity) be divided by the Product made by the multiplication of the two numbers given in the Queſtion, the Quotient will be the number ſought.

QUEST. 18.

There are two Numbers, the greater of which has ſuch proportion to the leſſer as 3 to 2, (or as r to s;) and the ſum of the ſaid numbers has ſuch proportion to the ſum of their Squares, as 1 to 13, (or as b to c.) What are the Numbers?

1. For the greater Number ſought put . .	a	a
2. Then, (according to *Queſt.* 2. in *Sect.* 4. *Chap.* 10.) the ſum of the two Numbers will be found	$\frac{5a}{3}$	$a + \frac{sa}{r}$
3. And (according to *Queſt.* 5. in the ſaid *Sect.* 4. *Chap.* 10) the ſum of the Squares of the two Numbers ſought will be . .	$\frac{13aa}{9}$	$aa + \frac{ssaa}{rr}$
4. Again, by the help of the latter Proportion given in the Queſtion, and of the ſum found in the ſecond ſtep, ſearch out the ſum of the Squares of the two numbers ſought; *viz.* ſay by the Rule of Three, If $1 . 13 :: \frac{5a}{3} . (\frac{65a}{3}$ Or, if $b . c :: a + \frac{sa}{r} . \frac{cra + csa}{br}$ whence the ſum of the ſaid Squares is found	$\frac{65a}{3}$	$\frac{cra + csa}{br}$

5. But the ſum of the Squares found out in the third ſtep muſt be equal to the ſum in the fourth; hence this Equation, *viz.*

$$\frac{13aa}{9} = \frac{65a}{3}$$

Or, . . . $$aa + \frac{ssaa}{rr} = \frac{cra + csa}{br}.$$

6. Which Equation, after due Reduction, will diſcover the greater of the two Numbers ſought, *viz.*

$$a = 15 = \frac{crr + crs}{brr + bss}.$$

7. Whence, by the help of the firſt proportion given in the Queſtion, the leſſer Number ſought will alſo be made known, *viz.*

$$10 = \frac{css + crs}{brr + bss}.$$

So the Numbers ſought are 15 and 10; for they are in the given Reaſon of 3 to 2; and their Sum 25 is to 325 the Sum of their Squares, as 1 to 13; as was preſcribed.

Moreover, the Letters in the latter parts of the two laſt Equations give a Canon to find out the Numbers required.

QUEST. 19.

There are two Numbers, the Greater of which has ſuch proportion to the Leſſer, as 3 to 2, (or as *r* to *s*;) and the Sum of the ſaid Numbers has ſuch proportion to the Product of their Multiplication, as 1 to 6, (or as *b* to *c*.) What are the numbers?

1. For the greater number ſought put . .	a	a
2. Then (according to *Queſt.* 2. in *Sect.* 4. *Chap.* 10.) the Sum of the two numbers will be	$\frac{5a}{3}$	$a+\frac{sa}{r}$
3. And (by *Queſt.* 4. in *Sect.* 4. *Chap.* 10.) the Product of their Multiplication is . .	$\frac{2aa}{3}$	$\frac{saa}{r}$
4. Again, by the help of the latter proportion given in the Queſtion, and of the Sum found in the ſecond ſtep, ſearch out the Product of the multiplication of the two numbers ſought; *viz.* ſay by the Rule of Three, If $1 \,.\, 6 :: \frac{5a}{3} \,.\, 10a$, Or, if $b \,.\, c :: a+\frac{sa}{r} \,.\, \frac{cra+csa}{br}$; whence the Product is found	$10a$	$\frac{cra+csa}{br}$

5. But the Products found out in the two laſt ſteps muſt be equal to one another; hence this Equation, *viz.*

$$\frac{2aa}{3}=10a,$$

Or, . . . $$\frac{saa}{r}=\frac{cra+csa}{br}.$$

6. Which Equation, after due Reduction, diſcovers the greater of the two Numbers ſought, *viz.*

$$a=15=\frac{cr+cs}{bs}.$$

7. Whence, by the help of the firſt Proportion given in the Queſtion, the leſſer number ſought will alſo be made known, *viz.*

$$10=\frac{cr+cs}{br}.$$

So the numbers ſought are found 15 and 10; but that they will ſolve the Queſtion the Proof will make manifeſt: For the greater is to the leſſer as 3 to 2; and their Sum 25, is to 150 the Product of their Multiplication, as 1 to 6; as was preſcribed.

Moreover, the two laſt Equations give a Canon to find out the Number ſought.

QUEST. 20.

There are two Numbers, the greater of which has ſuch Proportion to the leſſer as 2 to 1 (or as *r* to *s*,) and the ſum of the Squares of the ſaid Numbers is 125 (or *b*;) What are the Numbers?

1. For the greater number ſought put . . .	a	a
2. (Then according to *Queſt.* 1. in *Sect.* 4. *Chap.* 10.) the leſſer Number will be found	$\frac{a}{2}$	$\frac{sa}{r}$
3. Therefore the Sum of their Squares ſhall be	$\frac{5aa}{4}$	$aa+\frac{ssaa}{rr}$

4. Which

4. Which Sum muſt be equal to 125 (or *b*) the given ſum of the Squares; hence this Equation,	$\frac{5aa}{4} = 125$	$aa + \frac{ssaa}{rr} = b$
5. Which Equation, after due Reduction (according to *Sect.* 2, 5, and 7, of *Chap.* 12.) will diſcover the greater number ſought, *viz.*	$a = 10$	$a = \sqrt{\frac{rrb}{rr+ss}}$
6. But if *a* had been put for the leſſer number, it would by the like proceſs have been found	$= 5$	$= \sqrt{\frac{ssb}{rr+ss}}$

From the two laſt ſteps the numbers ſought are found 10 and 5, which will ſolve the Queſtion: For the greater is to the leſſer as 2 to 1, and the ſum of their Squares is 125; as was preſcribed.

Moreover, to find out the Numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

Multiply ſeverally the Squares of the Terms of the given Reaſon, by the given Sum of the Squares of the number ſought; then divide the Products ſeverally by the Sum of the Squares of the ſaid Terms; laſtly, extract the ſquare Root out of each Quotient, ſo ſhall theſe ſquare Roots be the Numbers ſought.

QUEST. 21.

There are two Numbers, the greater of which has ſuch proportion to the leſſer as 2 to 1, (or as *r* to *s*;) and the difference of the Squares is 75, (or *d*:) What are the Numbers?

1. For the greater Number ſought put	a	a
2. Then (according to *Queſt.* 1. in *Sect.* 4. *Chap.* 10.) the leſſer number will be	$\frac{a}{2}$	$\frac{sa}{r}$
3. Therefore the difference of their Squares is	$\frac{3aa}{4}$	$aa - \frac{ssaa}{rr}$
4. Which Difference muſt be equal to the given Difference 75 (or *d*;) hence this Equation, *viz.*	$\frac{3aa}{4} = 75$	$aa - \frac{ssaa}{rr} = d$
5. Which Equation, after due Reduction, diſcovers the greater Number, *viz.*	$a = 10$	$a = \sqrt{\frac{rrd}{rr-ss}}$
6. But if *a* had been put for the leſſer Number it would have been found by the like proceſs	$= 5$	$= \sqrt{\frac{ssd}{rr-ss}}$

So the Numbers ſought are 10 and 5, which will ſolve the Queſtion: For the greater is to the leſſer as 2 to 1, and the difference of their Squares is 75; as was preſcribed.

Moreover, to find out the numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

Multiply ſeverally the Squares of the Terms of the given Reaſon by the given Difference of the Squares, then divide the Products ſeverally by the Difference of the Squares of the ſaid Terms, laſtly extract the ſquare Root of each Quotient, ſo ſhall theſe ſquare Roots be the Numbers ſought.

QUEST. 22.

There are two numbers, the ſum of whoſe Squares is 125 (or *b*) and the Difference of their Squares is 75 (or *d*;) what are the Numbers?

1. For the greater number put	a	a
2 Then its Square will be	aa	aa
3. Which ſubtracted from 125 (or *b*) the given Sum, leaves the Square of the leſſer Number, to wit,	$125 - aa$	$b - aa$

4. And from the ſecond and third ſteps by ſubtracting the leſſer Square from the greater, their Difference is	$2aa - 125$	$2aa - b$
5. Which Difference muſt be equal to the given Difference 75 (or *d*,) whence this Equation ariſes, ...	$2aa - 125 = 75$	$2aa - b = d$
6 From which Equation after due Reduction, according to *Sect.* 3, 5, and 7. of *Chap.* 12. the greater Number ſought will be made known, *viz.*	$a = 10$	$a = \sqrt{\frac{b+d}{2}}$
7. But if *a* had been put for the leſſer Number ſought, it would by the like proceſs have been found	$= 5$	$= \sqrt{\frac{b-d}{2}}$

So the Numbers ſought are found 10 and 5, which will ſolve the Queſtion; for the ſum of their Squares is 125, and the difference of their Squares is 75, as was preſcribed.

Moreover, to find out the Numbers ſought, the two laſt ſteps of the literal Reſolution give this

CANON.

The ſquare Root of half the Sum of the given ſum and difference of the Squares of the two Numbers ſought, is equal to the greater Number, and the ſquare Root of half the difference of the ſaid given Sum and Difference gives the leſſer Number.

QUEST. 23.

There are two Numbers, the ſum of whoſe Squares is 340 (or *b*;) and the Product made by the multiplication of the two Numbers is equal to $\frac{6}{7}$ (or *c*) parts of the Square of the greater Number; what are the Numbers?

1. For the greater Number put	a	a
2. Then its ſquare is	aa	aa
3. And $\frac{6}{7}$ (or *c*) parts of that Square is	$\frac{6aa}{7}$	caa
4. Therefore alſo (according to the condition in the Queſtion) the Product of the multiplication of the two numbers ſought, ſhall be	$\frac{6aa}{7}$	caa
5. Which Product divided by the greater number *a* will give the leſſer number, to wit,	$\frac{6a}{7}$	ca
6. Therefore from the laſt ſtep the Square of the leſſer number is	$\frac{36aa}{49}$	$ccaa$
7. And by adding together the Squares in the ſecond and ſixth ſteps, their ſum will be .	$\frac{85aa}{49}$	$ccaa + aa$
8. Which ſum muſt be equal to the given ſum 340 (or *b*,) whence this Equation ariſes .	$\frac{85aa}{49} = 340$	$ccaa + aa = b$
9. From which Equation, after it is duly reduced according to *Sect.* 2, 5, and 7. of *Chap.* 12. the greater number ſought will be made known, *viz.*	$a = 14$	$a = \sqrt{\frac{b}{cc+1}}$
10. And from the ninth and fifth ſteps the leſſer number will alſo be diſcovered. . .	$= 12$	$= \sqrt{\frac{bcc}{cc+1}}$

So the two numbers ſought are found 14 and 12, which will ſolve the Queſtion; for the ſum of their Squares 196 and 144 is 340; alſo, 14 multiplied by 12 makes 168, which is equal to $\frac{6}{7}$ of the greater Square 196.

QUEST. 24.

A Merchant bought a certain Number of Yards of linnen Cloth at 12 pence (or *b*) *per* Yard; and if the number of pence paid for all the Cloth be multiplied by the number of Yards bought, the Product will be 30000, (or c.) The Queſtion is, to find the number of Yards bought.

1. For the number of Yards bought put . .	a	a
2. Then the number of pence paid for the whole Cloth will be	$12a$	ba
3. Which Number multiplied by a (the number of Yards bought, produces	$12aa$	baa
4. Which Product muſt, according to the Queſtion, be equal to 30000 (or c;) therefore	$12aa = 30000$	$baa = c$
5. From which Equation, after due Reduction, the number of Yards ſought will be diſcovered, *viz.*	$a = 50$	$a = \sqrt{\frac{c}{b}}$

So it is found that the Merchant bought 50 Yards of Cloth, which at 12 *d. per* Yard makes 600 *d.* this 600 multiplied by 50 (the Number of Yards bought,) produces 30000; as was preſcribed in the Queſtion.

QUEST. 25.

Two Merchants, *A* and *B*, were Co-partners in Traffic; *A* brought in a certain number of pounds, which continued in Company 4 (or c) Months, *B* brought in 100 (or b) pounds, which continued in Company ſuch a time, that if it be multiplied by the Stock of *A* it makes 50 (or d.) At the end of their Partnerſhip they had gained 60 Pounds, whereof *A* had 40 (or r) Pounds for his ſhare, and *B* the reſt, to wit, 20 (or s) Pounds. What was the Stock which *A* put in at firſt, and how many Months did the Stock of *B* continue in Company?

1 For the Stock of *A* put	a	a
2. Then multiplying that ſtock by the time it continued in Company, to wit, by 4 (or c,) it makes	$4a$	ca
3. Then divide 50 (or d) the Product given in the Queſtion, by a the (ſtock of *A*,) and the Quotient will give the time that the ſtock of *B* continued in Company, to wit, . . .	$\frac{50}{a}$	$\frac{d}{a}$
4. The ſtock of *B*, to wit, 100 *l.* (or b) multiplied by its time $\frac{50}{a}$ (or $\frac{d}{a}$) produces . .	$\frac{5000}{a}$	$\frac{bd}{a}$

5. Then according to the Nature of the Rule of Fellowſhip with Time, this Analogy will ariſe, *viz.* As the Product made by the mutual multiplication of the Stock and Time of *A*, is to the Product of the Stock and Time of *B*; ſo is the gain of *A* to the gain of *B*: *viz.*

$$\text{As, } 4a \;.\; \frac{5000}{a} :: 40 \;.\; 20,$$

$$\text{Or, } ca \;.\; \frac{bd}{a} :: r \;.\; s.$$

6. Which Analogy (according to *Sect.* 1. *Chap* 13.) may be converted into this Equation, *viz.*

$$80a = \frac{200000}{a},$$

$$\text{Or, } sca = \frac{rbd}{a}.$$

7. From which Equation, (after due Reduction according to *Sect.* 2, 5, and 7. of *Chap.* 12.) the Stock of *A* will be diſcovered, *viz.*

$$a = 50 = \sqrt{\frac{rbd}{sc}}.$$

8. And from the ſeventh and third ſteps, the Time that the Stock of *B* continued in Company will alſo be made known, *viz.*

$$\frac{50}{50} = 1 = \sqrt{\frac{scd}{rb}}.$$

9. So it is found that the Stock which *A* put in at firſt was 50 *l.* and the time during which the Stock of *B* continued in Company was one Month; as will appear by

The Proof.

$$50 \times 4 = 200$$
$$100 \times 1 = 100$$
$$\text{Then if} \ldots\ldots 300 \quad . \quad 60 :: \begin{cases} 200 \;.\; 40 \\ 100 \;.\; 20 \end{cases}$$

QUEST. 26.

Certain Noble-men made a Progreſs for their Pleaſure; every noble Man carried along with him the ſame Sum of Pounds; the Number of the Noble-men was equal to the number of Servants which attended upon each Noble-man; the number of Pounds that each Noble-man had was the double of the number of all their Servants; and the ſum of all their Mony was 3456 Pounds: the Queſtion is, to find out the Number of Noble-men; alſo, how many Pounds and Servants each Noble man had?

1. For the number of Noble-men put . . . . a
2. Then (according to the Queſtion) the number of Servants that attended upon each Noble-Man was alſo . . . . a
3. Therefore the Number of all the Servants was . . . . aa
4. Which laſt Number doubled gives the number of Pounds that each Nobleman had, to wit, ... $2aa$
5. And if the ſaid Number of Pounds be multiplied by the number of Noble-men, it produces the Sum of all their Money, to wit, . . . $2aaa$
6. Which ſum muſt be equal to the given ſum 3456, therefore . . $2aaa = 3456$
7. Therefore by taking the half of that Equation, there ariſes . . $aaa = 1728$
8. Laſtly, by extracting the Cubic Root of each part of the laſt Equation, the Number of Noble-men is diſcovered, to wit, . . . $a = 12$

So it is found that there were 12 Noble Men; alſo every one of them had 12 Servants and 288 Pounds, as will appear by

The Proof.

$$12 \times 12 = 144$$
$$144 \times 2 = 288$$
$$288 \times 12 = 3456.$$

QUEST. 27.

A Merchant bought as many Pounds of Pepper for one Crown as was half the number of Crowns he laid out, then in ſelling the Pepper he received for every 25 ℔ of Pepper as many Crowns as he paid for all the Pepper; and in concluſion he had 20 Crowns, The Queſtion is, to find how many Crowns he laid out.

1. For the Number of Crowns which the Merchant laid out, let there be put . . . . a.
2. Then the Number of Pounds of Pepper which he bought for one Crown was . . . . $\frac{a}{2}$
3. Whence the whole quantity of Pepper bought will be found $\frac{aa}{2}$, for, If $1 \;.\; \frac{a}{2} :: a \;.\; (\frac{aa}{2}$ . . . . $\frac{aa}{2}$

4. Then find how many Crowns the Merchant received for the total quantity of Pepper ſold; ſaying by the Rule of Three,

$$\text{If } 25 \;.\; a \;::\; \frac{aa}{2} \;.\; (\frac{aaa}{50};$$

whence the number of Crowns for which all the Pepper was ſold is found . . . . $\frac{aaa}{50}$

5. Which number of Crowns found out in the laſt ſtep, muſt be equal to 20 the number of Crowns given in the Queſtion; hence this Equation, ... $\frac{aaa}{50} = 20$

6. From which Equation, after it is reduced according to *Sect.* 2, and 7. of *Chap.* 12. there will come forth the firſt coſt of the Pepper, to wit, ... $a = 10$

So the number of Crowns which the Merchant laid out was 10, as will appear by the Proof; for firſt, the half of 10, to wit, 5, will be the number of Pounds of Pepper which he bought for 1 Crown; then ſay,

If 1 . 5 :: 10 . 50 || Pounds of Pepper bought,
If 25 . 10 :: 50 . 20 || Crowns received for Pepper ſold.

QUEST. 28.

There are two Numbers, the greater of which has ſuch proportion to the leſſer as 3 to 2, (or as r to s;) and the Sum of the Cubes of the two Numbers is 4375, (or b;) what are the numbers?

1. For the greater Number put	a	a
2 Then (according to *Queſt.* 1. in *Sect.* 4. of *Chap.* 10) the leſſer number will be found	$\frac{2a}{3}$	$\frac{sa}{r}$
3. Therefore from the firſt ſtep, the Cube of the greater number is	aaa	aaa
4. And from the ſecond ſtep the Cube of the leſſer Number is	$\frac{8aaa}{27}$	$\frac{sssaaa}{rrr}$
5. Therefore from the third and fourth ſteps, the Sum of the Cubes of both Numbers is	$\frac{35aaa}{27}$	$\frac{sssaaa}{rrr} + aaa$

6. Which Sum muſt be equal to the given Sum 4375, (or b;) whence this Equation ariſes, *viz.*

$$\frac{35aaa}{27} = 4375. \quad \text{Or, } \frac{sssaaa}{rrr} + aaa = b.$$

7. From which Equation, after due Reduction, (according to *Sect.* 2, 5, and 7. of *Chap.* 12.) the greater number ſought will be made known, *viz.*

$$a = 15 = \surd(3)\frac{rrrb}{sss + rrr}.$$

8. And from the ſeventh and ſecond ſteps, the leſſer number will alſo be diſcovered, to wit,

$$10 = \surd(3)\frac{sssb}{sss + rrr}.$$

So the numbers ſought are found 15 and 10, which will ſolve the Queſtion; for they are in the given Reaſon of 3 to 2; and the Sum of the Cubes of the ſaid 15 and 10, to wit, of 3375 and 1000 makes 4375; as was preſcribed.

Moreover, to find the Numbers ſought, the latter parts of the Equations in the ſeventh and eighth ſteps give this

CANON.

Multiply ſeverally the Cubes of the Terms of the given Reaſon (or Proportion) by the given Sum of the Cubes of the Numbers ſought; divide the Products ſeverally by the Sum of the Cubes of the ſaid Terms; laſtly, extract the Cubic Root of each of the Quotients, ſo theſe Roots ſhall be the Numbers ſought.

CHAP. XV.

Concerning the Reſolution of ſuch adfected or compounded Equations wherein their are two different Powers of the Quantity ſought, and thoſe Powers ſuch, that the higher of them is a Square whoſe Side or Square Root is the lower Power.

I. THE Equations treated of in this Chapter fall under three Heads or Forms hereunder ſpecified, which I ſhall firſt explain, and then ſhew how they may be Arithmetically reſolved.

Equations of the firſt Form.

$aa + 6a = 55.$	$aa + ca = b.$
$aaaa + 8aa = 48.$	$aaaa + daa = f.$
$aaaaaa + 4aaa = 837.$	$aaaaaa + gaaa = h.$

Equations of the ſecond Form.

$aa - 10a = 24.$	$aa - ba = k.$
$aaaa - 6aa = 27.$	$aaaa - paa = d.$
$aaaaaa - 2aaa = 48.$	$aaaaaa - maaa = g.$

Equations of the third Form.

$10a - aa = 24.$	$ca - aa = n.$
$5aa - aaaa = 4.$	$raa - aaaa = s.$
$9aaa - aaaaaa = 8.$	$daaa - aaaaaa = t.$

II. Every Equation which falls under any of the ſaid three Forms, conſiſts of three diſtinct Terms or Members, whereof two are unknown, and the third is known; of the two unknown Terms, one is a Square, (by which in this place I mean a ſquare number) which is called the higheſt Term in the Equation; and the other unknown Term is the Product made by the Multiplication of the ſquare Root of the ſaid ſquare number by ſome known number, which Product is called the middle Term; and the third or loweſt Term is a number purely known: So in this Equation $aa+6a=55$, the higheſt Term is aa, which may repreſent an unknown ſquare number whoſe Root is a; the middle term is $6a$, which is the Product of the Multiplication of the ſaid unknown Root a by the known number 6; and the loweſt Term (or known part of the ſaid Equation) is the number 55, which for diſtinction ſake is uſually called the Abſolute number given.

The like may be obſerved in this Equation $aa+ca=b$, where we may ſuppoſe b and c to repreſent two known numbers, and a ſome number unknown; then the higheſt Term is the Square aa; the middle Term is ca, to wit, the Product made by the Multiplication of a the Root of the ſaid ſquare aa by the known number c; and the loweſt Term of the ſaid Equation is the known Abſolute number b.

Again, in this Equation $5aa-aaaa=4$, the higheſt Term is the ſquare number $aaaa$; the middle term is $5aa$, to wit, the Product made by the Multiplication of aa the ſquare Root of the ſaid ſquare Number $aaaa$ into the known number 5; and the loweſt Term is the abſolute number 4.

III. In every Equation which falls under any of the three before-mentioned Forms, there are two different Powers or Degrees of the number ſought, and thoſe ſuch, that the Index or Exponent of the higher Power is the double of the Index of the lower: As in this Equation $aa+6a=55$, the Index or Number of Dimenſions in aa is 2, which is the double of 1 the Index of a (in the middle Term $6a$:) ſo alſo in this Equation $5aa-aaaa=4$, the Index of the higheſt Term $aaaa$ is 4, which is the double of 2 the Index of aa in the middle Term. Likewiſe in this Equation $aaaaaa+4aaa=837$, the Index of the higheſt Term $aaaaaa$ is 6, which is the double of 3 the Index of aaa in the middle Term. But in this Equation $aaa+6a=39$ the Index of the higheſt Term aaa is not the double of the Index of a in the middle Term, (for the Index of the former is 3, and of the latter 1;) and therefore the Equation laſt propoſed cannot be ranked under any of the three Forms aforeſaid, and conſequently it is not reſolvable by the following Rules of this Chapter, but belongs to the 10 and 11 Chapters of my ſecond Book.

IV. Known Numbers which are drawn into, or multiplied by ſome Degree or Power of the Number ſought are by *Vieta* and others called Coefficients, *viz.* Fellow-factors, or Copartners in Multiplication with unknown Powers: So in this Equation $aa+6a=55$ the Number 6 is called the Co-efficient, to wit, the Fellow-multiplier with the unknown Number a to make the Product $6a$. Likewiſe in this Equation $aa+ca=b$, we may ſuppoſe the Letters b and c to repreſent known Numbers, and the Letter a ſome unknown Number whoſe Co-efficient is c.

But ſometimes the Co-efficient will happen to be expreſs'd by many Letters, as in this Equation $aa+\frac{sca}{r}$ (or $\frac{sc}{r}a$) $=\frac{15sscc}{4rr}$, where a only is ſuppoſed to be unknown, and the known Number $\frac{sc}{r}$ is the Co-efficient, which ſignifies but one Number, to wit, the Quotient that ariſes, when the Product of the Number s multiplied by the number c is divided by the number r, *viz.* if $s=2$; $c=4$; and $r=1$, then $\frac{sc}{r}$ or 8 is the Co-efficient, and conſequently $\frac{sc}{r}a$ is the ſame with $8a$.

Likewiſe in this Equation $\frac{2r+s}{s}a$ (or $\frac{2ra+sa}{s}$) $-aa=\frac{2r}{s}$, the Co-efficient is $\frac{2r+s}{s}$, which is to be eſteemed but as one number; to wit, the Quotient that ariſes by dividing the Sum of $2r$ and s by s; ſo that if we ſuppoſe $r=3$ and $s=2$, then the Equation laſt propoſed may be expreſs'd thus, $4a-aa=3$.

Note. When no known number appears to be drawn into the middle Term of the Equation, then 1 (or Unity) muſt in that caſe be always taken for the Co-efficient; ſo in this Equation $aa+a=30$, the middle Term a implies $1a$, to wit, the Product of a multiplied by 1, and therefore 1 is the Co-efficient.

Note alſo. When the higheſt unknown Power or Degree is multiplied by any number greater than 1, then every Term or Member of the Equation muſt be divided by that number, to the end the ſaid higheſt unknown Power may be clear'd from any Co-efficient unleſs it be 1; as before has been ſhewn in *Sect.* 5. *Chap.* 12.

Theſe things being premiſed by way of Explication, I proceed to the Reſolution of Equations which fall under any of the three Forms before ſpecified.

V. *The Arithmetical Reſolution of Equations which fall under the firſt of the three Forms before ſpecified in* Sect. I. *of this* Chapter.

QUEST. 1.

1. What is the number repreſented by a in this Equation? . . . $aa+6a=55$
2. Which Equation, if c be aſſumed to ſignifie 6, and b 55, may be expreſs'd thus, ... $aa+ca=b$

RESOLUTION.

3. To reſolve the ſaid Equation imports the ſame thing as to ſolve this Queſtion, *viz.* There is an unknown number (repreſented by a) which is ſuch, that if to its Square you add the Product made by the Multiplication of that unknown number by 6, (or c,) the Sum will be 55, (or b;) what is that unknown number a? *Anſw.* 5; found out thus,
4. Let the Square of half the Co-efficient 6 (or c) be added to each part of the Equation propoſed, to the end its firſt part may be made a compleat Square, (according to *Sect.* 4. *Chap.* 9) whence this Equation ariſes,

$$aa+6a+9=64, \quad \text{or,} \quad aa+ca+\tfrac{1}{4}cc=b+\tfrac{1}{4}cc.$$

5. Then by extracting the ſquare Root of each part of the laſt Equation (according to *Sect.* 4 and 5. of *Chap.* 8.) this Equation ariſes;

$$a+3 = 8,$$

$$\text{Or,}\quad a+\tfrac{1}{2}c = \sqrt{:b+\tfrac{1}{4}cc:}$$

6. Wherefore by tranſpoſition (or equal ſubtraction) of 3, or $\frac{1}{2}c$, the number a ſought will be made known, *viz.*

$$a = 5 = \sqrt{:b+\tfrac{1}{4}cc:} - \tfrac{1}{2}c.$$

I ſay the number a ſought is 5, which will ſolve the Queſtion propoſed, as will appear by

The Proof.

If . . . . $a = 5$,
Then conſequently . . . . $aa = 25$,
And . . . . $6a = 30$;
Therefore . . . . $aa+6a = 55$.

Which was the Equation propoſed.

Note. Every Equation which falls under this firſt Form may be expounded by either of two Roots, whereof one is Affirmative or greater than nothing, and the other Negative or leſs than nothing. As in the Equation propoſed, to wit, $aa+6a=55$; foraſmuch as according to the Rules of Algebraical Multiplication, — multiplied by — produces +, and ſo in this Senſe the ſquare Root of 64 may be — 8 as well as + 8; therefore the ſquare Root of the Equation $aa+6a+9=64$ in the fourth ſtep may be this, to wit, ... $a+3 = -8$.

Whence, by tranſpoſition of +3, a Negative Root or value of a is diſcovered, to wit, ... } . . $a = -11$.

I ſay the Root a in the Equation $aa+6a=55$ may be expounded by -11. (beſides $+5$,) as will be manifeſt by

The Proof.

$$\left.\begin{array}{lr} \text{If} \ldots\ldots\ldots & a = -11 \\ \text{Then} \ldots\ldots\ldots & aa = +121 \\ \text{And} \ldots\ldots\ldots & 6a = -66, \\ \text{Therefore, as before,} & aa+6a = +55. \end{array}\right\}$$

Here the Rules of + and — in Algebraical Multiplication and Addition are to be reſpected.

Negative Roots are oftentimes of good uſe to find out Affirmative Roots, as hereafter will appear in *Chap.* 11. of the ſecond Book.

QUEST. 2.

1. What is the number repreſented by a in this Equation? : . $aaaa+8aa = 48$,
2. Which Equation, if d be put for 8, and f for 48, may be expreſs'd thus, ... } $aaaa+daa = f$.

RESOLUTION.

3. To reſolve the ſaid Equation imports the ſame thing as to ſolve this Queſtion, *viz.* There is an unknown number repreſented by a, which is ſuch, that if to its Biquadrate or ſquared Square you add the Product made by the Multiplication of the Square of that unknown number a by 8, (or d,) the Sum will be 48, (or f;) what is the unknown number a? *Anſw.* 2. found out in the ſame manner as before in *Queſt.* 1. *viz.*

4. Let the Square of half the Co-efficient 8 (or d) be added to each part of the Equation propoſed, to the end its former part may be made a compleat Square, according to *Sect.* 4. *Chap.* 9. whence this Equation ariſes;

$$aaaa+8aa+16 = 64,$$

$$\text{Or,}\quad aaaa+daa+\tfrac{1}{4}dd = f+\tfrac{1}{4}dd.$$

5. Then by extracting the ſquare Root of each part of the laſt Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariſes,

$$aa+4 = 8,$$

$$\text{Or,}\quad aa+\tfrac{1}{2}d = \sqrt{:f+\tfrac{1}{4}dd:}$$

6. Whence by equal ſubtraction or tranſpoſition of 4 (or $\frac{1}{2}d$) there will ariſe

$$aa = 4$$

$$\text{Or,}\quad aa = \sqrt{f+\tfrac{1}{4}dd:}-\tfrac{1}{2}d.$$

7. Therefore by extracting the ſquare Root of each part of the laſt Equation, the number a ſought, will be made known, *viz.*

$$a = 2 = \surd(2){:}\overline{\surd\overline{f+\tfrac{1}{4}dd}-\tfrac{1}{2}d}{:}$$

I ſay the number a ſought is 2, which will ſolve the Queſtion propoſed, as will appear by

The Proof.

If . . . . $a = 2,$
Then conſequently . . . . $aa = 4,$
And . . . . $aaaa = 16,$
Alſo . . . . $8aa = 32,$
Therefore . . . . $aaaa+8aa = 48.$
Which was the Equation propos'd to be reſolved.

QUEST. 3.

1. What is the number repreſented by a in this Equation? . . . . $aaaaaa+4aaa = 837.$
2. Which Equation, if g be put for 4, and b for 837, may be expreſs'd thus . . . . . $aaaaaa+gaaa = b.$

RESOLUTION.

3. To reſolve the ſaid Equation imports the ſame thing as to ſolve this Queſtion, *viz.* There is an unknown number repreſented by a, which is ſuch, that if to its cubed Cube or ſixth Power, you add the Product made by the Multiplication of the Cube of that unknown number by 4 (or g) the Sum will be 837, what is that unknown number a? *Anſw.* 3. found out in the ſame manner as before, *viz.*

4. By adding the ſquare of half the Co-efficient 4 (or g) to each part of the Equation propoſed, this Equation ariſes;

$$aaaaaa+4aaa+4 = 841.$$

Or, $aaaaaa+gaaa+\frac{1}{4}gg = b+\frac{1}{4}gg.$

5. And by extracting the ſquare Root of each part of the laſt Equation this ariſes;

$$aaa+2 = 29.$$

Or, $aaa+\frac{1}{2}g = \surd{:}\overline{b+\frac{1}{4}gg}{:}$

6. Whence by tranſpoſition of 2 (or $\frac{1}{2}g$) this Equation ariſes;

$$aaa \quad 27.$$

Or, $aaa = \surd{:}\overline{b+\frac{1}{4}gg}{:}-\frac{1}{2}g.$

7. Therefore by extracting the Cubic Root of each part of the laſt Equation the number a ſought will be made known, *viz.*

$$a = 3 = \surd(3){:}\overline{\surd\overline{b+\tfrac{1}{4}gg}-\tfrac{1}{2}g}{:}$$

I ſay the number a ſought is 3, which will ſolve the Queſtion propoſed, as will appear by

The Proof.

If . . . . $a = 3,$
Then conſequently . . . . $aaa = 27,$
And . . . . $aaaaaa = 729,$
Alſo . . . . $4aaa = 108,$
Therefore . . . $aaaaaa+4aaa = 837.$
Which was the Equation propos'd to be reſolved.

VI. From the Reſolution of the three laſt Queſtions the following Canon is deduced for the reſolving of all Equations which fall under the firſt of the three Forms before ſpecified in *Sect.* 1. of this *Chapter.*

CANON.

Add the ſquare of half the Co-efficient, or (which is the ſame thing) a quarter of the ſquare of the whole Co-efficient, to the given abſolute number.

Extract the ſquare Root of that Sum.

From the ſaid Square Root ſubtract half the Co-efficient, and reſerve the Remainder. Laſtly, when the unknown number which is multiplied by the Co-efficient in the middle Term of the Equation is expreſs'd by a ſingle Letter only, as *a*, then the Remainder before reſerved is the number ſought; but if the ſaid unknown number in the middle Term be a Square, as *aa*, then the Square Root of the Remainder reſerved is the number ſought; if a Cube, as *aaa*, then the Cubic Root of the ſaid Remainder ſhall be the number ſought; if any higher Power, then the Root for the kind muſt be extracted out of the ſaid Remainder, which Root ſhall be the number ſought.

An Example of the Canon.

1. Let the preceding *Queſt.* 1. be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . . . $aa+6a=55$
2. Or, what is the value of *a* in this Equation, . . $aa+ca=b$

RESOLUTION.

3. To the given abſolute number	55	$b.$
4. Add the Square of half the Co-efficient 6, to wit, the Square of 3, which is : . .	9	$\frac{1}{4}cc.$
5. The Sum is	64	$b+\frac{1}{4}cc$
6. The Square Root of that Sum is	8	$\surd\colon\overline{b+\frac{1}{4}cc\colon}$
7. From that Square Root ſubtract half the Co-efficient 6, to wit, ...	3	$\frac{1}{2}c.$
8. The Remainder is the number *a* ſought, to wit,	5	$\surd\colon\overline{b+\frac{1}{4}cc\colon}-\frac{1}{2}c.$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 1.

A ſecond Example of the Canon.

1. Let the preceding *Queſt.* 2. be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . . . . . $aaaa+8aa=48$
2. Or what is the value of *a* in this Equation, . . $aaaa+daa=f$

RESOLUTION.

3. To the given abſolute number	48	$f.$
4. Add the Square of half the Co-efficient 8, to wit, the Square of 4, which is . . .	16	$\frac{1}{2}dd$
5. The Sum is	64	$f+\frac{1}{4}dd.$
6. The ſquare root of that Sum is	8	$\surd\colon\overline{f+\frac{1}{4}dd\colon}$
7. From which ſquare root ſubtract half the Co-efficient 8, to wit, ...	4	$\frac{1}{2}d.$
8. The Remainder is the value of *aa*, to wit .	4	$\surd\overline{f+\frac{1}{4}dd}-\frac{1}{2}d$
9. Laſtly, the ſquare Root of the ſaid Remainder gives the number *a*, ...	2	$\surd(2)\colon\overline{\surd\overline{f+\frac{1}{4}dd}-\frac{1}{2}d\colon}$

Whence it is evident that the Anſwer is the ſame as was before found to *Queſt.* 2.

A third Example of the Canon.

1. Let the preceding *Queſt.* 3. be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . . $aaaaaa+4aaa=837.$
2. Or what is the value of *a* in this Equation, . . $aaaaaa+gaaa=h.$

RESOLUTION.

3. To the abſolute Number	837	$h.$
4. Add the Square of half the Coefficient 4, to wit,	4	$\frac{1}{4}gg.$
5. The Sum is	841	$h+\frac{1}{4}gg.$
6. The ſquare Root whereof is	29	$\surd\colon\overline{h+\frac{1}{4}gg\colon}$
7. From that ſquare Root ſubtract half the Co-efficient 4, to wit, ...	2	$\frac{1}{2}g.$

8. The Remainder is the value of *aaa*, to wit, . . 27 | $\sqrt{b+\frac{1}{4}gg}-\frac{1}{2}g$
9. Therefore the Cubic Root of that Remainder ſhall be the number *a* ſought, . . . 3 | $\sqrt[3]{\sqrt{b+\frac{1}{4}gg}-\frac{1}{2}g}$:

Whereby it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 3.

Example 4.

If . . . . $aa+a=b$ (or 35,) what is $a=$?

Anſw. . . . . . $a=\sqrt{b+\frac{1}{4}}-\frac{1}{2}=5\frac{4371}{10000}$, *&c.*

For the Co-efficient drawn into the middle Term *a* being 1, its half is $\frac{1}{2}$, the Square whereof is $\frac{1}{4}$, which added to the abſolute number 35 makes $35\frac{1}{4}$, whoſe Square Root is $5\frac{9371}{100000}$, *&c.* from which ſubtracting $\frac{1}{2}$, (or $\frac{5}{10}$) to wit; half the Co-efficient 1, the Remainder $5\frac{4371}{100000}$, *&c.* is the number *a* ſought, which here happens to be irrational, that is, inexpreſſible by any true number, but by continuing the extraction of the ſaid Square Root of the ſaid $35\frac{1}{4}$, you may approach infinitely near the exact number *a*.

Example 5.

If . . . . $aa+\frac{15}{2}a=\frac{143}{2}$, what is $a=$?

Anſw. . . . . . . $a=\sqrt{\frac{143}{2}+\frac{225}{16}}-\frac{15}{4}=\frac{11}{2}$.

The Learner muſt remember to reduce a Fraction to its leaſt Terms, before he goes about to extract any Root out of it.

Example 6.

If . . . . $\left\{\begin{array}{l} r=1, \\ s=2, \\ c=4, \end{array}\right.$

And if . . . . $aa+\frac{sc}{r}a=\frac{15sscc}{4rr}$;

What is . . . . $a=$?

Anſw. . . . . $a=\frac{3sc}{2r}=12$.

Example 7.

If . . $aada+\frac{13}{3}aa=\frac{87362}{405}$, what is $a=$?

Anſw. . . . $a=\frac{11}{3}$.

VII. *The Arithmetical Reſolution of Equations which fall under the ſecond of the three Forms before expreſſed in* Sect. 1. *of this* Chapter.

QUEST. 1.

1. What is the number repreſented by *a* in this Equation? . . . . $aa-10a=24$.
2. Which Equation, by aſſuming *b* to repreſent 10, and *k* to ſignifie 24, may be expreſs'd thus, ... $aa-ba=k$.

RESOLUTION.

3. Let the Square of half the Co-efficient 10 (or *b*) be added to each part of the Equation propoſed, to the end its firſt part may be made a compleat Square, (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariſes;

$$aa-10a+25=49,$$

Or, $aa-ba+\frac{1}{4}bb=k+\frac{1}{4}bb$.

4. Then by extracting the Square Root of each part of the laſt Equation (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariſes;

$$a-5=7,$$

Or, $a-\frac{1}{2}b=\sqrt{k+\frac{1}{4}bb}$:

5. Wherefore by equal addition of 5, or $\frac{1}{2}b$, the number *a* ſought will be made known, *viz.* $a=12=\frac{1}{4}b+\sqrt{k+\frac{1}{4}bb}$:

6. But foraſmuch as the Square Root of $aa-10a+25$ in the third ſtep may be $5-a$ as well as $a-5$, (for either of thoſe Roots being multiplied by it ſelf will produce

the ſame Square $aa-10a+25$,) therefore let $5-a$ be ſet inſtead of $a-5$ in the fourth ſtep; whence this Equation ariſes, *viz.*

$$5-a=7,$$

$$\text{Or,}\quad \tfrac{1}{2}b-a=\sqrt{:k+\tfrac{1}{4}bb:}$$

7. Therefore by tranſpoſition, another value of a ariſes, to wit,

$$a=-2=\tfrac{1}{2}b-\sqrt{:k+\tfrac{1}{4}bb:}$$

Which latter value of a is leſs than nothing, and ſuch it will always be, as may eaſily be proved from the laſt Equation. For $k+\frac{1}{4}bb$ is manifeſtly greater than $\frac{1}{4}bb$, and conſequently the Square Root of the former will be greater than the Square Root of the latter, *viz.* $\sqrt{:k+\frac{1}{4}bb:}$ is greater than $\frac{1}{2}b$, therefore $\frac{1}{2}b-\sqrt{:k+\frac{1}{4}bb:}$ (that is a) will be leſs than nothing, for if a greater Quantity be ſubtracted from a leſs, the Remainder will be a negative Quantity, that is leſs than nothing, as before has been ſhewn in *Algebraical Subtraction.* From the premiſes it is evident that the Equation propounded, to wit, $aa-10a=24$ (and likewiſe every Equation which falls under the ſecond form of Equations before-mentioned) is explicable by two Roots, whereof one is real or affirmative, whoſe value is before expreſs'd in the fifth ſtep; and the other negative or leſs than nothing, the value whereof is expreſs'd in the ſeventh ſtep.

I ſay the real or true number a ſought in the Queſtion propoſed is 12, as will appear by

The Proof.

If . . . . $a=12$,
Then conſequently . . . . . . $aa=144$,
And . . . . $10a=120$,
Therefore . . . . $aa-10a=24$.

Which was the Equation propoſed.

Moreover, according to the Rules of *Algebraical* Multiplication and Subtraction, the negative value of a, to wit -2 before found, will conſtitute the Equation firſt propoſed:

For if . . . . $a=-2$,
Then conſequently . . . $aa=+4$,
And . . . . $10a=-20$,
Therefore . . . . $aa-10a=+24$;

QUEST. 2.

1. What is the number repreſented by a in this Equation? . . . . $aaaa-6aa=27$
2. Which Equation, if p be put for 6, and d for 27, may be expreſs'd thus, ... $aaaa-paa=d$

RESOLUTION.

3. Let the Square of half the Co-efficient 6 (or p) be added to each part of the Equation propoſed, to the end its firſt part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariſes;

$$aaaa-6aa+9=36,$$

$$\text{Or,}\quad aaaa-paa+\tfrac{1}{4}pp=d+\tfrac{1}{4}pp.$$

4. Then by extracting the Square Root of each part of the laſt Equation (according to *Sect.* 4. and 5. of *Chap.* 8.) this Equation ariſes, *viz.*

$$aa-3=6,$$

$$aa-\tfrac{1}{2}p=\sqrt{:d+\tfrac{1}{4}pp:}$$

5. Whence, by equal Addition of 3 (or $\frac{1}{2}p$) there will ariſe

$$aa=9,$$

$$\text{Or,}\quad aa=\sqrt{:d+\tfrac{1}{4}pp:}+\tfrac{1}{2}p.$$

6. Wherefore by extracting the Square Root of each part of the laſt Equation, the number a ſought will be made known, *viz.*

$$a=3=\sqrt{(2):\sqrt{d+\tfrac{1}{4}pp}+\tfrac{1}{2}p:}$$

I ſay the number a ſought is 3, which will ſolve the Queſtion propoſed, as will appear by

The Proof.

If . . . . $a = 3$,
Then conſequently . . . . . . $aa = 9$,
And . . . . $aaaa = 81$,
Alſo . . . . $6aa = 54$,
Therefore . . . . $aaaa - 6aa = 27$.
Which was the Equation propoſed to be reſolved.

QUEST. 3.

1. What is the number repreſented by a in this Equation? . . . . $aaaaaa - 2aaa = 48$
2. Which Equation, if m be put for 2, and g for 48, may be expreſt thus, ... $aaaaaa - maaa = g$

RESOLUTION.

3. Let the Square of half the Co-efficient 2 (or m) be added to each part of the Equation propoſed, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariſes;

$$aaaaaa - 2aaa + 1 = 49,$$
Or, $$aaaaaa - maaa + \tfrac{1}{4}mm = g + \tfrac{1}{4}mm.$$

4. Then by extracting the Square Root of each part of the laſt Equation (according to *Sect.* 4, and 5 of *Chap.* 8.) this Equation ariſes;

$$aaa - 1 = 7,$$
Or, $$aaa - \tfrac{1}{2}m = \sqrt{}:\overline{g + \tfrac{1}{4}mm}:$$

5. Whence by equal Addition of 1 (or $\frac{1}{2}m$) there ariſes

$$aaa = 8,$$
Or, $$aaa = \sqrt{}:\overline{g + \tfrac{1}{4}mm}: + \tfrac{1}{2}m.$$

6. Wherefore by extracting the Cubic Root of each part of the laſt Equation, the number a ſought will be made known, *viz.*

$$a = 2 = \sqrt{}(3):\overline{\sqrt{g + \tfrac{1}{4}mm}. + \tfrac{1}{2}m}:$$

I ſay the number a ſought is 2, which will ſolve the Queſtion propoſed; as will appear by

The Proof.

If . . . . $a = 2$,
Then conſequently . . . . . $aaa = 8$,
And . . . . $aaaaaa = 64$,
Alſo . . . . $2aaa = 16$,
Therefore . . . . . $aaaaaa - 2aaa = 48$.
Which was the Equation propoſed to be reſolved.

VIII. From the Reſolution of the three laſt Queſtions the following Canon is deduced, for the reſolving of all Equations which fall under the ſecond of the three Forms before ſpecified, in *Sect.* 1. of this *Chap.*

CANON.

Add the Square of half the Co-efficient, or, (which is the ſame thing) a quarter of the Square of the whole Co-efficient, to the given Abſolute Number.

Extract the Square Root of that Sum.

To the ſaid Square Root add half the Co-efficient, and reſerve this Sum.

Laſtly, when the unknown number which is drawn into the Co-efficient in the middle term of the Equation is expreſt by a ſingle Letter only, as a, then the Sum before reſerved is the Number ſought; but if the ſaid unknown number in the middle term be a Square, as aa, then the Square Root of the Sum reſerved is the number ſought; if a Cube, as aaa, then the Cubic Root of the ſaid Sum ſhall be the number ſought; if any higher Power, then the Root for the kind muſt be extracted out of the ſaid Sum, which Root ſhall be the number ſought.

An Example of the ſaid Canon.

1. Let the preceding *Queſt.* 1. in *Sect.* 7. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . .	$aa - 10a = 24$
2. Or, what is the value of *a* in this Equation? . .	$aa - ba = k$

RESOLUTION.

3. To the given abſolute number . . .	24	$k.$
4. Add the Square of half the Co-efficient 10, to wit, the Square of 5, which is . . .	25	$\frac{1}{4}bb.$
5. The Sum is	49	$k+\frac{1}{4}bb.$
6. The Square Root of that Sum is	7	$\sqrt{:k+\frac{1}{4}bb:}$
7. To which Square Root add half the Co-efficient 10, to wit, ...	5	$\frac{1}{2}b.$
8. The Sum is the number *a* ſought, to wit,	12	$\sqrt{:k+\frac{1}{4}bb:}+\frac{1}{2}b.$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 1. in *Sect.* 7.

A Second Example of the Canon in Sect. 8.

1. Let the preceding *Queſt.* 2. in *Sect.* 7. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . .	$aaaa - 6aa = 27$
2. Or, What is the value of *a* in this Equation? . .	$aaaa - paa = d.$

RESOLUTION.

3. To the given abſolute number . . .	27	$d.$
4. Add the Square of half the Co-efficient 6, to wit, the Square of 3, which is . .	9	$\frac{1}{4}pp.$
5. The Sum is	36	$d+\frac{1}{4}pp.$
6. The Square Root of that Sum is . . .	6	$\sqrt{:d+\frac{1}{4}pp:}$
7. To which Square Root add half the Co-efficient 6, to wit, ...	3	$\frac{1}{2}p.$
8. The Sum is the value of *aa*, to wit, .	9	$\sqrt{:d+\frac{1}{4}pp:}+\frac{1}{2}p.$
9. Therefore the Square Root of the ſaid Sum ſhall be the number ſought, to wit, . .	3	$\sqrt{(2)}:\overline{\sqrt{d+\frac{1}{4}pp}+\frac{1}{2}p}:$

Whence it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 2. in *Sect.* 7.

A Third Example of the Canon in Sect. 8.

1. Let the Preceding *Queſt.* 3. in *Sect.* 7. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . .	$aaaaaa - 2aaa = 48,$
2. Or, What is the value of *a* in this Equation? . .	$aaaaaa - maaa = g.$

RESOLUTION.

3. To the given abſolute number . . .	48	$g.$
4. Add the Square of half the Co-efficient 2, to wit, the Square of 1, which is . .	1	$\frac{1}{4}mm.$
5. The Sum is	49	$g+\frac{1}{4}mm.$
6. The Square Root of that Sum is . . .	7	$\sqrt{:g+\frac{1}{4}mm:}$
7. To which Square Root add half the Co-efficient 2, to wit, ...	1	$\frac{1}{2}m.$
8. The Sum is the value of *aaa*, to wit, . .	8	$\sqrt{:g+\frac{1}{4}mm:}+\frac{1}{2}m.$
9. Therefore the Cubic Root of the ſaid Sum ſhall be the number *a* ſought, to wit, . .	2	$\sqrt{(3)}:\overline{\sqrt{g+\frac{1}{4}mm}+\frac{1}{2}m}:$

Whereby it is manifeſt that the Anſwer is the ſame as was before found to *Queſt.* 3. in *Sect.* 7.

Example 4.

If . . $aa - a = g$ (or 1122,) what is $a = ?$

Anſw. . . . $a = \sqrt{:}\overline{g+\frac{1}{4}}{:} + \frac{1}{2} = 34.$

Example 5.

If . . . : $aa - \frac{13}{5}a = 373\frac{17}{45}$, what is $a = ?$

Anſw. . . . . . . $a = 20\frac{2}{3}.$

Example 6.

If . . . . $\begin{cases} r = 1, \\ s = 2, \\ c = 4, \end{cases}$

And if . . . . $aa - \frac{sc}{r}a = \frac{15sscc}{4rr},$

What is . . . . $a = ?$

Anſw. . . . . $a = \frac{5sc}{2r} = 20.$

IX. *The Arithmetical Reſolution of Equations which fall under the laſt of the three Forms before expreſt in* Sect. I. *of this* Chapter.

QUEST. I.

1. What is the Number repreſented by a in this Equation? . . $10a - aa = 24,$
2. Which Equation, if c be aſſumed to ſignifie 10, and n put for 24, may be expreſt thus, ... $ca - aa = n.$

RESOLUTION.

3. Let the Equation propoſed, by tranſpoſition of its Terms, be reduced to an Equation of the ſecond of the three Forms before expreſt in *Sect.* 1. *viz.* Firſt by tranſpoſition of $-aa$, this Equation ariſes;

$$10a = 24 + aa,$$

Or, $$ca = n + aa.$$

4. Likewiſe by tranſpoſition of 24 (or n) this Equation ariſes;

$$10a - 24 = aa,$$

Or, $$ca - n = aa.$$

5. And from the laſt Equation by Tranſpoſition of $10a$ (or ca) there will ariſe

$$-24 = aa - 10a,$$

Or, $$-n = aa - ca.$$

6. Which laſt Equation, by tranſpoſing each part of it to the contrary Coaſt, may be expreſt thus;

$$aa - 10a = -24,$$

Or, $$aa - ca = -n.$$

7. Now let the following proceſs be made as before in the Reſolution of Equations of the ſecond Form (in *Sect.* 7.) *viz.* Let the Square of half the Co-efficient 10 (or c) be added to each part of the laſt Equation, to the end its former part may be made a compleat Square (according to *Sect.* 4. *Chap.* 9.) whence this Equation ariſes;

$$aa - 10a + 25 = 25 - 24 = 1,$$

Or, $$aa - ca + \tfrac{1}{4}cc = \tfrac{1}{4}cc - n.$$

8. Then by extracting the Square Root of each part of the laſt Equation, (according to *Sect.* 4, and 5. of *Chap.* 8.) this Equation ariſes, *viz.*

$$a - 5 = 1,$$

Or, ... $$a - \tfrac{1}{2}c = \sqrt{:}\overline{\tfrac{1}{4}cc - n}{:}$$

9. Whence by equal addition of 5 (or $\frac{1}{2}c$) one value of a will be made known, *viz.*

$$a = 6 = \tfrac{1}{2}c + \sqrt{:}\overline{\tfrac{1}{4}cc - n}{:}$$

10. But foraſmuch as the Square Root of $aa - 10a + 25$ in the ſeventh ſtep may be $5 - a$ as well as $a - 5$, (for either of thoſe Roots being multiplied into it ſelf, will

produce

produce $aa - 10a + 25$,) therefore let $5 - a$ be ſet inſtead of $a - 5$ in the eighth ſtep, whence this Equation will ariſe, *viz.*

$$5 - a = 1,$$

Or, $$\tfrac{1}{2}c - a = \sqrt{:\tfrac{1}{4}cc - n}:$$

11. Whence by due Tranſpoſition another value of a is diſcovered, to wit,

$$a = 4 = \tfrac{1}{2}c - \sqrt{:\tfrac{1}{4}cc - n}:$$

12, I ſay the Number a ſought may be either 6 or 4, for either of theſe numbers will conſtitute the Equation propoſed, as will appear by

The Proof.

If . . . . $a = 6$,
Then conſequently . . . . . . $aa = 36$,
And . . . . . : . . . . $10a = 60$,
Therefore . . . . . $10a - aa = 24$.

Which was the Equation propos'd to be reſolved.

Again,

If . . . . $a = 4$,
Then conſequently . . . . . $aa = 16$,
And . . . . $10a = 40$,
Therefore . . . . . . $10a - aa = 24$; as before.

13. But to the end that both the values of a before expreſt in the ninth and eleventh Equations may be real or Affirmative Numbers, (that is, each greater than nothing) the given Numbers in the Equation propoſed, and likewiſe in every Equation of the Third Form aforeſaid muſt be ſubject to this following

DETERMINATION.

The Abſolute number given muſt not exceed the Square of half the Co-efficient.

The Reaſon of this Determination is Evident by the ſaid ninth and eleventh Equations; for the latter part of each of them ſhews, that the given Abſolute Number is to be ſubtracted from the Square of half the Co-efficient, and therefore it ought to be leſs, or equal to the ſaid Square: Therefore when in any Equation of the third Form, the given Abſolute number exceeds the Square of half the Co-efficient that Equation is impoſſible, and likewiſe the Queſtion that produced it.

It is alſo evident by the ſaid ninth and eleventh Equations, That when it happens that $n = \tfrac{1}{4}cc$, then $\tfrac{1}{4}cc - n = 0$, and conſequently each value of a is equal to $\tfrac{1}{2}c$; *viz.* When the Abſolute number happens to be equal to the Square of half the Co-efficient, then the two values of a will be equal to one another, each value in that caſe being equal to half the Co-efficient: But when it happens that the Abſolute number is leſs than the Square of half the Co-efficient, then thoſe two Roots or values of a will be unequal. But here is to be noted, that although in this latter caſe the Equation be always explicable by either of thoſe two unequal Roots or Numbers, yet the Queſtion that produced the Equation will ſometimes be anſwered only by one of thoſe Roots or Numbers, (as hereafter will appear in *Queſt.* 10. *Chap.* 16. and by the latter way of reſolving the 16. *Queſt.* of the ſame *Chap.*)

QUEST. 2.

1. What is the Number repreſented by a in this Equation? . . . . $5aa - aaaa = 4$.
2. Which Equation, if r be put for 5, and s for 4, may be expreſt thus . . . . $raa - aaaa = s$.

RESOLUTION.

3. Let the Equation propos'd, by Tranſpoſition of its Terms (after the ſame manner as in the third, fourth, fifth, and ſixth ſteps of the preceding *Queſt.* 1. *Sect.* 9.) be reduced to an Equation of the ſecond of the three Forms before expreſt in *Sect.*1. ſo this Equation will ariſe, *viz.*

$$aaaa - 5aa = -4,$$

Or, $$aaaa - raa = -s.$$

4. Then by adding (as in the former Examples) the Square of half the Co-efficient 5 (or r) to each part of the laſt Equation, there ariſes

$$aaaa - 5aa + \tfrac{25}{4} = \tfrac{25}{4} - 4 = \tfrac{9}{4},$$

Or, $$aaaa - raa + \tfrac{1}{4}rr = \tfrac{1}{4}rr - s.$$

5. And by extracting the Square Root of each part of the laſt Equation this ariſes;

$$aa - \tfrac{5}{2} = \tfrac{3}{2},$$

Or, $$aa - \tfrac{1}{2}r = \sqrt{}:\overline{\tfrac{1}{4}rr - s}:$$

6. Whence by equal Addition of $\tfrac{5}{2}$ (or $\tfrac{1}{2}r$,) this Equation ariſes, *viz.*

$$aa = \tfrac{8}{2} \text{ or } 4,$$

Or, $$aa = \tfrac{1}{2}r + \sqrt{}:\overline{\tfrac{1}{4}rr - s}:$$

7. Therefore by extracting the Square Root of each part of the laſt Equation, one value of a will be made known, *viz.*

$$a = 2 = \sqrt{(2)}:\overline{\tfrac{1}{2}r + \sqrt{\tfrac{1}{4}rr - s}}:$$

8: But foraſmuch as the Square Root of $aaaa - 5aa + \tfrac{25}{4}$ in the fourth ſtep may be $\tfrac{5}{2} - aa$, as well as $aa - \tfrac{5}{2}$, (for either of thoſe Roots being multiplied by it ſelf produce $aaaa - 5aa + \tfrac{25}{4}$;) therefore let $\tfrac{5}{2} - aa$ be ſet inſtead of $aa - \tfrac{5}{2}$ in the fifth ſtep, whence this Equation will ariſe;

$$\tfrac{5}{2} - aa = \tfrac{3}{2},$$

Or, $$\tfrac{1}{2}r - aa = \sqrt{}:\overline{\tfrac{1}{4}rr - s}:$$

9. Whence by due Tranſpoſition this Equation ariſes;

$$aa = \tfrac{2}{2} \text{ or } 1,$$

Or, $$aa = \tfrac{1}{2}r - \sqrt{}:\overline{\tfrac{1}{4}rr - s}:$$

10. Wherefore by extracting the Square Root of each part of the laſt Equation, another value of a is diſcovered, to wit,

$$a = 1 = \sqrt{(2)}:\overline{\tfrac{1}{2}r - \sqrt{\tfrac{1}{4}rr - s}}:$$

I ſay the Number a ſought may be either 2 or 1, for either of theſe numbers will conſtitute the Equation propoſed, as will appear by

The Proof.

If . . . . $a = 2$,
Then conſequently . . . . . $aa = 4$,
And . . . . $aaaa = 16$,
Alſo . . . . $5aa = 20$,
Therefore . . . . $5aa - aaaa = 4$.

Which was the Equation propos'd to be reſolv'd.

Again, If . . . . $a = 1$,
Then . . . . $aa = 1$,
And . . . . $aaaa = 1$,
Alſo . . . . $5aa = 5$,
Therefore, ... $5aa - aaaa = 4$; as before.

QUEST. 3.

1. What is the number repreſented by a in this Equation? $9aaa - aaaaaa = 8.$
2. Which Equation, if d be put for 9, and t for 8 may be expreſt thus . . . . $daaa - aaaaaa = t.$

RESOLUTION.

3. Let the Equation propos'd, by tranſpoſition of its Terms (after the ſame manner as in the third, fourth, fifth and ſixth ſteps of the preceding *Queſt.* 1. *Sect.* 9.) be reduced to an Equation of the ſecond of the three forms before expreſt in *Sect.* 1. ſo this Equation will ariſe, *viz.*

$$aaaaaa - 9aaa = -8,$$

Or, $$aaaaaa - daaa = -t.$$

4. Then by adding the Square of half the Co-efficient 9 (or d) to each part of the laſt Equation, there ariſes.

$$aaaaaa - 9aaa + \tfrac{81}{4} = \tfrac{81}{4} - 8 = \tfrac{49}{4},$$

Or, $$aaaaaa - daaa + \tfrac{1}{4}dd = \tfrac{1}{4}dd - t.$$

5. And by extracting the Square Root of each part of the laſt Equation this ariſes,

$$aaa - \tfrac{9}{2} = \tfrac{7}{2},$$

Or, $$aaa - \tfrac{1}{2}d = \surd\overline{:\tfrac{1}{4}dd - t:}$$

6. Whence by equal addition of $\tfrac{9}{2}$ (or $\tfrac{1}{2}d$) this Equation ariſes;

$$aaa = \tfrac{16}{2} \text{ or } 8,$$

Or, $$aaa = \tfrac{1}{2}d + \surd\overline{:\tfrac{1}{4}dd - t:}$$

7. Therefore by extracting the Cubic Root of each part of the Equation, one value of a will be made known, *viz.*

$$a = 2 = \surd(3)\overline{:\tfrac{1}{2}d + \surd\overline{:\tfrac{1}{4}dd - t:}}$$

8. But foraſmuch as the Square Root of $aaaaaa - 9aaa + \tfrac{81}{4}$ in the fourth ſtep may be $\tfrac{9}{2} - aaa$ as well as $aaa - \tfrac{9}{2}$, (for either of theſe Roots being multiplied by it ſelf, will produce the ſame Square $aaaaaa - 9aaa + \tfrac{81}{4}$,) therefore let $\tfrac{9}{2} - aaa$ be ſet inſtead of $aaa - \tfrac{9}{2}$ in the fifth ſtep, whence this Equation will be made, *viz.*

$$\tfrac{9}{2} - aaa = \tfrac{7}{2},$$

Or, $$\tfrac{1}{2}d - aaa = \surd\overline{:\tfrac{1}{4}dd - t:}$$

9. Whence by due tranſpoſition this Equation ariſes, *viz.*

$$aaa = \tfrac{2}{2} = 1,$$

Or, $$aaa = \tfrac{1}{2}d - \surd\overline{:\tfrac{1}{4}dd - t:}$$

10. Wherefore by extracting the Cubic Root of each part of the laſt Equation, another value of a is made known, *viz.*

$$a = 1 = \surd(3)\overline{:\tfrac{1}{2}d - \surd\overline{\tfrac{1}{4}dd - t:}}$$

I ſay the Number a ſought is either 2 or 1, for either of theſe numbers will conſtitute the Equation propoſed; as will appear by

The Proof.

If . . . . $a = 2$,
Then conſequently . . . . . $aaa = 8$,
And . . . . $aaaaaa = 64$,
Alſo . . . . $9aaa = 72$,
Therefore . . . . $9aaa - aaaaaa = 8$.

Which was the Equation propoſed to be reſolved.

Again,

If . . . . $a = 1$,
Then conſequently . . . . . $aaa = 1$,
And . . . . $aaaaaa = 1$,
Alſo . . . . $9aaa = 9$,
Therefore . . . . $9aaa - aaaaaa = 8$; as before.

X. From the Reſolution of the three laſt Queſtions the following Canon is deduced for the reſolving all Equations which fall under the laſt of the three Forms before ſpecified in *Sect.* 1. of this *Chap.*

CANON.

From the Square of half the Co-efficient, or (which is the ſame thing) from a quarter of the Square of the whole Co-efficient, ſubtract the Abſolute number given.

Extract the Square Root of that Remainder.

Add the ſaid Square Root to half the Co-efficient, and alſo ſubtract it from half the Co-efficient, reſerving the Sum and Remainder.

Laſtly, when the unknown number which is multiplied by the Co-efficient in the middle term of the Equation is expreſt by a ſingle letter only, as a, then the Sum and Remainder before reſerved are the two Numbers ſought, each of which will conſtitute the Equation propoſed; but if the ſaid unknown number in the middle term be a Square, as aa, then the Square Root ſeverally extracted out of the Sum and Remainder reſerved ſhall be the two Numbers ſought; if a Cube, as aaa, then the Cubic Root ſeverally extracted out of the ſaid Sum and Remainder ſhall be the two Numbers ſought; if any higher Power, then the Root for the kind muſt be extracted ſeverally out of the ſaid Sum and Remainder, which Roots ſhall be the two Numbers ſought.

An

An Example of the ſaid Canon.

1. Let the preceding *Queſt.* 1. in *Sect.* 9. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . $10a - aa = 24$,
2. Or, What is the value of *a* in this Equation? . . . . $ca - aa = n$.

RESOLUTION.

3. From the Square of half the Co-efficient 10, to wit, the Square of 5, which is . . .	25	$\frac{1}{4}cc$.
4. Subtract the given abſolute number . . .	24	n.
5. The remainder is	1	$\frac{1}{4}cc - n$.
6. The Square Root of that remainder is .	1	$\surd:\overline{\frac{1}{4}cc - n}:$
7. To which Square Root add half the Co-efficient 10, to wit, ...	5	$\frac{1}{2}c$.
8. The Sum is the greater value of *a* ſought, to wit, ...	6	$\frac{1}{2}c + \surd:\overline{\frac{1}{4}cc - n}:$
9. But ſubtracting the ſaid Square Root from half the Co-efficient, the remainder is the leſſer value of *a*, to wit	4	$\frac{1}{2}c - \surd:\overline{\frac{1}{4}cc - n}:$

Either of which Numbers 6 and 4 found out in the two laſt ſteps will conſtitute the Equation propoſed, as before has been proved in the *Anſwer* to *Queſt.* 1. in *Sect.* 9. of this *Chap.*

A Second Example of the Canon in Sect. 10.

1. Let the preceding *Queſt.* 2. in *Sect.* 9. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . . . $5aa - aaaa = 4$
2. Or, What is the value of *a* in this Equation? . . . . $raa - aaaa = s$.

RESOLUTION.

3. From the Square of half the Co-efficient 5, to wit, the Square of $\frac{5}{2}$, which is .	$\frac{25}{4}$	$\frac{1}{4}rr$.
4. Subtract the given abſolute number .	4	s.
5. The remainder is	$\frac{9}{4}$	$\frac{1}{4}rr - s$.
6. The Square Root of that remainder is . .	$\frac{3}{2}$	$\surd:\overline{\frac{1}{4}rr - s}:$
7. To which Square Root add half the Co-efficient 5, to wit, ...	$\frac{5}{2}$	$\frac{1}{2}r$.
8. The Sum is the greater value of *aa*, to wit,	4	$\frac{1}{2}r + \surd:\overline{\frac{1}{4}rr - s}:$
9. But ſubtracting the ſaid Square Root from half the Co-efficient, the remainder is the leſſer value of *aa*, to wit, ...	1	$\frac{1}{2}r - \surd:\overline{\frac{1}{4}rr - s}:$
10. Therefore the Square Root of the Sum in the 8*th* ſtep is the greater value of *a*, to wit,	2	$\surd(2):\overline{\frac{1}{2}r + \surd\overline{\frac{1}{4}rr - s}}:$
11. And the ſquare root of the remainder in the ninth ſtep is the leſſer value of *a*, to wit,	1	$\surd(2):\overline{\frac{1}{2}r - \surd\overline{\frac{1}{4}rr - s}}:$

Either of which numbers 2 and 1 found out in the two laſt ſteps will conſtitute the Equation propoſed, as before has been proved in the *Anſwer* to *Queſt.* 2. in *Sect.* 9 of this *Chap.*

A Third Example of the Canon in Sect. 10.

1. Let the preceding *Queſt.* 3. in *Sect.* 9. of this *Chap.* be here repeated, *viz.* What is the number repreſented by *a* in this Equation? . . . . $9aaa - aaaaaa = 8$
2. Or, What is the value of *a* in this Equation? . . . $daaa - aaaaaa = t$

RESOLUTION.

3. From the Square of half the Co-efficient 9, to wit, the Square of $\frac{9}{2}$, which is, . .	$\frac{81}{4}$	$\frac{1}{4}dd$.
4. Subtract the given abſolute number . .	8	t.

5. The

5. The remainder is	$\frac{49}{4}$	$\frac{1}{4}dd - t:$
6. The Square Root of that remainder is . .	$\frac{7}{2}$	$\sqrt{:\frac{1}{4}dd - t:}$
7. To which Square Root add half the Co-efficient 9, to wit, ...	$\frac{9}{2}$	$\frac{1}{2}d.$
8. The ſum is the greater value of *aaa*, to wit,	8	$\frac{1}{2}d + \sqrt{:\frac{1}{4}dd - t:}$
9. But ſubtracting the ſaid Square Root from half the Co-efficient, the remainder is the leſſer value of *aaa*, to wit, ...	1,	$\frac{1}{2}d - \sqrt{:\frac{1}{4}dd - t:}$
10. Therefore the Cubic Root of the ſum in the eight ſtep is the greater value of *a*, to wit,	2	$\sqrt{(3)}\overline{\frac{1}{2}d + \sqrt{\frac{1}{4}dd - t}:}$
11. And the Cubic Root of the remainder in the ninth ſtep is the leſſer value of *a*, to wit,	1	$\sqrt{(3)}\overline{:\frac{1}{2}d - \sqrt{\frac{1}{4}dd - t}:}$

Either of which numbers 2 and 1 found out in the two laſt ſteps will conſtitute the Equation propoſed, as before has been proved in the *Anſwer* to *Queſt.* 3. in *Sect.* 9. of this *Chap.*

Example 4.

1. If b, d, f, g repreſent ſuch known Numbers that bf is greater than dg; and,

2. If $\frac{bg + 2bf + df}{bg + dg + bf + df}a - aa = \frac{bf - dg}{bg + dg + bf + df};$
What is a equal to?

Anſw. a is equal to 1, and alſo to $\frac{bf - dg}{bg + dg + bf + df}.$

Which values of a are alſo found out by the Canon in the Tenth *Section* of this *Chap.* but I ſhall leave the Operation as an exerciſe for the induſtrious Learner, and in the next place ſhew the uſe of the Rules before delivered in this fifteenth *Chap.* in the Reſolution of various Arithmetical Queſtions.

CHAP. XVI.

Various Arithmetical Queſtions, producing Equations that fall under ſome of the three Forms in Sect. 1. *of the foregoing* Chap. 15. *and are reſolvable by their reſpective Canons in* Sect. 6, 8, *and* 10. *of the ſame* Chap.

QUEST. 1.

THere are two Numbers whoſe difference is 16 (or c,) and the Product of their Multiplication is 36 (or b;) what are the Numbers?

RESOLUTION.	*Numeral.*	*Literal.*
1. For the leſſer of the two numbers ſought put	a	a
2. Then by adding to the ſaid leſſer number the given difference 16 (or c,) the greater number ſought will be	$a + 16$	$a + c$
3. Therefore from the two laſt ſteps the Product made by the mutual Multiplication of the two Numbers ſought will be . . .	$aa + 16a$	$aa + ca$

4. Which Product muſt be equal to the given Product 36 (or b) whence this Equation ariſes, *viz.* $aa + 16a = 36$,
Or, $aa + ca = b$.

5. Which Equation being reſolved by the Canon in *Sect.* 6. of *Chap.* 15. the value of a; or the leſſer number ſought by this Queſtion will be diſcovered, *viz.*

$$a = 2 = \sqrt{:b + \tfrac{1}{4}cc:} - \tfrac{1}{2}c.$$

6. To which leſſer number adding the given difference 16 (or c) the greater number ſought will alſo be made known, *viz.*

$$2+16=18=\sqrt{:b+\tfrac{1}{4}cc:}+\tfrac{1}{2}c.$$

Otherwiſe thus,

1. For the greater of the two numbers ſought put	a	a
2. Then by ſubtracting from the ſaid greater number the given difference 16, (or c) the leſſer number ſought will be	$a-16$	$a-c$
3. Therefore from the two laſt ſteps, the Product made by the mutual Multiplication of the two numbers ſought will be . .	$aa-16a$	$aa-ca$

4. Which Product muſt be equal to the given Product 36, (or b;) whence this Equation ariſes, *viz.*

$$aa-16a=36, \quad \text{or,} \quad aa-ca=b.$$

5. Which Equation being reſolved by the Canon in *Sect.* 8. of *Chap.* 15. the value of a, to wit, the greater number ſought will be diſcovered, *viz.*

$$a=18=\sqrt{:b+\tfrac{1}{4}cc:}+\tfrac{1}{2}c.$$

6. And by ſubtracting from the ſaid greater number the given difference 16 (or c,) the leſſer number ſought will alſo be diſcovered, *viz.*

$$18-16=2=\sqrt{:b+\tfrac{1}{4}cc:}-\tfrac{1}{2}c.$$

From either of thoſe ways of Reſolution, the numbers ſought are found 18 and 2, which will ſolve the Queſtion propoſed; for their difference is 16, and the Product of their Multiplication is 36, as was preſcribed.

Moreover, the two laſt ſteps of each Reſolution by Literal Algebra give one and the ſame Canon to ſolve the Queſtion propoſed.

CANON.

To the given Product add the ſquare of half the given difference, and extract the ſquare Root of that ſum; then to the ſaid ſquare Root adding half the given difference, and from the ſaid ſquare Root ſubtracting the ſaid half difference, the ſum and Remainder ſhall be the two numbers ſought.

Therefore the difference and the Rectangle (or Product of the Multiplication) of any two numbers being ſeverally given, the numbers themſelves ſhall alſo be given by the ſaid Canon.

QUEST. 2.

There are three numbers in Geometrical proportion continued; the difference of the extremes, that is, of the firſt and third is 16 (or c.) and the mean is 6 (or m;) what are the extreme Proportionals?

RESOLUTION.

1. For the leſſer of the two extreme Proportionals ſought put	a	a
2. Then by adding to the ſaid leſſer extreme the given difference of the extremes, to wit, 16 (or c,) the greater extreme will be	$a+16$	$a+c$
3. Therefore the Rectangle contained under the extreme Proportionals,) to wit, the Product made by their mutual Multiplication) ſhall be	$aa+16a$	$aa+ca$

4. Which Rectangle (or Product) muſt (by *Sect.* 1. *Chap.* 13.) be equal to the ſquare of the given mean Proportional 6 (or m), hence this Equation;

$$aa+16a=36, \quad \text{or,} \quad aa+ca=mm.$$

5. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, or the leſſer of the two extreme Proportionals ſought will be made known, *viz.*

$$a=2=\sqrt{:mm+\tfrac{1}{4}cc:}-\tfrac{1}{2}c.$$

6. To which leſſer extreme Proportional adding 16 (or c) the given difference of the extremes, the greater of the two extreme Proportionals will alſo be diſcovered, *viz.*

$$2+16 = 18 = \sqrt{:mm+\tfrac{1}{4}cc:} + \tfrac{1}{2}c.$$

I ſay the two extreme Proportionals ſought are 2 and 18, between which the given number 6 is a mean Proportional; for, as 2 is to 6, ſo is 6 to 18.

Moreover, the two laſt ſteps of the Reſolution give the following Canon to find out the extreme Proportionals ſought.

CANON.

To the Square of the given mean Proportional add the Square of half the given difference of the extremes, and extract the ſquare Root of that Sum; then to the ſaid ſquare Root adding half the ſaid difference, and from the ſaid ſquare Root ſubtracting the ſame half difference, the Sum and Remainder ſhall be the extreme Proportionals ſought.

Therefore if of three numbers in continual proportion the mean be given, as alſo the difference of the extremes, the extremes ſhall be given ſeverally by the ſaid Canon.

QUEST. 9.

There are two numbers whoſe Sum is 20 (or c,) and the Product of their Multiplication is 36 (or n;) what are the numbers?

RESOLUTION.

1. For one of the numbers ſought put . . .	a	a
2. Then by ſubtracting that number from the given Sum 20 (or c,) the Remainder will be the other number ſought, to wit, . .	$20-a$	$c-a$
3. Therefore the Product of the Multiplication of thoſe two numbers will be . . .	$20a-aa$	$ca-aa$

4. Which Product muſt be equal to the given Product 36 (or n,) whence this Equation ariſes, *viz.* $20a-aa = 36$,

Or, $ca-aa = n$.

5. Which Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers ſought by this Queſtion will be diſcovered, *viz.*

$$a = \begin{cases} 18 = \tfrac{1}{2}c + \sqrt{:\tfrac{1}{4}cc - n:} \\ 2 = \tfrac{1}{2}c - \sqrt{:\tfrac{1}{4}cc - n:} \end{cases}$$

I ſay the numbers ſought are 18 and 2, for their Sum is 20, and the Product of their Multiplication is 36, as was preſcribed.

Moreover, if the two values of a, which are expreſs'd by Letters in the laſt ſtep of the Reſolution, be expreſs'd by Words, they will give the following Canon to ſolve the Queſtion propoſed.

CANON.

From the Square of half the given Sum ſubtract the given Product, and extract the ſquare Root of the Remainder; then to the ſaid half Sum adding the ſaid ſquare Root, and from the ſaid half Sum ſubtracting the ſame ſquare Root, the Sum and Remainder ſhall be the two numbers ſought.

Therefore the Sum and Rectangle (or Product of the Multiplication) of any two numbers being ſeverally given, the numbers themſelves ſhall alſo be given ſeverally by the ſaid Canon.

QUEST. 4.

There are three numbers in continual proportion; the ſum of the extremes is 20, (or c,) and the mean proportional is 6, (or m;) what are the extremes?

RESOLUTION.

1. For one of the two extreme proportionals ſought put	a	a

2. Then by ſubtracting that extreme from 20 (or c(the given Sum, the Remainder will be the other extreme, to wit,	$20-a$	$c-a$
3. Therefore the Rectangle contained under the extreme proportionals, (to wit, the Product of their Multiplication) ſhall be	$20a-aa$	$ca-aa$

4. Which Rectangle (or Product) muſt (according to *Sect.* 1. *Chap.* 13.) be equal to the Square of the given mean Proportional 6 (or m,) whence this Equation ariſes, *viz.*

$$20a-aa=36,$$

Or, $ca-aa=mm.$

5. Which Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers ſought by this Queſtion will be diſcovered, *viz.*

$$a=\left\{\begin{array}{l} 18=\frac{1}{2}c+\sqrt{:\frac{1}{4}cc-mm:} \\ 2=\frac{1}{2}c-\sqrt{:\frac{1}{4}cc-mm:} \end{array}\right.$$

I ſay the two extreme Proportionals ſought are 18 and 2, between which the given number 6 is a mean Proportional; for, as 18 is to 6, ſo is 6 to 2.

Moreover, if the two values of a which are expreſs'd by Letters in the laſt ſtep of the Reſolution be expreſs'd by words, they will give the following Canon to find out the extreme Proportionals ſought.

CANON.

From the Square of half the given Sum of the extreme Proportionals ſubtract the Square of the given mean, and extract the ſquare Root of the Remainder; then to the ſaid half Sum adding the ſaid Square Root, and from the ſaid half Sum ſubtracting the ſame ſquare Root, the Sum and Remainder ſhall be the two extreme Proportionals ſought.

Therefore if of three Numbers in continual proportion the mean be given, as alſo the Sum of the extremes, the extremes themſelves ſhall be given ſeverally by the ſaid Canon.

QUEST. 5.

There are two Numbers whoſe difference is 15, (or d,) and if the Product of the Multiplication of the ſaid two Numbers be divided by 2, (or c,) the Quotient will give the Cube of the leſſer Number; what are the Numbers?

RESOLUTION.

1. For the leſſer Number ſought put	a	a
2. To which adding the given difference 15 (or d,) the Sum ſhall be the greater Number, to wit,	$a+15$	$a+d$
3. Therefore the Product of the Multiplication of the two Numbers is	$aa+15a$	$aa+da$
4. Which Product being divided by 2 (or c) the Quotient will be	$\frac{aa+15a}{2}$	$\frac{aa+da}{c}$
5. From the firſt ſtep the Cube of the leſſer Number is	aaa	aaa

6. Which Cube muſt (as the Queſtion requires) be equal to the Quotient in the fourth ſtep, whence this Equation;

$$aaa=\frac{aa+15a}{2},$$

Or, $aaa=\frac{aa+da}{c}.$

7. Which Equation being duly reduced (according to *Sect.* 2, 4, 3, 5 of *Chap.* 12.) there will ariſe $aa-\frac{1}{2}a=\frac{15}{2},$

Or, $aa-\frac{1}{c}a=\frac{d}{c}.$

8. Therefore the laſt Equation being reſolved by the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the leſſer number ſought will be diſcovered, *viz.*

$$a=3=\sqrt{:\frac{d}{c}+\frac{1}{4cc}:}+\frac{1}{2c}.$$

9. To which leſſer number adding the given difference 15 (or *d*) the Sum ſhall be the greater number ſought, to wit,

$$3 + 15 = 18 = \sqrt{:\frac{d}{c} + \frac{1}{4cc}:} + \frac{1}{2c} + d.$$

10. I ſay the two numbers ſought are 3 and 18, which will ſatisfie the conditions in the Queſtion, for their difference is 15, and if the Product of their Multiplication 54 be divided by 2, the Quotient is 27, which is the Cube of the leſſer number 3, as was required.

11. But if the Equation in the eighth ſtep be expreſs'd by words, it will give the following Canon to find out the leſſer number ſought, to which adding the given difference, the greater number is alſo given.

CANON.

Divide the given difference by the given Diviſor, alſo divide 1 (or Unity) by the quadruple of the Square of the given Diviſor, add thoſe two Quotients together, and extract the ſquare Root of the Sum; then to this ſquare Root add the Quotient that ariſes by dividing 1 by the double of the given Diviſor; ſo ſhall the Sum be the leſſer of the two numbers ſought, which increaſed with their given difference will give the greater number.

QUEST. 6.

There are two numbers whoſe difference is 2 (or *d*,) and the Sum of their Squares is 130 (or *c*;) what are the numbers?

RESOLUTION.

1. For the leſſer number ſought put . . .	a	a
2. Then to that leſſer number adding the given difference 2 (or *d*) the Sum ſhall be the greater number, to wit, . . .	$a+2$	$a+d$
3. Therefore from the firſt ſtep the Square of the leſſer number is	aa	aa
4. And from the ſecond ſtep the Square of the greater number is	$aa+4a+4$	$aa+2da+dd$
5. Therefore from the two laſt ſteps the Sum of the Squares of the two numbers ſought is	$2aa+4a+4$	$2aa+2da+dd$

6. Which Sum muſt be equal to the given Sum of the Squares 130 (or *c*,) whence this Equation ariſes, *viz.*

$$2aa + 4a + 4 = 130,$$

Or, $$2aa + 2da + dd = c.$$

7. Which Equation, after due Reduction according to the Rules of the twelfth *Chap.* will give this Equation, *viz.* $aa + 2a = 63$,

Or, $$aa + da = \tfrac{1}{2}c - \tfrac{1}{2}dd.$$

8. Therefore the Equation in the laſt ſtep being reſolved according to the Canon in *Sect.* 6. *Chap.* 15. the value of *a*, to wit, the leſſer number ſought by the Queſtion will by made known, *viz.*

$$a = 7 = \sqrt{:\tfrac{1}{2}c - \tfrac{1}{4}dd:} - \tfrac{1}{2}d.$$

9. To which leſſer number adding the given difference 2 (or *d*) the Sum ſhall be the greater number ſought, to wit,

$$7 + 2 = 9 = \sqrt{:\tfrac{1}{2}c - \tfrac{1}{4}dd:} + \tfrac{1}{2}d.$$

10. I ſay the two numbers ſought are 9 and 7; for their difference is 2, and the Sum of their Squares is 130, as was preſcribed by the Queſtion.

11. Moreover, from the eighth and ninth ſtep ariſes this

CANON.

From half the given Sum ſubtract the Square of half the given difference, and extract the ſquare Root of the Remainder; then from this ſquare Root ſubtract half the given difference, the Remainder ſhall be the leſſer number ſought, to which adding the given difference the Sum ſhall be the greater Number.

QUEST.

QUEST. 7.

There are two Numbers whoſe Sum is 14 (or b,) and the Sum of their Squares is 100 (or c,) what are the Numbers?

RESOLUTION.

1. For one of the Numbers ſought put . . .	a	a
2. Which ſubtracted from the given Sum 14 (or b) leaves the other Number . . .	$14 - a$	$b - a$
3. The Square of the firſt number is . . .	aa	aa
4. The Square of the other Number is . . .	$aa - 28a + 196$	$aa - 2ba + bb$
5. The Sum of the ſaid Squares is	$2aa - 28a + 196$	$2aa - 2ba + bb$

6. Which Sum muſt be equal to 100 (or c) the given Sum of the Squares, whence this Equation ariſes, *viz.* $2aa - 28a + 196 = 100$,
Or, $2aa - 2ba + bb = c$.

7. Which Equation, after due Reduction, according to the Rules of the twelfth *Chap.* will give this following Equation;

$$14a - aa = 48,$$

Or, $$ba - aa = \frac{1}{2}bb - \frac{1}{2}c.$$

8. Which Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers ſought by this Queſtion, will be diſcovered, *viz.*

$$a = \begin{cases} 8 = \frac{1}{2}b + \sqrt{:\frac{1}{2}c - \frac{1}{4}bb}. \\ 6 = \frac{1}{2}b - \sqrt{:\frac{1}{2}c - \frac{1}{4}bb}. \end{cases}$$

9. I ſay the Numbers ſought are 8 and 6; for their Sum is 14, and the Sum of their Squares is 100, as was preſcribed.

10. Moreover, if the two values of a which are expreſs'd by Letters in the eighth ſtep be expreſs'd by words there will ariſe this

CANON.

From half the given Sum of the Squares ſubtract the Square of half the given Sum of the two numbers, and extract the ſquare Root of the Remainder; then adding the ſaid ſquare Root to the ſaid half Sum of the Numbers, the Sum of this Addition ſhall be the greater Number; but ſubtracting the ſaid ſquare Root from the ſaid half Sum of the Numbers, the Remainder ſhall be the leſſer Number.

QUEST. 8.

There are three Numbers in Geometrical proportion continued, and ſuch, that if the difference between the ſum of the extremes and the mean be multiplied by the ſum of the extremes, the Product will be 1120 (or b;) but if the ſaid difference be multiplied by the ſum of all the three Proportionals, the Product will be 1456 (or c;) what are the Proportionals?

RESOLUTION.

1. For the difference of the Sum of the Extremes and Mean put	a	a
2. Then, according to the Queſtion, the ſum of the extremes is	$\frac{1120}{a}$	$\frac{b}{a}$
3. From which ſum if the difference in the firſt ſtep be ſubtracted, the Remainder will be the mean proportional, to wit, . . .	$\frac{1120}{a} - a$	$\frac{b}{a} - a$
4. Therefore from the two laſt ſteps the ſum of all three proportionals is	$\frac{2240}{a} - a$	$\frac{2b}{a} - a$

5. But (according to the Queſtion) if the ſum of all the three proportionals be multiplied by the difference of the ſum of the extremes and the mean, the Product muſt be equal to 1456 (or c;) therefore from the firſt and fourth ſteps this following Equation ariſes, *viz.*

$$2240 - aa = 1456,$$

Or, $$2b - aa = c.$$

6. Which Equation being reduced according to the Rules of the twelfth *Chap.* the value of *a* will be diſcovered, *viz.*

$$a = 28 = \sqrt{:2b-c.}$$

7. Therefore from the ſixth and ſecond ſteps, the Sum of the extremes is alſo known, *viz.*

$$40 = \frac{b}{\sqrt{:2b-c:}} = \text{the Sum of the extremes.}$$

8. And from the ſixth and third ſteps, the mean proportional is alſo given, *viz.*

$$12 = \frac{c-b}{\sqrt{:2b-c:}} = \text{the mean.}$$

9. Laſtly, the Sum of the extremes of three continual proportionals being given 40, as alſo the mean 12, the extremes ſhall alſo be given ſeverally by the Canon of the fourth *Queſtion* of this *Chap.* to wit, 4 and 36; therefore the three continual proportionals ſought are 4, 12 and 36, which will ſatisfie the conditions in the Queſtion propoſed, as will appear by

The Proof.

I. $\overline{4,\ 12,\ 36}$ are ÷; for, $4 \times 36 = 12 \times 12.$

II. $\overline{4+36-12}$ into $\overline{36+4} = 1120.$

III. $\overline{4+36-12}$ into $\overline{4+12+36} = 1456.$

QUEST. 9.

There are two Numbers whoſe Sum is 10 (or *b*,) and the Sum of their Cubes is 520 (or *c*;) what are the Numbers?

RESOLUTION.

1. For one of the Numbers ſought put	a	a
2. Then by ſubtracting that Number from the given Sum 10 (or *b*,) the other Number remains, to wit, ...	$10-a$	$b-a$
3. The Cube of the former is	aaa	aaa

4. And from the ſecond ſtep the Cube of the latter Number is

$$1000 - 300a + 30aa - aaa,$$
$$\text{Or,}\quad bbb - 3bba + 3baa - aaa.$$

4. Therefore the Sum of the two Cubes in the third and fourth ſteps is

$$1000 - 300a + 30aa,$$
$$\text{Or,}\quad bbb - 3bba + 3baa.$$

5. Which Sum muſt be equal to 520 (or *c*) the given Sum of the Cubes, whence this Equation ariſes, *viz.* $1000 - 300a + 30aa = 520,$

$$\text{Or,}\quad bbb - 3bba + 3baa = c.$$

6. Which Equation, after due Reduction according to the Rules of the twelfth *Chap.* will give this Equation; $16 = 10a - aa,$

$$\text{Or,}\quad \frac{bbb-c}{3b} = ba - aa.$$

7. Therefore the laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of *a*, which are the numbers ſought by this Queſtion, will be diſcovered, *viz.*

$$a = \begin{cases} \frac{1}{2}b + \sqrt{:\frac{c}{3b} - \frac{bb}{12}:} = 8. \\ \frac{1}{2}b - \sqrt{:\frac{c}{3b} - \frac{bb}{12}:} = 2. \end{cases}$$

8. I ſay the two Numbers ſought are 8 and 2; for their Sum is 10, and the Sum of their Cubes is 520, as was preſcribed.

9. Moreover, if the two values of *a* which are expreſs'd by Letters in the ſeventh ſtep be expreſs'd by words, they will give this

CANON.

From the Quotient that ariſes by dividing the given Sum of the two Cubes, by the triple of the given Sum of their ſides, ſubtract $\frac{1}{12}$ of the Square of the laſt mentioned

Sum, and extract the ſquare Root of the Remainder; then adding the ſaid ſquare Root to half the ſaid Sum of the Sides of the two Cubes, and alſo ſubtracting the ſaid ſquare Root from the ſaid half Sum, the Sum and Remainder ſhall be the Sides or numbers ſought.

QUEST. 10.

There are two numbers whoſe Sum is 10 (or b,) and the proportion which their difference beareth to the Sum of their Squares is as 2 to 29, (or as r to s; what are the Numbers?

RESOLUTION.

1. For the greater number ſought put . . .	a	a
2. Which ſubtracted from the given Sum 10 (or b) leaves the leſſer number . . .	$10-a$	$b-a$
3. Therefore the difference of the two numbers is	$2a-10$	$2a-b$
4. And from the firſt ſtep the ſquare of the greater number is	aa	aa

5. And from the ſecond ſtep the Square of the leſſer number is

$$100-20a+aa,$$

Or, $$bb-2ba+aa.$$

6. And from the two laſt ſteps the Sum of the Squares of the two numbers ſought is

$$100-20a+2aa,$$

Or, $$bb-2ba+2aa.$$

7. Then according to the Queſtion, the difference in the third ſtep muſt be to the ſum of the ſquares in the ſixth ſtep as 2 to 29, (or as r to s;) *viz.*

$$2\ .\ 29\ ::\ 2a-10\ .\ 100-20a+2aa,$$

Or, $$r\ .\ s\ ::\ 2a-b\ .\ bb-2ba+2aa.$$

8. Which Analogy may be converted into this following Equation, (according to the Theorem in *Chap.* 1. *Sect.* 13.) *viz.*

$$200-40a+4aa=58a-290,$$

Or, $$rbb-2rba+2raa=2sa-sb.$$

9. Which Equation, after due Reduction according to the Rules in the 12 *Chap.* will produce this Equation; $\frac{245}{2}=\frac{49}{2}a-aa,$

Or, $$\frac{rbb+sb}{2r}=\frac{s+rb}{r}a-aa.$$

10. Therefore by reſolving the Equation in the laſt ſtep according to *Sect.* 10. *Chap.* 15. the two values of a, or the two Roots of that Equation will be made known, *viz.*

$$a=\begin{cases}\frac{35}{2}=\frac{s}{2r}+\frac{b}{2}+\surd:\overline{\frac{ss}{4rr}-\frac{bb}{4}}:\\ 7=\frac{s}{2r}+\frac{b}{2}-\surd:\overline{\frac{ss}{4rr}-\frac{bb}{4}}:\end{cases}$$

11. The leſſer of which two Roots or Numbers, to wit 7, is the greater number ſought by this Queſtion; and conſequently, the ſaid 7 being ſubtracted from the given ſum 10, the Remainder 3 is the leſſer number ſought.

I ſay 7 and 3 will ſolve the Queſtion, for their ſum is 10; and their difference 4 is to the ſum of their ſquares 58, as 2 to 29; which was preſcribed.

12. *Note.* Altho the value of a in the Equation in the ninth ſtep may be either $\frac{35}{2}$ or 7, (for that Equation may be expounded by $\frac{35}{2}$ as well as 7,) yet 7 only, to wit, the leſſer value of a, ſhall be the greater number ſought by this Queſtion.

For that the greater value of a, to wit, $\frac{s}{2r}+\frac{b}{2}+\surd:\overline{\frac{ss}{4rr}-\frac{bb}{4}}:$ can never be equal to either of the two numbers ſought, I prove thus; Firſt, it is manifeſt by each of the values of a expreſs'd by Letters in the tenth ſtep, That if $\frac{s}{2r}=\frac{b}{2}$, then conſequently $\frac{ss}{4rr}=\frac{bb}{4}$, and the two values of a are equal one to the other, each being equal to $\frac{s}{2r}+\frac{b}{2}$, that is, b; and therefore in this firſt caſe, neither of the two values of a can poſſibly be equal to either of the two numbers ſought; for that which is equal to the ſum of two numbers muſt needs be greater than either of them.

Secondly, If $\frac{s}{2r}\sqsubset\frac{b}{2}$, which is a neceſſary Determination to make the Queſtion poſſible, then the greater value of a, that is, $\frac{s}{2r}+\frac{b}{2}+\sqrt{:\frac{ss}{4rr}-\frac{bb}{4}:}$ is manifeſtly greater than b the given ſum of the two numbers ſought, and therefore it cannot be equal to either of them. Wherefore the ſaid greater value of a cannot in any caſe be equal to either of the two numbers ſought. Which was to be proved.

But the ſaid leſſer value of a is the greater of the two numbers ſought, and conſequently they are given ſeverally by this following

CANON.

13. From the Quotient that ariſes by dividing the Square of the latter term of the given Reaſon by the Quadruple of the Square of the firſt Term, ſubtract a quarter of the Square of the given Sum of the two numbers ſought, and extract the ſquare Root of the Remainder; then ſubtract that ſquare Root from the Sum of the Quotient that ariſes by dividing the latter Term of the given Reaſon by the double of the firſt, and the half of the given ſum of the two numbers, ſo the Remainder ſhall be the greater number ſought; which ſubtracted from the ſaid given ſum leaves the leſſer number.

14 From the premiſes this following Queſtion may eaſily be ſolved, *viz.* The ſum of two numbers being given, ſuppoſe $\frac{4}{5}$ (or b,) and their difference being equal to the ſum of their Squares, to find the numbers.

Firſt, ſuppoſe $r=s=1$; (becauſe the Terms of the Proportion in this Queſtion are equal to one another,) then the two values of a before expreſs'd in the tenth ſtep will be converted into theſe, *viz.*

$$a=\frac{6}{5}=\frac{1+b}{2}+\sqrt{:\frac{1-bb}{4}},$$

$$a=\frac{3}{5}=\frac{1+b}{2}-\sqrt{:\frac{1-bb}{4}}.$$

The leſſer of which values of a, to wit, $\frac{3}{5}$, is the greater of the two numbers ſought, and therefore the ſaid $\frac{3}{5}$ being ſubtracted from $\frac{4}{5}$ the given ſum, leaves $\frac{1}{5}$ for the leſſer number. I ſay $\frac{3}{5}$ and $\frac{1}{5}$ will ſolve the Queſtion, for their difference $\frac{2}{5}$ is equal to the Sum of their Squares.

QUEST. 11.

There are two numbers, the Product of whoſe Multiplication is 48 (or p,) and the difference of their Squares is 28 (or d;) what are the numbers?

RESOLUTION.

1. For the greater number put	a	a
2. Then dividing 48 (or p) by a, the Quotient is the leſſer number, to wit, ...	$\frac{48}{a}$	$\frac{p}{a}$
3. From the firſt ſtep the ſquare of the greater number is	aa	aa
4. And from the ſecond ſtep the Square of the leſſer number is	$\frac{2304}{aa}$	$\frac{pp}{aa}$
5. Therefore the difference of the ſaid Squares is	$\frac{aaaa-2304}{aa}$	$\frac{aaaa-pp}{aa}$

6. Which difference muſt be equal to the given difference of the ſquares, whence this Equation ariſes, *viz.*

$$\frac{aaaa-2304}{aa}=28,$$

Or, $$\frac{aaaa-pp}{aa}=d.$$

7. Which Equation, after due Reduction according to the Rules of the twelfth *Chap.* will produce this;

$$aaaa - 28aa = 2304,$$

Or, $$aaaa - daa = pp.$$

8. Therefore by reſolving the laſt Equation according to the Canon in *Sect.* 8. *Chap.* 15. the value of *a*, to wit, the greater number ſought will be diſcovered, *viz.*

$$a = 8 = \sqrt{(2)}:\overline{\sqrt{pp+\tfrac{1}{4}dd}+\tfrac{1}{2}d}:$$

Whence the greater number is found 8, by which if the given Product 48 be divided, the Quotient 6 is the leſſer Number ſought.

I ſay, the Numbers 8 and 6 will ſolve the Queſtion; for the Product of their Multiplication is 48, and the difference of their Squares 64 and 36 is 28, as was preſcribed.

Moreover, the Equation in the eighth ſtep gives a Canon to find the greater of the two numbers ſought, by the help whereof and the given Product the leſſer number ſhall be alſo given.

CANON.

9 To the Square of the given Product add the Square of half the given difference of the Squares, and extract the Square Root of that Sum; then to the ſaid Square Root add the ſaid half difference, and extract the Square Root of this Sum, ſo ſhall the laſt Square Root be the greater of the two Numbers ſought; laſtly, by the ſaid greater number divide the given product of the multiplication of both numbers, and the Quotient ſhall be the leſſer Number.

QUEST. 12.

There are two Numbers the Product of whoſe Multiplication is 48 (or *p*,) and the Sum of their Squares is 100 (or *c*;) what are the Numbers?

RESOLUTION.

1. For one of the numbers ſought put	a	a
2. Then dividing 48 (or *p*) by *a*, the Quotient will give the other number, to wit, . .	$\frac{48}{a}$	$\frac{p}{a}$
3. From the firſt ſtep, the Square of one of the Numbers is	aa	aa
4. And from the ſecond ſtep the Square of the other Number is	$\frac{2304}{aa}$	$\frac{pp}{aa}$
5. Therefore the Sum of the ſaid Squares is .	$\frac{aaaa+2304}{aa}$	$\frac{aaaa+pp}{aa}$

6. Which Sum muſt be equal to the given Sum of the Squares, whence this Equation ariſes, *viz.*

$$\frac{aaaa+2304}{aa} = 100,$$

Or, $$\frac{aaaa+pp}{aa} = c$$

7. From which Equation, after due Reduction by the Rules in *Chap.* 12, this will ariſe,

$$2304 = 100aa - aaaa,$$

$$pp = caa - aaaa.$$

8. Which laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. the two values of *a*, which are the Numbers ſought, will be diſcovered, *viz.*

$$a = \begin{cases} 8 = \sqrt{(2)}:\overline{\tfrac{1}{2}c+\sqrt{\tfrac{1}{4}cc-pp}}: \\ 6 = \sqrt{(2)}:\overline{\tfrac{1}{2}c-\sqrt{\tfrac{1}{4}cc-pp}}: \end{cases}$$

9. I ſay, 8 and 6 are the Numbers required; for the Product of their Multiplication is 48, and the Sum of their Squares 64 and 36 is 100, as was preſcribed. From the laſt ſtep alſo ariſes this

CANON.

From the Square of half the given Sum of the Squares of the two numbers ſought ſubtract the Square of the given Product of their Multiplication, and extract the ſquare Root of the Remainder, then to half the ſaid Sum add the ſaid Square Root, and from the ſaid half Sum ſubtract the ſaid Square Root; laſtly, extract the Square Root of the Sum of that Addition, and alſo of the Remainder of the latter Subtraction, ſo ſhall theſe two Square Roots be the numbers ſought by the Queſtion propos'd.

QUEST. 13.

There are two Numbers whoſe Sum is 14 (or *b*,) and if the Sum of their Squares be multiplied by the Sum of their Cubes, the Product is 72800 (or *c*;) what are the Numbers?

RESOLUTION.

1. For one of the Numbers ſought put . . .	$a+7$	$a+\frac{1}{2}b$
2. Then, that their Sum may be 14 (or *b*,) the other number muſt be	$-a+7$	$-a+\frac{1}{2}b$
3. The Square of the firſt Number is . . .	$aa+14a+49$	$aa+ba+\frac{1}{4}bb$
4. The Square of the latter Number is . . .	$aa-14a+49$	$aa-ba+\frac{1}{4}bb$
5. Therefore the Sum of their Squares is . . .	$2aa+98$	$2aa+\frac{1}{2}bb$

6. Again, the Cube of the firſt Number will be

$$aaa+21aa+147a+343,$$

Or, $$aaa+\tfrac{3}{2}baa+\tfrac{3}{4}bba+\tfrac{1}{8}bbb.$$

7. And the Cube of the latter Number will be

$$-aaa+21aa-147a+343,$$

Or, $$-aaa+\tfrac{3}{2}baa-\tfrac{3}{4}bba+\tfrac{1}{8}bbb.$$

8. Therefore the Sum of the Cubes in the two laſt ſteps is

$$42aa+686, \quad \text{Or,} \quad 3baa+\tfrac{1}{4}bbb.$$

9. Which Sum of the Cubes in the laſt ſtep being Multiplied by the Sum of the Squares in the fifth ſtep, produces $84aaaa+5488aa+67228$,

Or, $$6baaaa+2bbbaa+\tfrac{1}{8}bbbbb.$$

10. Which Product in the laſt ſtep muſt be equal to 72800 (or *c*) the Product given in the Queſtion, whence this Equation ariſes, *viz.*

$$84aaaa+5488aa+67228=72800,$$

Or, $$6baaaa+2bbbaa+\tfrac{1}{8}bbbb=c$$

11. And from that Equatien, after due Reduction according to the Rules of the twelfth *Chapter*, this will ariſe; $aaaa+\frac{196}{3}aa=\frac{199}{3}$,

Or, $$aaaa+\tfrac{1}{3}bbaa=\frac{c}{6b}-\tfrac{1}{48}bbbb.$$

12. Which Equation being reſolved by the Canon in *Sect* 6. of *Chap.* 15. the value of *a* will be diſcovered, *viz.*

$$a=1=\sqrt{(2)}:\overline{\sqrt{\overline{\frac{c}{6b}+\tfrac{1}{144}bbbb.}}-\tfrac{1}{6}bb:}$$

13. Therefore from the twelfth, firſt and ſecond ſteps the two numbers ſought are made known:

$$7+1=8=\tfrac{1}{2}b+\sqrt{(2)}:\overline{\sqrt{\overline{\frac{c}{6b}+\tfrac{1}{144}bbbb.}}-\tfrac{1}{6}bb:}$$

$$7-1=6=\tfrac{1}{2}b-\sqrt{(2)}:\overline{\sqrt{\overline{\frac{c}{6b}+\tfrac{1}{144}bbbb.}}-\tfrac{1}{6}bb:}$$

I ſay the numbers ſought are 8 and 6; for their Sum is 14, and if 100 the Sum of their Squares be multiplied by 728, the Sum of their Cubes, the Product will be 72800, as was preſcribed.

Moreover, the thirteenth ſtep gives a Canon to find out the Numbers ſought.

CANON.

Divide the given Product by ſix times the given Sum; then to the Quotient add $\frac{1}{144}$ of the Biquadrate of the given Sum, and extract the Square Root of the Sum of that addition; then from the ſaid Square Root ſubtract $\frac{1}{6}$ of the Square of the given Sum, and extract the Square Root of the Remainder; laſtly, add this Square Root to half the given Sum and Subtract it from the ſaid half Sum, ſo ſhall the Sum and Remainder be the two numbers ſought.

QUEST. 14.

There are two numbers the Product of whoſe Multiplication is 20 (or b,) and the ſum of their Cubes is 189 (or c;) what are the numbers:

RESOLUTION.

1. For one of the Numbers ſought put . . .	a	a
2. Then, by dividing the given Product 20 (or b) by a, the other Number will be :	$\frac{20}{a}$	$\frac{b}{a}$
3. Therefore from the firſt ſtep, the Cube of the firſt Number is	aaa	aaa
4. And from the ſecond ſtep the Cube of the other number is	$\frac{8000}{aaa}$	$\frac{bbb}{aaa}$
5. Therefore the Sum of the ſaid Cubes is .	$\frac{aaaaaa+8000}{aaa}$	$\frac{aaaaaa+bbb}{aaa}$

6. Which ſum muſt be equal to 189 (or c) the Sum given in the Queſtion, whence this Equation ariſes, *viz.*

$$\frac{aaaaaa+8000}{aaa} = 189,$$

Or, $$\frac{aaaaaa+bbb}{aaa} = c.$$

7. Which Equation being reduced according to *Sect.* 2, 3, and 5. of *Chap.* 12. there will ariſe $8000 = 189aaa - aaaaaa$,

Or, $bbb = caaa - aaaaaa$.

8. And by reſolving the Equation in the laſt ſtep by the Canon in *Sect.* 10. *Chap.* 15. the two values of a, which are the numbers ſought by this Queſtion, will be made known, *viz.*

$$a = \begin{cases} 5 = \sqrt{(3)}:\overline{\tfrac{1}{2}c+\sqrt{\overline{\tfrac{1}{4}cc - bbb}}}: \\ 4 = \sqrt{(3)}:\tfrac{1}{2}c-\sqrt{\overline{\tfrac{1}{4}cc - bbb}}: \end{cases}$$

9. I ſay, the numbers ſought are 5 and 4; for the Product of their Multiplication is 20, and the Sum of their Cubes 125 and 64 is 189, as was preſcribed.

Moreover, from the two values of a expreſt by Letters in the eighth ſtep, the following Canon ariſes to find out the number ſought.

CANON.

From the Square of half the given Sum ſubtract the Cube of the given Product, and extract the Square Root of the Remainder; then add the ſaid ſquare Root to half the given Sum, and alſo ſubtract it from the ſaid half Sum; laſtly, extract the Cubic Root of the Sum of that Addition, and likewiſe extract the Cubic Root of the latter Remainder, ſo ſhall theſe Cubic Roots be the Numbers ſought.

QUEST. 15.

There are two numbers the Product of whoſe Multiplication is 20 (or b,) and the difference of their Cubes is 61 (or d;) what are the numbers?

RESOLUTION.

1. For the greater of the two Numbers ſought put	a	a
2. Then, by dividing the given Product 20 (or b) by a, the leſſer number will be . .	$\frac{20}{a}$	$\frac{b}{a}$
3. Therefore from the firſt ſtep the Cube of the greater number is	aaa	aaa
4. And from the ſecond ſtep the Cube of the leſſer number is	$\frac{8000}{aaa}$	$\frac{bbb}{aaa}$
5. Therefore from the two laſt ſteps, the difference of the Cubes of the two Numbers ſought is	$\frac{aaaaaa-8000}{aaa}$	$\frac{aaaaaa-bbb}{aaa}$

6. Which difference muſt be equal to 61 (or d) the difference given in the Queſtion, whence this Equation ariſes, *viz.*

$$\frac{aaaaaa-8000}{aaa} = 61, \quad \text{Or,} \quad \frac{aaaaaa-bbb}{aaa} = d.$$

7. Which Equation, after due Reduction, (according to *Sect.* 2, 3, and 5. of *Chap.* 12.) will give this that follows, *viz.*
$$aaaaaa - 61aaa = 8000,$$
Or, $$aaaaaa - daaa = bbb.$$

8. Therefore by reſolving the Equation in the laſt ſtep by the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the greater Number ſought will be made known, *viz.*
$$a = 5 = \sqrt{(3)}:\tfrac{1}{2}d+\sqrt{\tfrac{1}{4}dd+bbb}:$$

9. Whence the greater Number ſought is found 5, by which if the given Product 20 be divided, the Quotient will give 4 for the leſſer number required.

I ſay the Numbers 5 and 4 will ſolve the Queſtion propoſed; for the Product of their Multiplication is 20, and the difference of their Cubes 125 and 64 is 61, as was preſcribed.

Moreover, the Equation in the eighth ſtep gives a Canon to find out the greater of the two numbers ſought, by the help whereof and the given Product the leſſer number is alſo given.

CANON.

To the Square of half the given difference add the Cube of the given Product, and extract the Square Root of the Sum of that Addition; then add the ſaid Square Root to half the given difference and extract the Cubic Root of this Sum, ſo ſhall the ſaid Cubic Root be the greater of the two numbers ſought; by which greater number if the given Product be divided the Quotient ſhall be the leſſer number ſought.

QUEST. 16.

A Merchant having bought certain Cloths, ſells them at $17\frac{1}{4}$ *l.* (or b) the Cloth, and then found that by every 100 *l.* (or c) that he had laid out, he gained as many Pounds as he paid for one Cloth; what was the firſt coſt of a Cloth?

RESOLUTION.

1. For the firſt coſt of one Cloth put . .	a	a
2. Which firſt coſt being ſubtracted from the money for which the Merchant ſold one Cloth, there will remain the gain of one Cloth, to wit, ...	$17\frac{1}{4} - a$	$b - a$

3. Then find what was gained in laying out 100 *l.* (or c,) *viz.* ſay by the Rule of Three,
$$\text{If } a \;.\; 17\tfrac{1}{4} - a \;::\; 100 \;.\; \frac{1725 - 100a}{a},$$
$$\text{Or, } a \;.\; b - a \;::\; c \;.\; \frac{cb - ca}{a}.$$
Whence the gain of 100 *l.* is found $\frac{1725 - 100a}{a}$, or $\frac{cb - ca}{a}$.

4. But according to the Queſtion the gain of 100 *l.* (or c) muſt be equal to the firſt coſt of one Cloth, therefore from the firſt and third ſteps this Equation ariſes, *viz.*
$$a = \frac{1725 - 100a}{a}, \quad \text{Or,} \quad a = \frac{cb - ca}{a}.$$

5. Which Equation, after due Reduction (according to *Sect.* 2, and 3. of *Chap.* 12.) will give this that follows, *viz.*
$$aa + 100a = 1725,$$
Or, $$aa + ca = cb.$$

6. Therefore by reſolving the Equation in the laſt ſtep by the Canon in *Sect.* 6. *Chap.* 15. the value of a; to wit, the firſt coſt of a Cloth will be diſcovered, *viz.*
$$a = 15 = \sqrt{:cb+\tfrac{1}{4}cc:} - \tfrac{1}{2}c.$$

I ſay the firſt coſt of a Cloth was 15 *l.* as will appear by the Proof: For if a Cloth be bought for 15 *l.* and ſold for $17\frac{1}{4}$ *l.* the gain is $2\frac{1}{4}$ *l.* Then if 15 *l.* gain $2\frac{1}{4}$ *l.* it will follow that 100 *l.* will gain 15 *l.* which is equal to the firſt coſt of a Cloth; as was preſcribed.

Another

Another way of reſolving the preceding Queſt. 16.

1. Let the ſame things be given as before, then for the gain of one Cloth put . .	a	a
2. Which gain being Subtracted from the Money for which one Cloth was ſold, will leave the firſt coſt of a Cloth, to wit, . .	$17\frac{1}{4} - a$	$b - a$

3. Then find what was gained in laying out 100 *l.* (or *c*,) and ſay by the Rule of Three,

$$\text{If} \quad 17\tfrac{1}{4} - a \;.\; a :: 100 \;.\; \frac{100a}{17\frac{1}{4} - a},$$

$$\text{Or,} \quad b - a \;.\; a :: c \;.\; \frac{ca}{b - a}.$$

Whence the gain of 100 *l.* is found $\frac{100a}{17\frac{1}{4} - a}$ or $\frac{ca}{b - a}$.

4. But, according to the Queſtion, the gain of a 100 *l.* (or *c*) muſt be equal to the firſt coſt of one Cloth; therefore from the ſecond and third ſteps this Equation ariſes, *viz.*

$$\frac{100a}{17\frac{1}{4} - a} = 17\tfrac{1}{4} - a, \qquad \text{Or,} \quad \frac{ca}{b - a} = b - a.$$

5. Which Equation, after due Reduction according to *Sect.* 2, and 3. of *Chap.* 12. will give this that follows, *viz.*

$$\tfrac{269}{2}a - aa = \tfrac{4761}{16}, \qquad \text{Or,} \quad ca + 2ba - aa = bb.$$

6. Therefore by reſolving the Equation in the laſt ſtep by the Canon in *Sect* 10 *Chap.* 15. the two values of *a*, or the two Roots of that Equation will be made known, *viz.*

$$a = \begin{cases} \frac{529}{4} = \frac{1}{2}c + b + \sqrt{:\frac{1}{4}cc + cb:} \\ \frac{9}{4} = \frac{1}{2}c + b - \sqrt{:\frac{1}{4}cc + cb:} \end{cases}$$

The leſſer of which two Roots or Numbers, to wit, $\frac{9}{4}$ or $2\frac{1}{4}$ *l.* is the gain of a Cloth, which ſubtracted from $17\frac{1}{4}$ *l.* leaves 15 *l.* for the firſt coſt of a Cloth, as before.

Note. Although the value of *a* in the Equation in the fifth ſtep may be either $\frac{529}{4}$ or $\frac{9}{4}$, (for that Equation may be expounded by $\frac{529}{4}$ as well as $\frac{9}{4}$,) yet $\frac{9}{4}$ only, to wit, the leſſer value of *a* ſhall be the gain of a Cloth; for $\frac{529}{4}$ is greater than $17\frac{1}{4}$, and conſequently the gain of one Cloth would exceed the Money for which one Cloth was ſold. Which abſurdity appears alſo by the greater value of *a* as 'tis expreſt by Letters in the ſixth ſtep, for $\frac{1}{2}c + b + \sqrt{:\frac{1}{4}cc + cb:}$ is manifeſtly greater than *b*.

QUEST. 17.

Each of two Captains, whereof one had a leſſer number of Soldiers in his Company by 40 (or *b*) than the other, diſtributed equally among the Soldiers of his own Company 1200 (or *c*) Crowns, whereby it happened that the Soldiers of the leſſer Company had 5 (or *d*) Crowns a piece more than the Soldiers of the greater Company; the Queſtion is to find the number of Soldiers in each Company, and how many Crowns each Soldier received.

RESOLUTION.

1. For the number of Soldiers in the leſſer Company put	a	a
2. To which adding 40 (or *b*) the ſum will give the number of Soldiers in the greater Company, to wit, ...	$a+40$	$a+b$
3. Then if 1200 (or *c*.) Crowns be equally divided among the Soldiers of the leſſer Company, the Quotient or ſhare of every Soldier will be	$\frac{1200}{a}$	$\frac{c}{a}$
4. Likewiſe, if 1200 (or *c*) Crowns be equally divided among the Soldiers of the greater Company, the Quotient or ſhare of every Soldier will be	$\frac{1200}{a+40}$	$\frac{c}{a+b}$
5. To which latter Quotient adding 5 (or *d*) Crowns, the ſum is	$\frac{5a+1400}{a+40}$	$\frac{da+db+c}{a+b}$

6. But according to the Queſtion the Sum in the laſt ſtep muſt be equal to the Quotient in the third ſtep, whence this Equation ariſes, *viz.*

$$\frac{5a+1400}{a+40} = \frac{1200}{a}, \quad \text{Or,} \quad \frac{da+db+c}{a+b} = \frac{c}{a}.$$

7. From which Equation after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. this will ariſe, *viz.* $aa+40a = 9600$,

Or, $$aa + ba = \frac{bc}{d}.$$

8. Therefore the Equation in the laſt ſtep being reſolved by the Canon in *Sect.* 6. *Chap.* 15. the value of a, to wit, the number of Soldiers in the leſſer Company will be diſcovered, *viz.*

$$a = 80 = \sqrt{:\frac{bc}{d} + \frac{bb}{4}:} - \tfrac{1}{2}b.$$

From the eighth, firſt, and ſecond ſteps it is evident that the leſſer Company conſiſted of 80, and the greater 120 Soldiers; which numbers will ſatisfie the Conditions in the Queſtion. For the difference of the two Companies is 40 Soldiers; alſo $\frac{1200}{80} = 15$, and $\frac{1200}{120} = 10$; whence it is manifeſt that the Soldiers of the leſſer Company received 15 Crowns a piece, the Soldiers of the greater Company 10 Crowns a piece, and conſequently the Soldiers of the leſſer Company had 5 Crowns a piece more than the Soldiers of the greater Company, as was preſcribed.

QUEST. 18.

Two Merchants ſell Linnen-Cloth in this manner, *viz.* each ſells 60 (or b) Ells, and the firſt Merchant ſelling 2 (or c) Ells leſs for one pound than the ſecond, receives for his 60 Ells 5 (or d) pounds more than the ſecond Merchant for his 60 Ells. The Queſtion is to find how many Ells each Merchant ſold for 1 Pound?

RESOLUTION.

1. For the number of Ells which the firſt Merchant ſold for 1 *l.* put	a	a
2. To which number of Ells adding 2 (or c,) the Sum will be the number of Ells which the latter Merchant ſold for 1 *l.* to wit, . .	$a+2$	$a+c$
3. Then find how much Money the firſt Merchant received for his 60 Ells, *viz.* ſay by the Rule of Three, If $a \,.\, 1 :: 60 \,.\, \frac{60}{a}$; Or, $a \,.\, 1 :: b \,.\, \frac{b}{a}$. whence the firſt Merchants total Mony is found	$\frac{60}{a}$	$\frac{b}{a}$
4. Find likewiſe how much Money the latter Merchant received for his 60 Ells, *viz.* ſay, If $a+2 \,.\, 1 :: 60 \,.\, \frac{60}{a+2}$; Or, $a+c \,.\, 1 :: b \,.\, \frac{b}{a+c}$ whence the latter Merchants total Money is found	$\frac{60}{a+2}$.	$\frac{b}{a+c}$
5. To which latter Sum of Money adding 5 (or d) pounds, the Sum will be . .	$\frac{5a+70}{a+2}$	$\frac{da+dc+b}{a+c}$

6. But according to the Queſtion the Sum of Money it in the laſt ſtep muſt be equal to the Sum in the third ſtep, whence this Equation ariſes, *viz.*

$$\frac{5a+70}{a+2} = \frac{60}{a}, \quad \text{Or,} \quad \frac{da+dc+b}{a+c} = \frac{b}{a}:$$

7. Which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. will give this that follows, *viz.* $aa + 2a = 24,$

Or, $aa + ca = \frac{bc}{d}.$

8. Which Equation in the laſt ſtep being reſolved by the Canon in *Sect. 6. Chap.* 15. the value of a, to wit the Number of Ells which the firſt Merchant ſold will be made known, *viz.* $a = 4 = \sqrt{:\frac{bc}{d}+\frac{cc}{4}:} - \frac{1}{2}c.$

I ſay the firſt Merchant ſold 4 Ells for 1 Pound, and the ſecond 6 Ells for 1 Pound, as will appear by the Proof. For if 4 Ells give 1 Pound, then 60 Ells will give 15 Pounds Again, if 6 Ells give one Pound, then 60 Ells will give 10 Pounds. Whence it is manifeſt that the firſt Merchant ſold his 60 Ells for 5 Pounds more than the ſecond ſold his 60 Ells, and ſold 2 Ells leſs for 1 pound than the ſecond Merchant ſold for one Pound.

QUEST. 19.

Two Societies, whereof one exceeds the other by 4 (or b) men, divide two equal ſums of Crowns; the Men of the leſſer Society have 8 (or c) Crowns a piece more than thoſe of the greater: And the number of Crowns which each Society receives exceeds the number of Men of both Societies by 172 (or d.) The Queſtion is, to find the number of Men in each Society, and the number of Crowns which each Society had?

RESOLUTION.

1. For the number of Men of the leſſer Society put	a	a
2. To which number adding 4 (or b,) the ſum will be the Number of Men of the greater Society, to wit, ...	$a+4$	$a+b$
3. Then, according to the Queſtion, if 172 (or d) be added to the Sum of the Men of both Societies, it will give the number of Crowns ſhared by each Society, to wit, . .	$2a+176$	$2a+b+d$
4. Which number of Crowns being divided by (a) the number of Men of the leſſer Society, the Quotient or ſhare of every Man in that Society will be	$\frac{2a+176}{a}$	$\frac{2a+b+d}{a}$
5. Likewiſe if the ſame number of Crowns before expreſt in the third ſtep be divided by $a+4$, (or $a+b$, the number of Men of the greater Society,) the Quotient will give the ſhare of every man in this Society to wit,	$\frac{2a+176}{a+4}$	$\frac{2a+b+d}{a+b}$
6. To which Quotient in the laſt ſtep adding 8 (or c) the Sum will be	$\frac{10a+208}{a+4}$	$\frac{2+ab+d+ca+cb}{a+b}$

7. But, according to the Queſtion, the ſum in the laſt ſtep muſt be equal to the Quotient in the fourth ſtep, whence this Equation ariſes, *viz.*

$$\frac{10a+208}{a+4} = \frac{2a+176}{a}, \quad \text{Or,} \quad \frac{2a+b+d+ca+cb}{a+b} = \frac{2a+b+d}{a}$$

8. From which Equation, after due Reduction according to *Sect.* 2, 3, and 5. of *Chap.* 12. this Equtſion will ariſe, *viz.* $aa + 3a = 88,$

Or, $aa + \frac{cb-2b}{c}a = \frac{bb+bd}{c}.$

9. Therefore by reſolving the laſt Equation according to the Canon in *Sect. 6. Chap.* 15. the value of a, to wit, the number of Men in the leſſer Society will be diſcover'd, *viz.*

$$a = 8 = \sqrt{:\frac{cbd+\frac{1}{4}ccbb+bb}{cc}:} - \frac{b}{2} + \frac{b}{c}.$$

10. Laſtly, from the ninth, firſt, ſecond, and third ſteps, it is manifeſt that the number of men in the leſſer Society was 8, that of the greater 12, and the number of Crowns divided by each Society 192; which numbers will ſatisfie the Conditions in the

Queſtion as will appear by the Proof: For $\frac{192}{8} = 24$, and $\frac{192}{12} = 16$; whence it is evident that the Men of the leſſer Society had 8 Crowns a piece more than thoſe of the greater; alſo 192, the number of Crowns which each Society divided, exceeded 20 the number of Men in both Societies by 172, and 12 the number of Men in the greater Society exceeded 8 the number of Men in the leſſer by 4; as was preſcribed.

QUEST. 20.

A Graſier having bought certain Oxen for 270 (or b) Pounds, finds, that if he had paid that ſum for 5 (or c) Oxen fewer, every Ox would have coſt him $\frac{3}{4}$ *l.* (or d) more than he paid for an Ox: What was the number of Oxen bought?

RESOLUTION.

1. For the number of Oxen bought put . .	a	a
2. Then find out the coſt of an Ox, and ſay, If $a \;.\; 270 :: 1 \;.\; \frac{270}{a}$; Or, $a \;.\; b :: 1 \;.\; \frac{b}{a}$. whence the price of an Ox is : . . .	$\frac{270}{a}$	$\frac{b}{a}$
3. Subtract 5 (or c) from the number of Oxen bought, and then find what the reſt would coſt a piece, ſaying, If $a - 5 \;.\; 270 :: 1 \;.\; \frac{270}{a-5}$ Or, $a - c \;.\; b :: 1 \;.\; \frac{b}{a-c}$. Whence the price of an Ox is found . .	$\frac{270}{a-5}$	$\frac{b}{a-c}$

4. Then according to the Queſtion, the laſt mentioned price of an Ox muſt exceed that in the ſecond ſtep by $\frac{3}{4}$ *l.* (or d;) therefore if the former price be ſubtracted from the latter, the remainder muſt be equal to $\frac{3}{4}$ or d;) whence this Equation ariſes, *viz.*

$$\frac{270}{a-5} - \frac{270}{a} = \frac{3}{4}; \qquad \text{Or,} \quad \frac{b}{a-c} - \frac{b}{a} = d.$$

5. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will give this that follows,

$$aa - 5a = 1800,$$

$$\text{Or,} \quad aa - ca = \frac{bc}{d}.$$

6. Therefore the Equation in the laſt ſtep being reſolved by the Canon in *Sect.* 12. *Chap.* 15. the value of a, to wit the Number of Oxen bought will be diſcovered, *viz.*

$$a = 45 = \sqrt{:\frac{bc}{d} + \frac{cc}{4}:} + \tfrac{1}{2}c.$$

I ſay the Number of Oxen bought was 45, and every Ox coſt 6 Pounds, as will appear by the Proof: For firſt, $\frac{270}{45} = 6$; then from 45 Oxen ſubtracting 5, the remaining 40 Oxen valued at 270 *l.* will yield $6\frac{3}{4}$ *l.* a piece, which exceeds the former price 6 *l.* by $\frac{3}{4}$ *l.* as was preſcribed.

QUEST. 21.

A Merchant buyes linnen Clothes of two ſorts, *viz.* 90 (or b) Ells of one ſort, together with 40 (or c) Ells of a worſer ſort for 42 (or d) Pounds; and he finds that in laying out 1 Pound upon each ſort he has $\frac{1}{3}$ (or m) of an Ell more of the worſer ſort than the other: What was the price of an Ell of each ſort.

RESOLUTION.

1. For the Number of Ells of the better ſort of Cloth which the Merchant bought for 1 *l.* put	a	a
2. Then according to the Queſt. the number of Ells of the worſer ſort bought for 1 *l.* will be	$a + \frac{1}{3}$	$a + m$

3. Find

3. Find the coſt of all the Ells of the worſer ſort, and ſay, If $a+\frac{1}{3}$. 1 :: 40 : $\frac{40}{a+\frac{1}{3}}$; Or, $a+m$. 1 :: c : $\frac{c}{a+m}$. whence the ſaid full Coſt is found . . .	$\frac{40}{a+\frac{1}{3}}$	$\frac{c}{a+m}$
4. Find likewiſe the coſt of all the Ells of the better ſort, and ſay, If a . 1 :: 90 : $\frac{90}{a}$; Or, a . 1 :: b : $\frac{b}{a}$. whence the ſaid full Coſt is	$\frac{90}{a}$	$\frac{b}{a}$
5. Then the two ſums of Money found out in the third and fourth ſteps being added together will give the full coſt of both ſorts of Cloth, to wit, ...	$\frac{130a+30}{aa+\frac{1}{3}a}$	$\frac{ca+ba+bm}{aa+ma}$

6. Which total Coſt expreſs'd in the laſt ſtep, muſt (according to the Queſtion) be equal to 42 (or d;) whence this Equation ariſes, *viz.*

$$42 = \frac{130a+30}{aa+\frac{1}{3}a}; \quad \text{Or,} \quad d = \frac{ca+ba+bm}{aa+ma}.$$

7. Which Equation, after due Reduction (according to the Rules in *Chap.* 12.) will give this that follows, *viz.* $aa-\frac{58}{21}a = \frac{5}{7}$,

$$\text{Or,} \quad aa-\frac{c+b-dm}{d}a = \frac{mb}{d}.$$

In which laſt Equation, if inſtead of the known Co-efficient $\frac{c+b-dm}{d}$ we take f, that Equation may be expreſs'd thus;

$$aa-fa = \frac{mb}{d}.$$

8. Therefore by reſolving the laſt Equation according to the Canon in *Sect.* 8. *Chap.* 15. the value of a, to wit, the number of Ells of the better ſort of Cloth which were bought for 1 *l.* will be diſcovered, *viz.*

$$a = 3 = \sqrt{:\frac{mb}{d}+\frac{ff}{4}:} + \tfrac{1}{2}f.$$

Thus it is found that 3 Ells of the better ſort of Cloth did coſt 1 *l.* and conſequently 1 Ell coſt $\frac{1}{3}$ *l.* and 90 Ells 30 *l.* which ſubtracted from 42 *l.* (the full coſt of both ſorts,) leaves 12 *l.* for the full coſt of 40 Ells of the worſt ſort; and conſequently 1 Ell coſt $\frac{3}{10}$ *l*, and at this rate 1 *l.* will buy $3\frac{1}{3}$ Ells, which is more by $\frac{1}{3}$ of an Ell than was bought of the better ſort of Cloth for 1 *l.* Therefore all the Conditions in the Queſtion are ſatisfied.

QUEST. 22.

A Merchant having Spices, to wit, 80 ℔ weight (or b) of Mace, and 100 ℔ weight (or c) of Cloves, ſells both Quantities for 65 (or d) Pounds in Money; whereby it happened that he ſold a quantity of Mace for 10 *l.* (or m,) and the like quantity of Cloves with 60 ℔ weight (or n) more of Cloves for 20 *l.* (or r.) The Queſtion is, to find how many ℔ weight of Mace he ſold for 10 *l.*

RESOLUTION.

1. Let the number of ℔ weight of Mace that the Merchant ſold for 10 *l.* be repreſented by	a	a
2. To which number adding 60, the ſum will give the number of ℔ weight of Cloves that he ſold for 20 *l.* to wit, ...	$a+60$	$a+n$

3. Then

3. Then find how much Money 80 ℔ weight of Mace was ſold for, and ſay, If $a \;.\; 10 :: 80 \;.\; \frac{800}{a}$; Or, $a : m :: b \;.\; \frac{mb}{a}$. whence the Money for which the ſaid 80 ℔ of Mace was ſold is	$\frac{800}{a}$	$\frac{mb}{a}$
4. Find likewiſe how much Money 100 ℔ weight of Cloves was ſold for, and ſay, If $a+60 \;.\; 20 :: 100 \;.\; \frac{2000}{a+60}$; Or, $a+n : r :: c \;.\; \frac{rc}{a+n}$. whence the Money for which the ſaid 100 ℔ of Cloves was ſold is	$\frac{2000}{a+60}$	$\frac{rc}{a+n}$
5. The ſum of both the ſaid ſums of Money found out in the third and fourth ſteps is	$\frac{2800a+48000}{aa+60a}$	$\frac{mba+mbn+rca}{aa+na}$

6. Which Sum in the laſt ſtep muſt (according to the Queſtion) be equal to 65 *l.* (or *d*,) hence this Equation ariſes, *viz.*

$$65 = \frac{2800a+48000}{aa+60a}; \qquad \text{Or,} \quad d = \frac{mba+mbn+rca}{aa+na}.$$

7. Which Equation, after due Reduction (according to *Sect.* 2. 3, 5. *Chap.* 12.) will give this following Equation, *viz.* $aa+\frac{220}{13}a = \frac{9600}{13}$,

$$\text{Or,} \quad aa+\frac{dn-mb-rc}{d}a = \frac{mbn}{d}.$$

In which laſt Equation if we take f inſtead of the known Co-efficient $\frac{dn-mb-rc}{d}$, and g inſtead of the known number $\frac{mbn}{d}$, that Equation may be expreſs'd thus,

$$aa + fa = g.$$

8. Therefore by reſolving the laſt Equation according to the Canon in *Sect. 6. Chap.* 15. the value of a, to wit, the number of ℔ weight of Mace that was ſold for 10 *l.* will be made known, *viz.*

$$a = 20 = \sqrt{:g+\tfrac{1}{4}ff:} - \tfrac{1}{2}f.$$

Thus it is found that 20 ℔ weight of Mace was ſold for 10 *l.* and conſequently 80 ℔ weight for 40 *l.*

Moreover, adding 60 to 20 (before found,) the Sum 80 is the number of ℔ weight of Cloves that was ſold for 20 *l.* and conſequently 100 ℔ of Cloves was ſold for 25 *l.* which added to 40 *l.* (the price of 80 ℔ of Mace,) makes 65 *l.* the preſcribed ſum of Money for both Quantities of Spices ſold.

QUEST. 23.

Two Merchants entred into Partnerſhip; the firſt brought in a certain ſum of Pounds which continued in Company 12 (or b) Months, and the ſecond put in 20 *l.* (or c) for 17 (or d) Months; they gained together $18\frac{3}{4}$ *l.* (or m) whereof the firſt Merchant had 26 *l.* (or n) for his Principal and Gain. It is required to find how many Pounds the firſt Merchant brought into the common Stock?

RESOLUTION.

1. For the firſt Merchant's Stock put . . .	a	a
2. Which Stock being multiplied by the time it continued in Company, produces . . .	$12a$	ba
3. The ſecond Merchant's Stock being multiplied by the time it remained in Company, produces	510	cd

4. Then proceeding with thoſe two Products according to the Rule of Fellowſhip with Time, find the gain of the firſt Merchant, and ſay,

$$\text{If} \quad 12a+510 \;.\; 18\tfrac{3}{4} :: 12a \;.\; \frac{225a}{12a+510};$$

$$\text{Or,} \quad ba+cd \;.\; m :: ba \;.\; \frac{mba}{ba+cd}.$$

Whence the gain of the firſt Merchant is found $\frac{225a}{12a+510}$ or $\frac{mba}{ba+cd}$.

5. Which gain added to the firſt Merchants Stock a, gives for the Sum of his Stock and gain,

$$\frac{12aa+735a}{12a+510}; \qquad \text{Or,} \quad \frac{baa+cda+mba}{ba+cd}.$$

6. Which Sum muſt be equal to the 26 *l.* (or n) given in the Queſtion, whence this Equation ariſes, *viz.*

$$\frac{12aa+735a}{12a+510} = 26; \qquad \text{Or,} \quad \frac{baa+cda+mba}{ba+cd} = n.$$

7. Then by reducing that Equation according to the Rules in *Chap.* 12. there will ariſe,

$$aa + 35\tfrac{1}{4}a = 1105,$$

$$\text{Or,} \quad aa + \frac{cd+mb-nb}{b}a = \frac{ncd}{b}.$$

8. Which laſt Equation being reſolved by the Canon in *Sect.* 6. of the 15 *Chap.* the value of a, to wit, the firſt Merchant's Stock will be found 20 Pounds, *viz.* If inſtead of the known Co-efficient $\frac{cd+mb-nb}{b}$ we take f, and g inſtead of the given number $\frac{ncd}{b}$; Then by the ſaid Canon,

$$a = 20 = \sqrt{:g+\tfrac{1}{4}ff:} - \tfrac{1}{2}f.$$

Whence the firſt Merchants Stock is found 20 *l.* The Proof may be made by the Rule of Fellowſhip with Time, in manner following.

$$\begin{array}{l} 20 \times 12 = 240 \\ 30 \times 17 = \underline{510} \\ \qquad\qquad\;\; 750 \;.\; 18\tfrac{3}{4} :: \left\{ \begin{array}{l} 240 \;.\; 6 \\ 510 \;.\; 12\tfrac{3}{4}. \end{array} \right. \end{array}$$

QUEST. 24.

Two Merchants entred into Partnerſhip, the firſt put in a certain number of Pounds for 3 (or b) Months; the ſecond put in 50 *l.* (or c) more than the firſt for 5 (or d) Months: They gained together 140 *l.* (or m,) whereof the firſt Merchant had ſuch part, that if 60 *l.* (or n) be added to it, the Sum will be equal to the Stock wherewith he entred Partnerſhip: What was the Stock and gain of each Merchant?

RESOLUTION.

1. For the Stock of the firſt Merchant put . .	a	a
2. To which adding 50 *l.* (or c,) the Sum will give the ſecond Merchant's Stock, to wit,	$a+50$	$a+c$
3. Then multiplying the firſt Merchant's Stock by the time it remained in Company, the Product is	$3a$	ba
4. Likewiſe by multiplying the ſecond Merchant's Stock by the time it continued in Company, the Product is	$5a+250$	$da+dc$

5. Then proceeding with thoſe two Products according to the Rule of Fellowſhip with Time, find the firſt Merchant's Gain, and ſay,

$$\text{If} \quad 8a+250 \;.\; 140 :: 3a \;.\; \frac{420a}{8a+250};$$

$$\text{Or,} \quad ba+da+dc \;.\; m :: ba \;.\; \frac{mba}{ba+da+dc}.$$

Whence the gain of the firſt Merchant is found $\frac{420a}{8a+250}$; Or, $\frac{mba}{ba+da+dc}$.

6. To which gain add 60 (or n,) ſo the Sum will be

$$\frac{900a+15000}{8a+250}; \qquad \text{Or,} \qquad \frac{mba+nba+nda+ndc}{ba+da+dc}.$$

7. But, according to the Queſtion, the Sum in the laſt ſtep muſt be equal to (a) the firſt Merchant's Stock, whence this Equation ariſes;

$$\frac{900a+15000}{8a+250} = a = \frac{mba+nba+nda+ndc}{ba+da+dc}.$$

8. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will produce this following Equation, *viz.* $aa-81\frac{1}{4}a = 1875$,

$$\text{Or,} \qquad aa - \frac{mb+nb+nd-dc}{b+d}a = \frac{ndc}{b+d}.$$

9. In which Equation the value of a, to wit, the firſt Merchant's Stock, will be diſcovered by the Canon in *Sect.* 8, *Chap.* 15. viz. $a = 100\,l.$ And conſequently from the premiſes the ſecond Merchant's Stock was 150 *l.* the gain of the firſt 40 *l.* and the gain of the ſecond 100 *l.* All which will be evident by the following Proof wrought by the Rule of Fellowſhip with Time.

$$\begin{array}{l} 100 \times 3 = 300 \\ 150 \times 5 = \underline{750} \\ \qquad\quad 1050 \;.\; 140 :: \left\{ \begin{array}{l} 300 \;.\; 40 \\ 750 \;.\; 100. \end{array} \right. \end{array}$$

QUEST. 25.

A Citizen having bought a Houſe for a certain ſum of Pounds, ſells it for 64 *l.* (or d,) and finds that his loſs in 100 Pounds (or c) was equal to a fourth part (or m) of the Money that he paid for the Houſe. What number of Pounds did the Citizen pay for the Houſe?

RESOLUTION.

1. For the number of pounds which the Citizen paid for the Houſe, put	a	a
2. Then will the whole loſs by ſale of the Houſe be	$a-64$	$a-d$

3. Find how much was loſt by 100 *l.* (or c,) and ſay,

$$\text{If} \quad a \;.\; a-64 :: 100 \;.\; \frac{100a-6400}{a};$$

$$\text{Or,} \quad a \;.\; a-d :: c \;.\; \frac{ca-cd}{a}.$$

Whence the loſs *per Cent.* is found $\frac{100a-6400}{a}$; Or, $\frac{ca-cd}{a}$.

4. But according to the Queſtion the loſs *per Cent.* was equal to $\frac{1}{4}$ part of the Money which the Citizen paid for the Houſe, therefore from the firſt and third ſteps this Equation ariſes, *viz.*

$$\frac{100a-6400}{a} = \frac{a}{4}; \qquad \text{Or,} \qquad \frac{ca-cd}{a} = ma.$$

5. Which Equation, after due Reduction according to the Rules in *Chap.* 12. will give

$$400a-aa = 25600; \qquad \text{Or,} \qquad \frac{c}{m}a-aa = \frac{cd}{m}.$$

6. Therefore by reſolving the ſaid Equation according to the Canon in *Sect.* 10. *Chap.* 15. both the values of a will be diſcovered, either of which will ſolve the Queſtion; which values or numbers are theſe following, *viz.*

$$a = \left\{ \begin{array}{l} 320 = \frac{c}{2m} + \surd{:}\frac{cc-4cdm}{4mm}: \\ 80 = \frac{c}{2m} - \surd{:}\frac{cc-4cdm}{4mm}: \end{array} \right.$$

I ſay either of the numbers 320 and 80 will ſatisfie the Conditions in the Queſtion, as will be evident by the Proof: For if a Houſe coſt 320 *l.* and be ſold for 64 *l.* the loſs is 256 *l.* and 100 *l.* at that rate of loſs will loſe 80, which is $\frac{1}{4}$ part of the firſt Coſt 320 *l.*

Again, if a Houſe coſt 80 *l.* and be ſold for 64 *l.* the loſs is 16 *l.* and 100 *l.* at this rate of loſs will loſe 20 *l.* which is likewiſe $\frac{1}{4}$ part of the firſt Coſt 80 *l.*

QUEST. 26.

Two Merchants entred into Partnerſhip; the Sum of their Stocks was 165 (or *b*) Pounds: the firſt Merchant's Stock continued in Company 12 (or *c*) Months, and the Stock of the ſecond 8 (or *d*) Months: they gained a certain ſum of Pounds, which together with their Stocks they divided between themſelves in ſuch manner, that the firſt Merchant received 67 (or *f*) Pounds for his Stock and Gain, and the ſecond 126 (or *g*) Pounds for his Stock and Gain. It is deſired to find out each Merchant's Stock and Gain.

RESOLUTION.

1. For the firſt Merchant's Stock put . . .	a	a
2. Then, by ſubtracting that Stock (a) from 165 (or b,) there remains the ſecond Merchant's Stock; to wit, ...	$165-a$	$b-a$
3. And if you ſubtract (a) the firſt Merchant's Stock from 67 (or f) the ſum of his Stock and Gain, there will remain his Gain only; to wit, ...	$67-a$	$f-a$
4. Likewiſe, if you ſubtract the ſecond Merchant's Stock (in the ſecond ſtep) from 126 (or g) the Sum of his Stock and Gain, there will remain his Gain only; to wit,	$a-39$	$a+g-b$

5. Now according to the Nature of the Rule of Fellowſhip with Time, the Gain of the firſt Merchant $67-a$ muſt be in ſuch proportion to $a-39$ the Gain of the ſecond, as the Product of the firſt Merchant's Stock a multiplied by its time 12 Months, is to the Product of the ſecond Merchant's Stock $165-a$ multiplied by its time 8 Months: Hence this Analogy, *viz.*

$$67-a \;.\; a-39 \;::\; 12a \;.\; 1320-8a,$$

That is, $$f-a \;.\; a+g-b \;::\; ca \;.\; db-da.$$

6. Which Analogy, by comparing the Product made by the Multiplication of the Means one into the other, to the Product of the Extremes, produces this Equation, *viz.*

$$12aa-468a = 8aa-1856a+88440,$$

That is, $$caa+cga-cba = daa-dba-dfa+dbf.$$

7. From which Equation after due Reduction this ariſes, *viz.*

$$aa+347a = 22110,$$

That is, $$aa+\frac{db+df+cg-cb}{c-d}a = \frac{dbf}{c-d}.$$

8. Wherefore by reſolving the laſt Equation according to the Canon in *Sect. 6. Chap.* 15. the value of *a*, that is, the number of Pounds expreſſing the firſt Merchant's Stock will be found 55; which ſubtracted from 165 *l.* the ſum of both their Stocks, leaves 110 *l.* for the ſecond Merchant's Stock: then each of their Stocks being ſubtracted from their reſpective Stock and Gain, *viz.* 55 *l.* from 67 *l.* and 110 *l.* from 126 *l.* there remains 12 *l.* for the Gain of the firſt Merchant, and 16 *l.* for the Gain of the ſecond; whence the total Gain was 28 *l.* Which numbers will ſolve the Queſtion, as may eaſily be proved by the Rule of Fellowſhip with Time; thus,

$$\begin{array}{r} 55\times 12 = 660 \\ 110\times 8 = 880 \\ \hline 1540 \end{array} \;.\; 28 \;::\; \left\{\begin{array}{l} 660 \;.\; 12 \\ 880 \;.\; 16. \end{array}\right.$$

QUEST. 27.

A certain Foot-man *A* departs from *London* towards *Lincoln*, and at the ſame time another Foot-man *B* departs from *Lincoln* toward *London*, each keeping the ſame Road. When they met, *A* ſays to *B*, I find that I have travelled 20 (or *c*) miles more than you, and have gone as many miles in $6\frac{2}{3}$ (or *d*) days, as you have gone

Miles in all hitherto: 'Tis true ſaith *B*, I am not ſo good a Foot man as you, but I find that at the end of 15 (or *f*) days hence, I ſhall be at *London*, if I travel as many Miles in every one of thoſe 15 days, as I have done in every day hitherto. The Queſtion is, to find how many Miles thoſe two Cities are diſtant one from another, and how many Miles each Foot-man had travelled when they met one another.

RESOLUTION.

1. For the deſired diſtance between the two Cities put	a	a
2. Then foraſmuch as the number of Miles each Foot-man had travelled when they met, being added together make the Sum (a,) and the difference between thoſe two numbers was 20 (or c,) for *A* had travelled 20 Miles more than *B*: Therefore (by the Theorem at the end of *Queſt.* 1. *Chap.* 14.) the number of Miles which *A* had travelled was	$\frac{1}{2}a+10$	$\frac{1}{2}a+\frac{1}{2}c$
3. And (by the ſame Theorem) the number of Miles which *B* had travelled was . .	$\frac{1}{2}a-10$	$\frac{1}{2}a-\frac{1}{2}c$
4. Then ſay, If in $6\frac{2}{3}$ days *A* had travelled $\frac{1}{2}a-10$ Miles, how many Miles did he travel in one day? ſo by the Rule of Three, you will find	$\frac{\frac{1}{2}a-10}{6\frac{2}{3}}$	$\frac{\frac{1}{2}a-\frac{1}{2}c}{d}$
5. Say again, If in 15 days *B* muſt travel $\frac{1}{2}a+10$ Miles, (that is, all the Miles which *A* had travelled,) how many Miles muſt *B* travel in one day? ſo you will find . .	$\frac{\frac{1}{2}a+10}{15}$	$\frac{\frac{1}{2}a+\frac{1}{2}c}{f}$
6. Say again, If $\frac{\frac{1}{2}a+10}{15}$ Miles were travelled by *B* in one day, in how many days did he travel $\frac{1}{2}a-10$ Miles? ſo you will find	$\frac{7\frac{1}{2}a-150}{\frac{1}{2}a+10}$	$\frac{\frac{1}{2}fa-\frac{1}{2}fc}{\frac{1}{2}a+\frac{1}{2}c}$
7. Say again, If $\frac{\frac{1}{2}a-10}{6\frac{2}{3}}$ Miles were travelled by *A* in one day, in how many days did he travel $\frac{1}{2}a+10$ Miles? ſo you will find	$\frac{3\frac{1}{3}a+66\frac{2}{3}}{\frac{1}{2}a-10}$	$\frac{\frac{1}{2}da+\frac{1}{2}dc}{\frac{1}{2}a-\frac{1}{2}c}$

8. But the numbers of days found out in the two laſt ſteps muſt be equal to one another; for when *A* and *B* met, each had travelled the ſame number of days, becauſe they began their Journey at one and the ſame time: Hence this Equation ariſes, *viz.*

$$\frac{3\frac{1}{3}a+66\frac{2}{3}}{\frac{1}{2}a-10}=\frac{7\frac{1}{2}a-150}{\frac{1}{2}a+10};$$

That is, $$\frac{\frac{1}{2}da+\frac{1}{2}dc}{\frac{1}{2}a-\frac{1}{2}c}=\frac{\frac{1}{2}fa-\frac{1}{2}fc}{\frac{1}{2}a+\frac{1}{2}c}.$$

9. In which Equation, if you double both the Numerators and Denominators, and then reduce the Equation reſulting, to a common Denominator, and caſt away the common Denominator, the new Numerators being compared to one another will give this following Equation, *viz.*

$$\frac{20}{3}aa+\frac{800}{3}a+\frac{8000}{3}=15aa-600a+6000;$$

That is, $$daa+2dca+dcc=faa-2fca+fcc.$$

10. Which laſt Equation duly reduced gives this that follows, *viz.*

$$104a-aa=400,$$

That is, $$\frac{2dc+2fc}{f-d}a-aa=cc.$$

11. Wherefore by reſolving the Equation in the laſt ſtep according to the Canon in *Sect.* 10. *Chap.* 15. the two values of a will be found theſe, *viz.*

$$a=100=\frac{dc+fc+\sqrt{4dfcc}}{f-d}$$

$$a=4=\frac{dc+fc-\sqrt{4dfcc}}{f-d}.$$

12. But altho by either of thoſe values of a, to wit, 100 and 4, the Equation in the tenth ſtep may be expounded, yet the greater value only is the deſired number of Miles expreſſing the diſtance between the two Cities; for 'tis evident by the Queſtion, that 20 is but part of the number of Miles between the two Cities, and therefore 4 the leſſer value of a is much leſs than the ſaid Diſtance: Wherefore 100 the greater value of a is the deſired number of Miles between the two Cities. And conſequently the ſecond, third, fourth and fifth ſteps being reſolved into numbers, will ſhew, that when the two Foot-men *A* and *B* met one another, *A* had travelled 60 Miles, and *B* 40 Miles: Alſo, *A* travelled 6 Miles, and *B* 4 Miles every day; as will eaſily appear by the Proof.

13. But the numbers in this Queſtion muſt not be given at random, for the Denominator of the Fraction $\frac{2dc+2fc}{f-d}$ in the Equation in the tenth ſtep ſhews that the number d muſt be leſs than the number f, otherwiſe the Queſtion is impoſſible; as may eaſily be infer'd from the literal Equation in the ninth ſtep: for if in that Equation d be ſuppoſed greater than f, then conſequently dcc is greater than fcc, and after due tranſpoſition this Equation will ariſe, *viz.* $dcc-fcc=faa-daa-2dca-2fca$; where if d be greater than f, then the firſt part of the Equation will be a real Quantity, that is, greater than nothing, and the latter part leſs than nothing; but to affirm that a Quantity greater than nothing is equal to a Quantity leſs than nothing is abſurd; the like abſurdity will follow if we ſuppoſe $d=f$.

14. Having ſhew'd that d muſt neceſſarily be leſs than f, I ſhall prove that the leſſer value of a, as it is expreſs'd by Letters in the eleventh ſtep can never be equal to the whole diſtance between the two Cities. For if we ſhould ſuppoſe the leſſer value to be equal to the ſaid diſtance, it muſt neceſſarily be greater than c, which the Queſtion ſhews to be but part of the ſaid diſtance: But from that Suppoſition, it will follow by undeniable conſequence, that d is greater than f, which is contrary to what has been before proved. Now to prove the ſaid conſequence;

15. Suppoſe the leſſer value of a to exceed c, viz. $\frac{dc+fc-\sqrt{4dfcc}}{f-a} \sqsubset c$

16. Then by multiplying each part by $f-d$, it follows that . . . . $dc+fc-\sqrt{4dfcc} \sqsubset fc-dc$

17. And by adding $\sqrt{4dfcc}$ to each part, . . $dc+fc \sqsubset fc-dc+\sqrt{4dfcc}$

18. And by adding dc to each part, ... $2dc+fc \sqsubset fc+\sqrt{4dfcc}$

19. And by ſubtracting fc from each part, . : $2dc \sqsubset \sqrt{4dfcc}$

20. And by ſquaring each part, ... $4ddcc \sqsubset 4dfcc$

21. And by dividing each part by $4dcc$. . . $d \sqsubset f$

22. Thus from a Suppoſition that the leſſer value of a in the eleventh ſtep is greater than c, it follows by juſt conſequence that d is greater than f, which is impoſſible, for it has before been proved that d muſt be leſs than f. And becauſe the Series of Inferences deduced from the ſaid Suppoſition ends in an impoſſibility, therefore that which was ſuppoſed cannot be true; *viz.* The leſſer value of a is not greater than c, and conſequently it cannot be equal to the diſtance between the two Cities. Which was to be proved.

23. Again, by ſuppoſing d to be leſs than f, as it ought to be, to the end the Queſtion may be poſſible, we may prove the leſſer value of a to be leſſer than c, by returning backwards from the 21 ſtep to the 15, in this manner, *viz.*

24. Suppoſe . . . . $d \sqsupset f$

25. Then by multiplying each part by $4dcc$, . $4ddcc \sqsupset 4dfcc$

26. And by extracting the ſquare Root out of each part, ... : . . . $2dc \sqsupset \sqrt{4dfcc}$

27. And by adding fc to each part, . . . $2dc+fc \sqsupset fc+\sqrt{4dfcc}$

28. And by ſubtracting dc from each part, . . $dc+fc \sqsupset fc-dc+\sqrt{4dfcc}$

29. And by ſubtracting $\sqrt{4dfcc}$ from each part, $dc+fc-\sqrt{4dfcc} \sqsupset fc-dc$

30. Wherefore by dividing each part by $f-d$, it is manifeſt that the leſſer value of a is leſs than c, viz. $\frac{dc+fc-\sqrt{4dfcc}}{f-d} \sqsupset c$

Which was to be proved. Wherefore the leſſer value of a cannot poſſibly be equal to the diſtance between the two Cities, for the ſaid diſtance muſt neceſſarily be greater than part of it ſelf.

31. But it may be objected, That altho f be greater than d, yet how does it appear that $dc+fc$ is greater than $\sqrt{4dfcc}$, to the end that this may be ſubtracted from that, as the leſſer value of a requires, to make it ſelf a poſſible Root of the Equation in the tenth ſtep? In anſwer to this Objection, I ſhall in the next place prove that $dc+fc$ is greater than $\sqrt{4dfcc}$.

32. Foraſmuch as theſe Quantities are Proportionals, (for the Product of the Extremes is equal to the Product of the means,) $dd : df :: df . ff$

33. Therefore (*per* 25 Prop. 5. Elem. *Euclid.*) $dd+ff \sqsubset 2df$

34. And by multiplying all in the laſt ſtep by cc, $ddcc+ffcc \sqsubset 2dfcc$

35. And by adding $2dfcc$ to each part, . . . $ddcc+ffcc+2dfcc \sqsubset 4dfcc$

36. Wherefore by extracting the ſquare Root out of each part in the laſt ſtep, . . . $dc+fc \sqsubset \sqrt{4dfcc}$.

Which was to be proved.

CHAP. XVII.

Concerning Arithmetical PROGRESSION.

I. *Arithmetical Progreſſion* is, when many numbers (or other Quantities of one and the ſame kind) proceed by a common difference or exceſs; as in theſe, 2, 4, 6, 8, 10, 12, 14, *&c.* here 2 is the common difference betwixt 2 and 4, 4 and 6, 6 and 8, 8 and 10, *&c.* So 1, 2, 3, 4, 5, 6, *&c.* are in Arithmetical Progeſſion, 1 being the common difference: Likewiſe 3, 7, 11, 15, 19, *&c.* or 19, 15, 11, 7, and 3, where 4 is the common difference.

II. Arithmetical Progreſſion is either continued, as in the Examples above expreſs'd, where every two terms that ſtand next to one another, have one common difference; or elſe diſcontinued or interrupted, as in theſe numbers, 3, 5: 9, 11, where 5 exceeds 3 by 2, and ſo does 11 exceed 9; but 9 does not exceed 5 by 2, for the exceſs of 9 above 5 is 4. In like manner 18, 14: 21, 17, are in Arithmetical Progreſſion diſcontinued.

III. For the better Manifeſtation of the following Propoſitions concerning Arithmetical Progreſſion, let there be a rank of numbers in a continued Arithmetical Progreſſion, as, 3, 7, 11, 15, 19, 23, 27, *&c.* which numbers may be repreſented by a, b, c, d, e, f, g, *&c.* Alſo, let 105 the ſum of all the Terms of the Progreſſion be repreſented by Z; the common exceſs or difference 4 by X; and the number of Terms 7 by T: all which are here orderly expreſs'd underneath.

Quantities in Arithmetical Progreſſion continued:

$$\left\{\begin{array}{rcccl} 3 & = & a & = & a \\ 7 & = & b & = & a + \mathrm{X}. \\ 11 & = & c & = & a + 2\mathrm{X}. \\ 15 & = & d & = & a + 3\mathrm{X}. \\ 19 & = & e & = & a + 4\mathrm{X}. \\ 23 & = & f & = & a + 5\mathrm{X}. \\ 27 & = & g & = & a + 6\mathrm{X}. \end{array}\right.$$

The Sum of all the Terms is . . $105 = \mathrm{Z} = \mathrm{Z}$

The common difference is . $4 = \mathrm{X} = \mathrm{X}$

The number of Terms is . . $7 = \mathrm{T} = \mathrm{T}$.

IV. Whence it is manifeſt, that if a be put for the firſt and leaſt Term of an Arithmetical Progreſſion continued, and X for the common difference, then (according to the Definition in *Sect.* 1.) the ſecond Term ſhall be $a+\mathrm{X}$, the third $a+2\mathrm{X}$, the fourth $a+3\mathrm{X}$, the fifth $a+4\mathrm{X}$, *&c.* Moreover, according to the Suppoſitions in *Sect.* 3. $a=a$. $b=a+\mathrm{X}$. $c=a+2\mathrm{X}$. $d=a+3\mathrm{X}$. $e=a+4\mathrm{X}$, *&c.*

V. Therefore it follows, that the laſt and greateſt Term of every Arithmetical Progreſſion continued is compos'd of the firſt (to wit, the leaſt) term, and of the Product of the common difference multiplied by a number leſs by 1 (or Unity) than the number of Terms; as g, or $a+6X$ is compos'd of the firſt Term a and the Product of X multiplied by 6, which is leſs by 1 than 7 the number of Terms.

VI. Therefore the firſt and laſt Terms, as alſo the number of Terms being ſeverally given, the common difference ſhall be alſo given; for if the firſt, (to wit, the ſmalleſt) Term be ſubtracted from the laſt, and the Remainder be divided by a number leſs by 1 (or Unity) than the number of Terms, the Quotient is the common difference, *viz.* $\frac{g-a}{T-1}=X$.

VII. It is alſo manifeſt from *Sect.* 3. That if the firſt (to wit, the leaſt) Term be equal to the common difference, then the laſt Term is equal to the Product of the common difference (or firſt Term) multiplied by the number of Terms, *viz.* If $a=X$, then $g=X+6X=7X$.

VIII. Therefore in an Arithmetical Progreſſion continued whoſe firſt or leaſt Term is equal to the common difference, if the laſt Term and the number of Terms be ſeverally given, the firſt Term (or the common difference) ſhall alſo be given: For if the laſt Term be divided by the number of Terms, the Quotient is the firſt Term or common difference; as, if $a=X$, then $g=X+6X=7X$; therefore $\frac{7X}{7}=X=a$.

IX. It is alſo manifeſt from *Sect.* 7. That when the common difference divides any Term juſt without any Remainder, then the common difference is the ſame with the leaſt Term in that Progreſſion, and the Quotient is the number of Terms; but if any number remain after the Diviſion is finiſhed, then that Remainder is the leaſt Term, and the Quotient increaſed with 1 (or Unity) gives the number of Terms (*per* Sect. 4, & 5.) Therefore if any term greater than the leaſt be given, as alſo the common difference, the leaſt term, as alſo the number of terms in that Progreſſion ſhall alſo be given; as if 27 be ſome term greater than the leaſt, and 3 the common difference, by dividing 27 by 3, the Quotient 9 is the number of terms, and the leaſt term is equal to the common difference 3; as in this Progreſſion, 3,6,9,12,15,18,21,24,27.

But if 27 be given as before, and 4 be preſcribed for the common difference, then 27 divided by 4 gives 6 in the Quotient, and there remains 3 for the leaſt term, and 7 (to wit 6+1) is the number of terms; as in this Progreſſion, 3, 7, 11, 15, 19, 23, 27.

X. If three Numbers, ſuppoſe a,b,c, be in a continued Arithmetical Progreſſion, *viz.* If the Exceſs of c above b be equal to the Exceſs of b above a, the Sum of the Extremes, that is, of the firſt and laſt terms ſhall be equal to the double of the mean or middle term; *viz.* $a+c=2b$. For,

1. By Suppoſition, ... $c-b=b-a$,
2. Therefore by adding b to each part, it gives . . . . $c=2b-a$,
3. And by adding a to each part of the laſt Equation . . . $a+c=2b$.

Which was to be proved.

XI. If four Numbers; ſuppoſe a, b, c, d, be in Arithmetical Progreſſion whether continued or interrupted, *viz.* If the exceſs of b above a be equal to the exceſs of d above c, the Sum of the Extremes ſhall be equal to the Sum of the Means, *viz.* $a+d=b+c$. For,

1. By Suppoſition, ... $d-c=b-a$,
2. Therefore by equal addition of a, ... $a+d-c=b$,
3. Therefore by equal addition of c, ... $a+d=b+c$.

Which was to be proved.

XII. If there be as many numbers as you pleaſe in a continued Arithmetical Progreſſion, the Sum of the Extremes is equal to the Sum of any two Means equally diſtant from the Extremes, and alſo to the double of the Mean when the number of Terms is odd.

Let a,b,c,d,e,f, be in Arithmetical Progreſſion continued, and increaſing from a; I ſay the Sum of the Extremes a and f is equal to the Sum of any two terms equally diſtant from the extremes, that is, to the Sum of b and e, and to the Sum of c and d. For,

1. By Suppoſition, in regard of the continued Progreſſion, . $f-e=b-a$,
2. Therefore by equal addition of e and a to each part, . . $a+f=b+e$,
3. Again, by ſuppoſition . . . . $c-b=e-d$,

4. Therefore by equal addition of d and b, to each part $c+d = b+e$,
5. Therefore from the ſecond and fourth ſteps (*per* 1. *Axiom.* 1. *Elem. Euclid.*) $a+f = c+d = b+e$.

Which was to be proved.

And if more numbers were propos'd the Demonſtration would not be otherwiſe; therefore the firſt part of the Theorem is manifeſt.

But if the number of Terms be odd as in this continued Progreſſion, a, b, c, d, e, f, g, then the Sum of the Extremes a and g is equal to the double of the middle Term d, *viz.* $a+g = 2d$; which I prove thus:

1. By ſuppoſition, in regard of the continued Progreſſion, ... $d-c = e-d$,
2. And conſequently by equal addition of c and d, . . $2d = c+e$,
3. But by what has been proved concerning the firſt part of the Theorem in this twelfth *Sect.* . . . $a+g = c+e$,
4. Therefore from the two laſt ſteps, (*per Axiom.* 1. *Elem.* 1. *Euclid.*) $a+g = 2d$.

Which was to be demonſtrated. Therefore the Theorem is every way manifeſt.

XIII. In every Arithmetical Progreſſion continued, the Sum of the Extremes multiplied by the number of terms produces the double of the Sum of all the terms.

The number of terms is either even or odd: Firſt, let there be an even number of terms, *viz.* ſuppoſe theſe ſix numbers a, b, c, d, e, f, to be in Arithmetical Progreſſion continued;

I ſay, ... $6a+6f = \left\{\begin{array}{l}2a+2b+2c+2d,\\ +2e+2f.\end{array}\right.$

DEMONSTRATION.

1. It is evident that . . . . $2a+2f = 2a+2f$,
2. And by *Sect.* 12. . . . . $2a+2f = 2b+2e$,
3. Likewiſe, by the ſame *Sect.* . . . . $2a+2f = 2c+2d$,
4. Therefore by adding the three laſt Equation together, $6a+6f = \left\{\begin{array}{l}2a+2b+2c,\\ +2d+2e+2f.\end{array}\right.$

Which was to be demonſtrated. And ſo of others when the number of terms is even.

Secondly, let there be an Arithmetical Progreſſion conſiſting of an odd number of terms, ſuppoſe theſe five, a, b, c, d, e.

I ſay, ... $5a+5e = 2a+2b+2c+2d+2e$.

DEMONSTRATION.

1. It is manifeſt that . . . . . . $2a+2e = 2a+2e$,
2. And by *Sect.* 12. . . . . $2a+2e = 2b+2d$,
3. Likewiſe by *Sect.* 12. . . . . $a+e = 2c$,
4. Therefore by adding the three laſt Equations together, ... $5a+5e = 2a+2b+2c+2d+2e$.

And ſo of others when the number of terms is odd.

XIV. Therefore from the laſt *Sect.* the firſt and laſt terms, as alſo the number of terms in an Arithmetical Progreſſion continued being given, the ſum of all the terms ſhall be alſo given: For if the ſum of the firſt and laſt terms be multiplied by the number of terms the Product is the double ſum of all the terms, and conſequently the half of that Product is the ſum it ſelf. For example, If a, b, c, d, e, f, g, be in Arithmetical Progreſſion continued, and T be put for the number of terms, alſo Z for their ſum (as before;) Then $Ta+Tg = 2Z$, and conſequently $\frac{1}{2}Ta+\frac{1}{2}Tg = Z$.

XV. Mr. *William Oughtred* in *Prob.* 4. *Chap.* 19. of his incomparable *Clavis Mathemat.* has very elegantly handled 20 Propoſitions about Arithmetical Progreſſion continued, which (for the more ample Illuſtration of the preceding Rules in this Book,) I ſhall explain in this *Section*, uſing his own Symbols, which are theſe, *viz.*

α	Stands for	The leaſt (or firſt) term.
ω		The greateſt (or laſt) term.
T		The number of Terms.
X		The common difference of the Terms.
Z		The ſum of all the terms.

Any three of theſe five things being given, the other two ſhall be alſo given, by the reſpective Canons of the following 20 Propoſitions, which Mr. *Oughtred* ſtates thus;

Given,	Sought,	By Propoſ.
$\alpha,\ \omega,\ T$	Z and X	1 and 2
$\alpha,\ \omega,\ X$	T and Z	3 and 4
$\alpha,\ \omega,\ Z$	T and X	5 and 6
$\alpha,\ T,\ X$	ω and Z	7 and 8
$\alpha,\ T,\ Z$	ω and X	9 and 10
$\alpha,\ X,\ Z$	ω and T	11 and 12
$\omega,\ T,\ X$	α and Z	13 and 14
$\omega,\ T,\ Z$	α and X	15 and 16
$\omega,\ X,\ Z$	α and T	17 and 18
T, X, Z	α and ω	19 and 20

PROP. I.

1. . . . . $\left\{\begin{array}{l}\alpha,\ \omega,\ T \text{ are given ſeverally;}\\ Z \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By *Sect.* 14. of this *Chap.* . . . . $T\omega + T\alpha = 2Z.$

Which Equation, if expreſs'd by words, gives this

CANON.

Multiply the Sum of the firſt and laſt Terms by the number of Terms, the Product ſhall be the double of the Sum of all the Terms, and conſequently the half of that Product is the required Sum of all the Terms.

Which Canon may be exemplified by the following (or any other) rank of numbers in Arithmetical Progreſſion continued, *viz.*

3, 7, 11, 15, 19, 23, 27.

PROP. II.

1. . . . . $\left\{\begin{array}{l}\alpha,\ \omega,\ T \text{ are given ſeverally;}\\ X \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By *Sect.* 6. of this ſeventeenth *Chap.* . . . . $\frac{\omega - \alpha}{T - 1} = X.$

Which Equation gives this following

CANON.

Divide the exceſs of the greateſt (or laſt) Term above the leaſt, by the number of Terms leſſened by 1 (or Unity,) and the Quotient is the common difference required.

Which Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion continued, *viz.*

3, 7, 11, 15, 19, 23, 27.

From the Equation in the ſecond ſtep of *Prop.* 1. and the Equation in the ſecond ſtep of *Prop.* 2. the Canons of all the following 18 Propoſitions are deduced.

PROP. III.

1. . . . . $\left\{\begin{array}{l}\alpha,\ \omega,\ X \text{ are given ſeverally;}\\ T \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. The Letters put for the things given and ſought, without any other Letter, are contained in the Equation in the ſecond ſtep of *Prop.* 2. therefore the work here is only to ſet T alone in that Equation, which may be done thus, *viz.*

3. By the Canon of *Prop.* 2. . . . . $\frac{\omega-\alpha}{T-1}=X,$

4. Therefore by multiplying each part of that Equation by T — 1, this ariſes, *viz.* . . . . $\omega-\alpha=TX-X,$

5. And by addition of X to each part of the laſt Equation, this ariſes; . . . . $\omega-\alpha+X=TX,$

6. Therefore each part of the laſt Equation being divided by X, the number T will be made known, *viz.* . . $\frac{\omega-\alpha}{X}+1=T.$

The laſt Equation gives this following

CANON.

From the laſt (to wit, the greateſt) Term ſubtract the firſt, and divide the Remainder by the common difference; then to the Quotient add 1 (or Unity) ſo ſhall the Sum be the required number of Terms.

This Canon may be exemplified by the following (or any other) Rank of Numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. 4.

1. . . . $\left\{\begin{array}{l}\alpha,\ \omega,\ X \text{ are given ſeverally;}\\ Z \text{ is required.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega+T\alpha=2Z,$

3. And by the Canon of *Prop.* 3. . . . . $\frac{\omega-\alpha}{X}+1=T,$

4. Now if inſtead of T in the firſt part of the Equation in the ſecond ſtep, you multiply into $\omega+\alpha$ that which in the laſt Equation is found equal to T, the former Equation will be converted into this, *viz.*

$$\frac{\omega\omega-\alpha\alpha}{X}+\omega+\alpha=2Z.$$

Which in words is this following

CANON.

From the Square of the greateſt (or laſt) Term ſubtract the Square of the leaſt (or firſt,) then dividing the Remainder by the common difference, and to the Quotient adding the Sum of the firſt and laſt Terms, the half of the Sum of this Addition ſhall be the required Sum of all the Terms.

The Canon may be exemplified by the following (or any other) Rank of Numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. V.

1. . . . . $\left\{\begin{array}{l}\alpha,\ \omega,\ Z, \text{ are given ſeverally;}\\ T \text{ is required.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega+T\alpha=2Z,$

3. Therefore by dividing each part of that Equation by $\omega+\alpha$, this ariſes, *viz.* . . . . $T=\frac{2Z}{\omega+\alpha},$

Which Equation gives this following

CANON.

Divide the double of the Sum of all the Terms by the Sum of the firſt and laſt Terms, the Quotient is the number of Terms ſought; as may be proved by this following (or any other) Rank of numbers in Arithmetical Progreſſion:

3, 7, 11, 15, 19, 23, 27.

PROP.

PROP. VI.

1. . . . . $\left\{\begin{array}{l} \alpha, \omega, Z \text{ are given ſeverally;} \\ \quad X \text{ is required.} \end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 4. . . . . $\frac{\omega\omega - \alpha\alpha}{X} + \omega + \alpha = 2Z,$

3. Which Equation multiplied by X produces, . . $\omega\omega - \alpha\alpha + \omega X + \alpha X = 2ZX,$

4. And by ſubtracting $\omega X + \alpha X$ from each part of the laſt Equation, this ariſes, *viz.* . . . . . $\omega\omega - \alpha\alpha = 2ZX - \omega X - \alpha X,$

5. Therefore by dividing each part of the laſt Equation by the Co-efficients that are drawn into X, you will find, ... $\frac{\omega\omega - \alpha\alpha}{2Z - \omega - \alpha} = X.$

Which laſt Equation gives this

CANON.

From the Square of the laſt Term ſubtract the Square of the firſt (to wit, the leaſt) Term; divide the Remainder by the exceſs whereby the double Sum of all the Terms exceeds the Sum of the firſt and laſt Terms, ſo ſhall the Quotient be the common difference required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion:

3, 7, 11, 15, 19, 23, 27.

PROP. VII.

1. . . . . $\left\{\begin{array}{l} \alpha, T, X \text{ are given ſeverally;} \\ \quad \omega \text{ is ſought.} \end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 2. . . . . $\frac{\omega - \alpha}{T - 1} = X,$

3. Therefore by multiplying each part of the ſaid Equation by T—1, this will be produced, . . $\omega - \alpha = TX - X,$

4. And by adding α to each part of the laſt Equation this ariſes, *viz.* . . . . $\omega = TX + \alpha - X.$

Which laſt Equation gives this

CANON.

To the Product made by the Multiplication of the number of Terms into the common difference, add the firſt (to wit, the leaſt) Term, and from the Sum ſubtract the ſaid difference, ſo ſhall the Remainder be the laſt Term ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. VIII.

1. . . . . $\left\{\begin{array}{l} \alpha, T, X \text{ are given ſeverally;} \\ \quad Z \text{ is ſought.} \end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega + T\alpha = 2Z,$

3. And by the Canon of *Prop.* 7. . . . . . . $TX + \alpha - X = \omega.$

4. Now to find an Equation that may conſiſt only of the things given and ſought in this *Prop.* 8. multiply each part of the Equation in the third ſtep by T, and there will be produced

$$TTX + T\alpha - TX = T\omega;$$

5. Then if inſtead of $T\omega$ in the ſecond ſtep, you take that which in the fourth is found equal to $T\omega$, the Equation in the ſecond ſtep will be reduced to this, to wit,

$$TTX + 2T\alpha - TX = 2Z,$$

That is, $\overline{TX + 2\alpha - X}$ into $T = 2Z$.

Which laſt Equation gives this

CANON.

6. To the Product of the Multiplication of the number of Terms by the common difference, add the double of the firſt (to wit, the leaſt) Term, and from the Sum of that Addition ſubtract the common difference; then multiply the Remainder by the number of Terms; ſo ſhall the Product be the double Sum of all the Terms, and conſequently the half of that Product is the required Sum of all the Terms.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. IX.

1. . . . . . $\left\{\begin{array}{l}\alpha,\ T,\ Z \text{ are given ſeverally;} \\ \quad\ \omega \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . . . $T\omega + T\alpha = 2Z,$
3. Therefore by equal ſubtraction of $T\alpha$, . . $T\omega = 2Z - T\alpha,$
4. Therefore by dividing each part of the laſt Equation by T, this ariſes; . . . $\omega = \frac{2Z - T\alpha}{T}.$

Which laſt Equation gives this

CANON.

From the double of the Sum of all the Terms ſubtract the Product of the Multiplication of the number of Terms by the firſt (to wit, the leaſt) Term, and divide the Remainder by the number of Terms; ſo ſhall the Quotient be the laſt Term ſought.

This Canon may be exemplified by the following (or any other) Rank of Numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. X.

1. . . . . . $\left\{\begin{array}{l}\alpha,\ T,\ Z \text{ are given ſeverally;} \\ \quad\ X \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 8. . . . . . $TTX + 2T\alpha - TX = 2Z,$
3. Therefore by equal ſubtraction of $2T\alpha$ from each part, this will ariſe; to wit, $TTX - TX = 2Z - 2T\alpha,$
4. And by dividing each part of the laſt Equation by $TT - T$, the common difference X will be made known, *viz.* . . $X = \frac{2Z - 2T\alpha}{TT - T}.$

Which laſt Equation gives this

CANON.

From the double Sum of all the Terms ſubtract the double Product made by the Multiplication of the number of Terms by the leaſt Term, and divide the Remainder by the exceſs of the Square of the Number of Terms above the number of Terms, ſo ſhall the Quotient be the common difference ſought.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XI.

1. . . . . . $\left\{\begin{array}{l}\alpha,\ X,\ Z \text{ are given ſeverally;} \\ \quad\ \omega \text{ is ſought}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 4. . : . . . $\frac{\omega\omega - \alpha\alpha}{X} + \omega + \alpha = 2Z,$

3. Therefore by multiplying that Equation by X, this will be produced; to wit, ... $\omega\omega - \alpha\alpha + X\omega + X\alpha = 2ZX$,

4. And by tranſpoſition of $-\alpha\alpha$, this ariſes; . . $\omega\omega + X\omega + X\alpha = 2ZX + \alpha\alpha$,

5. And from the laſt Equation by tranſpoſition of $X\alpha$ this ariſes; . . . . $\omega\omega + X\omega = 2ZX + \alpha\alpha - X\alpha$,

6. Which laſt Equation falling under the firſt of the three Forms in *Sect.* 1. *Chap.* 15 of this Book, the value of ω ſhall be given by the Canon in *Sect.* 6. of the ſame *Chap. viz.*

$$\omega = \sqrt{:\tfrac{1}{4}XX + 2ZX + \alpha\alpha - X\alpha:} - \tfrac{1}{2}X.$$

Which Equation gives this

CANON.

From the ſum of theſe three numbers, to wit, the Square of half the common difference; the double Product of the Multiplication of the ſum of all the terms by the common difference; and the Square of the firſt (to wit, the leaſt) term, ſubtract the Product of the firſt term multiplied by the common difference, and extract the ſquare Root of the Remainder; then from the ſaid ſquare Root ſubtract half the common difference, ſo ſhall this laſt Remainder be the laſt and greateſt term ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XII.

1. . . . . $\left\{ \begin{array}{l} \alpha,\ X,\ Z \text{ are given ſeverally;} \\ \qquad T \text{ is ſought.} \end{array} \right.$

RESOLUTION.

2. The Canon of *Prop.* 8. gives this Equation, . . . $XTT + 2\alpha T - XT = 2Z$,

3. Where in regard X is drawn into TT (which is the higheſt degree of the Quantity ſought,) let every term of the Equation be divided by X, whence this Equation will ariſe; . . . $TT + \frac{2\alpha T - XT}{X} = \frac{2Z}{X}$,

4. Now it muſt be diſcovered from the things given whether 2α exceeds X, or is leſs, or equal to X. Firſt then ſuppoſe $2\alpha \sqsubset X$, and then the laſt Equation may be expreſs'd thus;

$$TT + \frac{2\alpha - X}{X}T = \frac{2Z}{X}.$$

5. Which Equation falling under the firſt of the three Forms in *Sect.* 1. *Chap.* 15. the value of T ſhall be given by the Canon in *Sect.* 6. of the ſame *Chap. viz.*

$$T = \sqrt{:\frac{\alpha\alpha - \alpha X + \frac{1}{4}XX + 2ZX}{XX}:} - \frac{2\alpha - X}{2X}.$$

6. Secondly, If $2\alpha \sqsupset X$, then the Equation in the third ſtep ſhall be expreſs'd thus;

$$TT - \frac{X - 2\alpha}{X}T = \frac{2Z}{X}.$$

7. Which Equation falling under the ſecond of the three Forms in *Sect.* 1. *Chap.* 15. the value of T ſhall be given by the Canon in *Sect.* 8. of the ſame *Chap. viz.*

$$T = \sqrt{:\frac{\frac{1}{4}XX - \alpha X + \alpha\alpha + 2ZX}{XX}:} + \frac{X - 2\alpha}{2X}.$$

8. Laſtly, If $2\alpha = X$, then the Equation in the third ſtep will be expreſs'd thus;

$$TT = \frac{2Z}{X}; \qquad \text{Whence,} \quad T = \sqrt{\frac{2Z}{X}}.$$

The three Equations in the 5,7, and 8 ſteps give a threefold Canon to ſolve this 12 *Prop. viz.*

Canon I. *When the double of the leaſt term exceeds the common difference.*

9. To the Square of the exceſs of the leaſt term above half the common difference add the double Product of the Multiplication of the Sum of all the Terms by the common difference, divide the Sum of that Addition by the ſquare of the common difference and extract the ſquare Root of the Quotient; then from the double of the leaſt term ſubtract the common difference and divide the Remainder by the double of the common difference: laſtly, ſubtracting this Quotient from the ſquare Root before found, the Remainder ſhall be the number of terms ſought.

This Canon may be exemplified by the following or the like Series of Numbers in Arithmetical Progreſſion continued, where the double of the leaſt Term exceeds the common difference of the Terms:

3, 5, 7, 9, 11, 13, 15, *&c.*

Canon II. *When the double of the leaſt Term is leſs than the common-difference of the Terms.*

10. To the Square of the exceſs of half the common difference above the leaſt Term, add the double Product of the Multiplication of the Sum of all the Terms by the common difference; divide the Sum of that Addition by the Square of the common difference, and extract the ſquare Root of the Quotient; then from the common difference ſubtract the double of the leaſt Term, and divide the Remainder by the double of the common difference; laſtly, adding this Quotient to the ſquare Root before found, the Sum ſhall be the number of Terms ſought.

This Canon may be exemplified by the following or the like Rank of numbers in Arithmetical Progreſſion continued, where the double of the leaſt Term is leſs than the common difference:

2, 7, 12, 17, 22, 27, 32, 37.

Canon. III. *When the double of the leaſt Term is equal to the common difference of the Terms.*

11. Divide the double of the Sum of all the Terms by the common difference, ſo ſhall the ſquare Root of the Quotient be the number of Terms ſought.

This Canon may be exemplified by the following Rank of numbers in Arithmetical Progreſſion continued, where the double of the leaſt Term is equal to the common difference of the Terms:

3, 9, 15, 21, 27, 33, 39.

PROP. XIII.

1. . . . . $\left\{\begin{array}{l}\omega,\ T,\ X \text{ are given ſeverally;}\\ \alpha \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 7. . . . . $TX-X+\alpha=\omega$,

3. Therefore by tranſpoſition of TX—X, this Equation will ariſe, which makes known the value of α; $\alpha=\omega+X-TX$.

Which Equation gives this

CANON.

To the laſt, (that is, the greateſt) Term add the common difference, and from the Sum ſubtract the Product of the number of Terms multiplied by the common difference; ſo ſhall the Remainder be the firſt (or leaſt) Term ſought.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XIV.

1. . . . . $\left\{\begin{array}{l}\omega,\ T,\ X \text{ are given ſeverally;}\\ Z \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 1. . . . . $T\omega+T\alpha=2Z$,

3. And by the Canon of *Prop.* 13. . . . . $\omega+X-TX=\alpha$,

4. Which latter Equation if it be multiplied by T, will produce $T\omega+TX-TTX=T\alpha$,

5. Then if inſtead of $T\alpha$ in the Equation in the ſecond ſtep, you take that which in the fourth ſtep is found equal to $T\alpha$, the Equation in the ſecond ſtep will be converted into this; $2T\omega+TX-TTX=2Z$,

6. That is . . . . $\overline{2\omega+X-TX}\text{ into }T=2Z$.

Which Equation gives this

CANON.

To the double of the laſt (to wit, the greateſt) Term, add the common difference; from the Sum ſubtract the Product of the number of Terms multiplied by the common difference;

difference: then multiply the Remainder by the number of Terms, the Product ſhall be the double of the Sum of all the Terms, and conſequently the half of that Product is the required Sum of all the Terms.

This Canon may be exemplified by the following (or any other Rank) of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27, 31.

PROP. XV.

1. . . . . $\left\{\begin{array}{l}\omega,\ T,\ Z \text{ are given ſeverally;}\\ \alpha \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 9. . . . . $\frac{2Z-T\alpha}{T}=\omega,$

3. Therefore multiplying each part of that Equation by T, this will ariſe; . . . . $2Z-T\alpha=T\omega,$

4. And by tranſpoſition of $-T\alpha$ in the laſt Equation this will ariſe; . . . . $2Z=T\omega+T\alpha,$

5. Likewiſe by tranſpoſition of $T\omega$, this Equation ariſes, $2Z-T\omega=T\alpha,$

6. Therefore each part of the laſt Equation being divided by T, the value of α will be made known, *viz.* $\frac{2Z}{T}-\omega=\alpha.$

Which Equation gives this

CANON.

Divide the double Sum of all the Terms by the number of Terms, and from the Quotient ſubtract the laſt (to wit, the greateſt term; ſo ſhall the Remainder be the firſt and leaſt term ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVI.

1. . . . . $\left\{\begin{array}{l}\omega,\ T,\ Z \text{ are given ſeverally;}\\ X \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 14. . . . . $\overline{2\omega+X-TX}$ into $T=2Z,$

3. That is . . . . $2T\omega+TX-TTX=2Z,$

4. Therefore by due tranſpoſition this Equation will ariſe, $2T\omega-2Z=TTX-TX,$

5. Therefore by dividing all in the laſt Equation by $TT-T$, the value of X will be made known, *viz.* $\frac{2T\omega-2Z}{TT-T}=X.$

Which Equation gives this

CANON.

From the double Product of the Multiplication of the number of Terms by the greateſt Term, ſubtract the double of the Sum of all the Terms; divide the Remainder by the exceſs of the Square of the number of Terms above the number of Terms, ſo ſhall the Quotient be the common difference ſought.

This Canon may be exemplified by the following (or any other) Rank of numbers in Arithmetical Progreſſion continued:

3, 7, 11, 15, 19, 23, 27.

PROP. XVII.

1. . . . . $\left\{\begin{array}{l}\omega,\ X,\ Z \text{ are given ſeverally;}\\ \alpha \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 6. . . . . $\frac{\omega\omega-\alpha\alpha}{2Z-\omega-\alpha}=X,$

3. Therefore each part of that Equation being multiplied by $2Z-\omega-\alpha$, there will ariſe, ... $\omega\omega-\alpha\alpha=2XZ-X\omega-X\alpha,$

4. Whence by equal addition of $X\omega+X\alpha$ you will find, $\omega\omega+X\omega+X\alpha-\alpha\alpha=2ZX,$

Now before known Quantities can be ſeparated from unknown in the laſt Equation, we muſt diſcover from the things given in the Propoſition, whether $\omega\omega + X\omega$ be equal, greater, or leſs than $2ZX$? Firſt therefore,

5. Suppoſe . . . . $\omega\omega + X\omega = 2ZX$,
6. And then by ſetting $\omega\omega + X\omega$ in the place of $2ZX$ in the Equation in the fourth ſtep, there will ariſe, ... $\omega\omega + X\omega + X\alpha - \alpha\alpha = \omega\omega + X\omega$,
7. Whence by ſubtracting $\omega\omega + X\omega$ from each part, and by tranſpoſition of $-\alpha\alpha$, this Equation ariſes; . . . . . $X\alpha = \alpha\alpha$,
8. Which laſt Equation being divided by α, gives . . . . $X = \alpha$.

From the premiſes ariſes this

CANON I.

9. When the ſum of the Square of the laſt (to wit, the greateſt) term and the Product of the multiplication of the ſaid laſt term by the common difference of the terms is equal to the double of the Product made by the multiplication of the ſum and common difference of the terms, then the ſaid difference is equal to the firſt or leaſt term ſought.

This Canon may be exemplified by the following Series of numbers in Arithmetical Progreſſion continued:

2, 4, 6, 8, 10, 12, 14.

10. Secondly, ſuppoſe . . . . $\omega\omega + X_{\omega} \sqsubset 2ZX$.
11. Then from the Equation in the fourth ſtep, after due Reduction, there will ariſe, $\alpha\alpha - X\alpha = \omega\omega + X_{\omega} - 2ZX$,
12. In which laſt Equation all things are known but α, and the ſaid Equation falls under the ſecond of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the value of α, to wit, the firſt (or leaſt) term ſought ſhall be given by the Canon in *Sect.* 8. of the ſame *Chap. viz.*

$$\alpha = \tfrac{1}{2}X + \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

From the tenth and twelfth ſteps ariſes

CANON. II.

13. If the ſum of the Square of the laſt (to wit, the greateſt) term, and the Product of the multiplication of the ſaid laſt term by the common difference of the terms, exceeds the double of the Product made by the multiplication of the ſum and common difference of the terms; then to the ſum firſt mentioned add the Square of half the common difference; from this ſum ſubtract the double Product above mentioned, and extract the ſquare Root of the Remainder: laſtly, add the ſaid ſquare Root to half the common difference, ſo ſhall the Sum be the firſt (or leaſt) term ſought.

This Canon may be exemplified by the following Progreſſion:

3, 5, 7, 9, 11, 13.

14. Thirdly, ſuppoſe : . . . . $\omega\omega + X_{\omega} \sqsupset 2ZX$,
15. But in this third caſe, to the end a poſſible Equation may ariſe, this Determination is neceſſary, *viz.* . . . . $\omega\omega + X_{\omega} + \frac{1}{4}XX$, not $\sqsupset 2ZX$,
16. Then from the Equation in the fourth ſtep by tranſpoſition of $\omega\omega + X\omega$, this will ariſe; $X\alpha - \alpha\alpha = 2ZX - \omega\omega - X\omega$.
17. In which laſt Equation all things are known but α, and the Equation falls under the laſt of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the two values of α in that Equation ſhall be given by the Canon in *Sect.* 10. of the ſame *Chap. viz.*

$$\alpha = \tfrac{1}{2}X + \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

Or, $$\alpha = \tfrac{1}{2}X - \sqrt{:\omega\omega + X\omega + \tfrac{1}{4}XX - 2ZX:}$$

18. Whence it is manifeſt, that if in this third Caſe it happens that $\omega\omega + X_{\omega} + \frac{1}{4}XX = 2ZX$, then $= \frac{1}{2}X$; that is to ſay, the firſt (or leaſt) term ſought ſhall be equal to half the given difference of the terms. But if in the ſaid third Caſe it happens

that $\omega\omega+X\omega+\frac{1}{4}XX \sqsubset 2ZX$, then there will be two unequal Roots or values of α, to wit, thoſe above expreſs'd, by either of which the Equation in the ſixteenth ſtep may be expounded; yet (as may eaſily be apprehended) only one of thoſe values of α can be ſuch a firſt (or leaſt) term as will agree with the things given in the Propoſition: But which of thoſe two values of α is the leaſt term ſought, you may diſcover by the Proof formed thus, *viz.* Firſt, by the help of one of thoſe unequal values of α found out as above, together with the given laſt (to wit, the greateſt) term and the given common difference of the terms, you may find out (by the Canon of the third *Prop,*) the number of terms, (which muſt always be a whole number,) and then by the ſame value of α, together with the ſaid laſt term and the number of Terms you may by the Canon of *Prop.* 1. find out the ſum of all the terms; then if this ſum be equal to the ſum given in the *Propoſ.* propos'd, that value of α by which the Proof was made, is the leaſt term ſought. But if that Proof will not ſucceed, then the other value of α ſhall be the leaſt term ſought; as will be evident by the Proof made as before.

From the five laſt ſteps there will ariſe

CANON. III.

19. When the ſum of the Square of the laſt (to wit, the greateſt) term, and the Product of the Multiplication of the ſaid laſt term by the common difference, is leſs than the double of the Product made by the multiplication of the ſum and common difference of the terms; but the Aggregate of the ſum firſt mentioned and the ſquare of half the common difference is not leſs than the ſaid double Product; then from the ſaid Aggregate ſubtract the ſaid double Product and extract the ſquare Root of the Remainder, that done, add the ſaid ſquare Root to half the common difference of the terms, and alſo ſubtract the ſaid ſquare Root from the ſaid half difference, ſo the Sum or elſe the Remainder, (*viz.* ſuch of them, which by the Proof made according to the direction in the preceding eighteenth ſtep will be found to agree with the things given in the Propoſition,) ſhall be the firſt (or leaſt) term ſought.

This Canon may be exemplified by the two following Ranks of numbers in Arithmetical Progreſſion continued:

I.	2, 5, 8, 11, 14, 17.
II.	2, 7, 12, 17, 22, 27.

PROP. XVIII.

1. . . . . $\left\{\begin{array}{l}\omega,\ X,\ Z \text{ are given ſeverally;} \\ \quad T \text{ is required.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 14. . . . . . . $2\omega T+XT-XTT = 2Z.$

3. Therefore dividing every member of the ſaid Equation by X, (becauſe it is drawn into TT the higheſt degree of the number ſought,) this following Equation will ariſe, *viz.*

$$\frac{2\omega T+XT}{X} - TT = \frac{2Z}{X},$$

That is, $$\frac{2\omega+X}{X}T - TT = \frac{2Z}{X}.$$

4. In which all things are known but T, and the ſaid Equation falls under the laſt of the three Forms in *Sect.* 1. *Chap.* 15. Therefore the two values of T will be made known by the Canon in *Sect.* 10. of the ſame *Chap. viz.*

$$T = \frac{\omega+\frac{1}{2}X}{X} + \sqrt{:\frac{\omega\omega+\omega X+\frac{1}{4}XX-2ZX}{XX}:}$$

$$\text{Or, } T = \frac{\omega+\frac{1}{2}X}{X} - \sqrt{:\frac{\omega\omega+\omega X+\frac{1}{4}XX-2ZX}{XX}:}$$

5. But altho the Equation in the third ſtep may be expounded by either of the two Roots or values of T above expreſs'd in the fourth ſtep, yet only one of them can be the number of terms ſought; but which of the ſaid numbers, or values of T will ſolve the Propoſition you may diſcover thus: Firſt, If one of the two numbers or values of T before found out be a Fraction or a mixt number, that value cannot be the number of terms ſought; for the number of terms in an Arithmetical Progreſſion is alwnys a whole number. Secondly, If both the values of T happen to be whole numbers, then the true number of terms ſought may be diſcovered by this Proof; *viz.* Firſt, by the help of one of thoſe values of T in whole numbers, together with the given laſt (or greateſt) term, and the given common difference, find out (by the Canon of *Prop.* 13.) the firſt (to wit, the leaſt) term; and then by the ſame number T, together with the firſt and laſt terms, find out (by the Canon of *Prop* 1.) the ſum of all the terms; laſtly, If the ſum ſo found out be equal to the ſum given in the Propoſition propos'd, then that number or value of T by which the Proof was made ſhall be the true number of terms ſought. But if the Proof will not ſucceed to find out a number equal to the ſum firſt given, then the other value of T is the number of terms ſought; which will be evident by the Proof made therewith in the ſame manner as before.

From the premiſſes there ariſes this

CANON.

6. From the Square of the ſum of the laſt (to wit, the greateſt) term, and half the common difference, ſubtract the double of the Product of the Multiplication of the ſum of all the terms by the common difference; divide the Remainder by the ſquare of the ſaid difference, and extract the ſquare Root of the Quotient. That done, add the ſaid ſquare Root to the Quotient which ariſes by dividing the ſum of the laſt term and half the common difference by the difference it ſelf, and alſo ſubtract the ſaid ſquare Root from the ſaid Quotient; ſo the Sum, or elſe the Remainder (*viz.* ſuch of them which according to the preceding fifth ſtep will be found to agree with the things given in the *Propoſ.*) ſhall be the number of terms ſought.

This Canon may be exemplified by the three following Progreſſions; in the firſt of which the greater of the two values of T (in the fourth ſtep) is the number of terms ſought; but in each of the two latter Progreſſions the leſſer value of T is the number of terms ſought.

I.	2, 7, 12, 17, 22, 27, 32.
II.	2, 5, 8, 11, 14, 17, 20.
III.	12, 20, 28, 36, 44, 52, 60.

PROP. XIX.

1. . . . . $\left\{\begin{array}{l}T,\ X,\ Z \text{ are given ſeverally;}\\ \alpha \text{ is ſought.}\end{array}\right.$

RESOLUTION.

2. By the Canon of *Prop.* 10. . . . . $\dfrac{2Z-2T\alpha}{TT-T}=X,$

3. Therefore multiplying each part of that Equation by TT—T, this will be produced, to wit, . . $2Z-2T\alpha=TTX-TX$

4. In which laſt Equation all things are known but α, whoſe value after due Reduction of that Equation will be found out, *viz.* . . . . $\alpha=\dfrac{Z}{T}+\frac{1}{2}X-\frac{1}{2}TX,$

Which in words gives this

CANON.

5. Divide the given ſum of all the terms by the given number of terms, to the Quotient add half the given difference of the terms, and from the ſum of that addition ſubtract half the Product of the Multiplication of the ſaid number of terms by the common difference; ſo ſhall the Remainder be the firſt (to wit, the leaſt) term required.

This Canon may be exemplified by the following (or any other) Series of numbers in Arithmetical Progreſſion continued:

2, 7, 12, 17, 22, 27, 32.

PROP.

PROP. XX.

1. . . . . . T, X, Z are given ſeverally; ω is ſought.

RESOLUTION.

2. By the Canon of *Prop.* 16. . . . . $\frac{2T\omega - 2Z}{TT - T} = X$,

3. Therefore multiplying each part of that Equation by TT—T, this will be produced, to wit, . . $2T\omega - 2Z = TTX - TX$,

4. In which laſt Equation all things are known but ω, whoſe value, after due Reduction of that Equation, will be diſcovered, *viz.* . . . . $\omega = \frac{Z}{T} + \frac{1}{2}TX - \frac{1}{2}X$.

Which in words gives this

CANON.

5. Divide the given ſum of all the terms by the given number of terms; to the Quotient add half the Product of the Multiplication of the number of terms by the common difference given, and from the ſum of that Addition ſubtract half the ſaid difference; the Remainder ſhall be the laſt (to wit, the greateſt) term required.

This Canon may be exemplified by the following or any other Rank of numbers in Arithmetical Progreſſion continued:

2, 5, 8, 11, 14, 17, 20.

Queſtions to exerciſe ſome of the Canons of the preceding Propoſitions.

Queſt 1. Suppoſe 40 Stones be ſo placed in a ſtreight line, that the firſt is diſtant from a Basket one Yard, the ſecond two, the third three, and the reſt in the ſame exceſs; now if ſome Footman undertakes to go from the Basket to fetch into it every Stone one after another, how many Yards muſt he go to perform that work? *Anſw.* 1640 Yards.

Foraſmuch as the Footman muſt go 2 Yards (to wit, one forwards, and the ſame backwards,) to fetch the firſt Stone into the Basket; 4 Yards for the ſecond; 6 for the third, *&c.* here is an Arithmetical Progreſſion continued whoſe firſt (or leaſt) term is 2, the common difference of the terms is alſo 2, and the number of Terms is 40; therefore the ſum of all the terms, to wit, the number of Yards ſought will be found 1640, by the Canon of the preceding eighth *Prop.*

Queſt. 2. Two Footmen, *A* and *B*, depart at the ſame time from *London* towards *York*, and travel in this manner, *viz.* *A* travels 8 (or *c*) Miles every day; *B* travels 1 Mile the firſt day, 2 Miles the ſecond day, 3 Miles the third day, and ſo forward; travelling every day one Mile more than in the day next preceding: The Queſtion is, to find in how many days *B* will overtake *A*? *Anſw.* At the end of 15 days, found out by this following

RESOLUTION.

1. For the number of days that *B* had travelled when he overtook *A*, put a

2. Then to find how many Miles *B* had travelled when he overtook *A*, there is an Arithmetical Progreſſion continued wherein the firſt and leaſt term is 1, (to wit, 1 Mile which *B* travelled the firſt day,) alſo the common difference is 1, (for the Queſtion ſaith that *B* travelled every day 1 Mile more than in the day next preceding,) and the number of terms is *a*, (which we aſſumed for the number of days that *B* had travelled when he overtook *A*;) therefore the ſum of all the terms (or number of Miles that *B* had travelled) will by the Canon of the preceding *Prop* 8. be found to be . . . . $\frac{1}{2}aa + \frac{1}{2}a$

3. And becauſe *A* travelled 8 (or *c*) Miles daily, and had travelled the ſame number of days as *B* when *B* overtook *A*, therefore 8 (or *c*) multiplied by *a* produces the number of Miles that *A* had then travelled; to wit, ... ca

4. But when *B* overtook *A*, each had travelled the ſame number of Miles; therefore the numbers found out in the two laſt ſteps muſt be equal the one to the other, *viz.* . . . . $\frac{1}{2}aa+\frac{1}{2}a = ca$

5. Which Equation after due Reduction gives . . . . $a = 2c - 1$

Which in words is this

CANON.

From the double of the number of Miles that *A* travelled daily, ſubtract 1 (or Unity,) ſo ſhall the Remainder be the number of days ſought.

Whence the number of days required will be found 15; for the double of 8 is 16, from which ſubtracting 1, the Remainder 15 is the number of days ſought; *viz. B* will overtake *A* at the end of 15 days, as will be evident by

The Proof

If 15 be the number of terms, and 1 the firſt (or leaſt) term, as alſo the common difference of the terms of an Arithmetical Progreſſion continued; the ſum of all the terms will (*per* Canon of *Prop.* 8.) be found 120, being the number of Miles which *B* had travelled in 15 Days, (according to the Progreſſion of 1 Mile the firſt Day, 2 Miles the ſecond, 3 Miles the third, *&c.*) Alſo, *A* travelling 8 Miles every day, would in 15 days have travelled 120 Miles. Therefore the conditions in the Queſtion are ſatisfied.

Queſt. 3. A Merchant diſcharged a Debt of 1370 *l* by ſeveral Payments made in this manner, *viz.* the firſt payment was $1\frac{1}{2}$ *l.* the ſecond payment exceeded the firſt by $\frac{1}{6}$ *l.* the third exceeded the ſecond by the ſame exceſs, and the reſt of the payments in like manner. The Queſtion is, to find how many payments the Merchant made in diſcharging the ſaid Debt? *Anſw.* 120, found out thus:

There is given in the Queſtion $1\frac{1}{2}$, to wit, the firſt and leaſt term of an Arithmetical Progreſſion continued; alſo $\frac{1}{6}$ the difference of the terms, and 1370 the ſum of all the terms, to find the number of terms, which (by Canon 1 of the foregoing *Prop.* 12. of this *Chap.*) will be found 120.

Queſt. 4. If a Debt of 1370 *l.* was diſcharged by ſeveral Payments made in ſuch manner, that the ſecond payment exceeded the firſt by $\frac{1}{6}$ *l.* the third the ſecond, the fourth the third, *&c.* in the ſame exceſs, *viz.* every following payment exceeded the next preceding by $\frac{1}{6}$ *l.* and that the laſt payment was $21\frac{1}{3}$ *l.* What was the firſt (to wit, the leaſt) Payment, and how many ſeveral Payments did the Debitor make? *Anſw.* The firſt and leaſt Payment was $1\frac{1}{2}$ *l.* (found out by the Canon 2. of *Prop.* 17.) and the number of Payments was 120, found out by the Canon of *Prop.* 18.

Queſt. 5. A Footman travelled 124 Miles in 8 Days at this rate, *viz.* The ſecond Days journey exceeded the firſt by 3 Miles, the third the ſecond by 3 Miles, and ſo forward in that exceſs; How many Miles was his firſt Days journey, and how many his laſt? *Anſw.* 5, and 26 Miles; found out by the Canons of *Prop.* 19 and 20.

Queſt. 6. A Draper bought 20 Cloths for 20 Crowns a piece, and ſold the firſt Cloth for a certain number of Crowns; the ſecond for two Crowns more than the firſt; the third for two Crowns more than the ſecond; and ſo by increaſing the price of every following Cloth by two Crowns more than the next preceding Cloth, he ſold the laſt Cloth for 41 Crowns. It is deſired to find the number of Crowns for which he ſold the firſt Cloth, and what he gained or loſt by all the Cloths.

This Queſtion implies an Arithmetical Progreſſion, whoſe number of Terms is 20; the common difference of the Terms is 2; and the laſt Term is 41: Therefore by the Canon of *Prop.* 13. of this *Chap.* the firſt and leaſt term will be found 3; and then by the Canon of *Prop.* 1. (or by the Canon of *Prop.* 14.) the ſum of all the terms will be found 440. Whence it is manifeſt that the Draper gained 40 Crown by the 20 Cloths; for he bought them for 400 Crowns, and ſold them for 440.

Queſt. 7. One diſtributed 456 Pence among a certain number of poor Perſons in this manner, *viz.* To the firſt he gave 6 Pence, to the laſt 51 Pence; the number of Pence given to the ſecond exceeded that given to the firſt, the third the ſecond, and ſo forward to the laſt by an equal exceſs. The Queſtion is, to find how many poor perſons there were; and how many Pence every one between the firſt and laſt received?

To ſolve this Queſtion, an Arithmetical Progreſſion muſt be conceived, whoſe firſt Term is 6; the laſt Term is 51; and the ſum of all the Terms 456: then by the Canon of *Prop.* 5. the number of Terms will be found 16; and by the Canon of *Prop.* 6. the common difference of the Terms will be found 3; wherefore there were 16 poor Perſons: and if this Arithmetical Progreſſion, to wit, 6, 9, 12, *&c.* be continued to the ſixteenth Term incluſive, it will ſhew the number of Pence which every one of the poor Perſons received; and all thoſe 16 Terms or Numbers being added together, make the given ſum 456.

Queſt. 8 A Stationer ſold 7 (or t) Reams of Paper, the particular prices whereof were certain numbers of Shillings in Arithmetical Progreſſion; the price of the ſecond Ream, that is, of that next above the cheapeſt, was 8 (or b) Shillings; and the price of the laſt or deareſt Ream was 23 (or c) Shillings: what was the price of each Ream?

RESOLUTION.

1. For the price of the cheapeſt or firſt Ream put	a	a
2. Then becauſe the price of the ſecond Ream was 8, (or b,) therefore by ſubtracting a from 8, (or b,) there remains the common difference of the Terms of the Progreſſion, *viz.*	$8-a$	$b-a$
3. Then by the help of the leaſt term, the common difference of the terms, and the number of terms, ſeek (by the Canon of *Prop.* 7. of this *Chap.*) the laſt and greateſt term, which will be found .	$48-5a$	$2a-ta+tb-b$

4. Which greateſt Term laſt found out muſt be equal to 23 (or c,) hence this Equation ariſes, *viz.*

$$48-5a=23;\quad \text{Or,}\quad 2a-ta+tb-b=c.$$

5. From which Equation after due Reduction this ariſes, *viz.*

$$a=5=\frac{tb-b-c}{t-2}.$$

Which in words is this

CANON.

From the Product of the price of the ſecond Ream of Paper (to wit, of that next above the cheapeſt, multiplied by the number of Reams, ſubtract the ſum of the prices of the ſecond and laſt Reams; then divide the Remainder by the exceſs of the number of Reams above 2: ſo ſhall the Quotient be the price of the firſt (or cheapeſt) Ream

Whence, by the help of the numbers given in the Queſtion, theſe following numbers in Arithmetical Progreſſion will be diſcovered, which ſolve the Queſtion, *viz.* 5, 8, 11, 14, 17, 20, 23.

Queſt. 9. One being asked what were the ſeveral ages of his five (or t) Children, anſwered, that the age of the eldeſt exceeded that of the ſecond by 2 (or x) Years; and by the ſame exceſs the ſecond exceeded the third, the third the fourth, the fourth the fifth or youngeſt Child's age; and if the age of the eldeſt Child were multiplied by the age of the youngeſt it would produce 128 (or c) Years. It's deſired to find out the age of every one of the five Children.

The numbers ſought by the Queſtion are in Arithmetical Progreſſion.

RESOLUTION.

1. For the age of the youngeſt Child (being the leaſt Term of the Arithmetical Progreſſion in the Queſtion,) put	a	a
2. Then by the help of a, x and t, *viz.* the age of the youngeſt Child, the common difference of their ages, and the number of Children, ſeek (by the Canon of *Prop.* 7. of this *Chap.*) the age of the eldeſt, that is, the greateſt Term of the Progreſſion, ſo you will find	$a+8$	$a+tx-x$
3. Therefore the Product of the multiplication of the firſt and laſt Terms of the Progreſſion is .	$aa+8a$	$aa+txa-xa$

4. Which Product muſt be equal to 128 (or c,) the Product given in the Queſtion; hence this Equation, *viz.* $aa+8a=128$; Or, $aa+txa-xa=c$.

5. Wherefore, by reſolving the laſt Equation according to the Canon in *Sect.* 6. *Chap.* 15. the value of a, that is, the age of the youngeſt Child will be diſcovered, *viz.*

$$a=8=\frac{\sqrt{:ttxx-2txx+xx+4c:}-tx-x}{2}.$$

Which in words is this

CANON.

From the Product of the number of Children multiplied into the common difference of their ages ſubtract the ſaid difference; then to the Square of the Remainder add four times the Product of the age of the eldeſt Child multiplied into the age of the youngeſt, and extract the ſquare Root of the ſum of that Addition: then from the ſaid ſquare Root ſubtract the Product of the common difference of their Ages multiplied into the exceſs of the number of Children above Unity; ſo the half of the Remainder ſhall be the age of the youngeſt Child.

Whence theſe five numbers are diſcovered, *viz.* 8, 10, 12, 14, 16; which ſhew the number of Years expreſſing the age of every one of the five Children: for the Product of the firſt and laſt numbers is 128, and the common difference is 2, as was required.

Queſt. 10. If the ſum of 6 (or t) numbers or terms in Arithmetical Progreſſion be 48 (or z,) and the Product of the common difference multiplied into the leaſt Term be equal to the number of Terms; what are the Numbers of that Progreſſion?

RESOLUTION.

1. For the common difference of the Terms put . .	a	a
2. Then according to the condition in the Queſtion, if the number of Terms be divided by the common difference, the Quotient is the leaſt Term, to wit,	$\frac{6}{a}$	$\frac{t}{a}$
3. Now by the help of the common difference, the leaſt Term, and the number of Terms, ſeek (by by eighth *Prop.* of this *Chap.*) the double ſum of all the Terms, ſo you will find . : . . : . .	$30a+\frac{72}{a}$	$tta+\frac{2tt}{a}-ta$

4. Which double Sum muſt be equal to twice 48, the Sum given in the Queſtion; hence this Equation ariſes, *viz.*

$$30a+\frac{72}{a}=96;$$

That is, . . $tta+\frac{2tt}{a}-ta=2z.$

5. Which Equation duly reduced gives

$$\frac{16}{5}a-aa=\frac{12}{5};$$

That is, . . $\frac{2z}{tt-t}a-aa=\frac{2t}{t-1}:$

6. Wherefore by reſolving the laſt Equation according to the Canon in *Sect.* 10. *Chap.* 15 the two values of a will be found theſe, *viz.*

$$a=2=\frac{z+\sqrt{:zz+2ttt-2tttt}}{tt-t}:$$

$$a=\frac{6}{5}=\frac{z-\sqrt{:zz+2ttt-2tttt}}{tt-t}:$$

7. Each of which values of a, to wit, 2 and $\frac{6}{5}$ may be taken for the common difference ſought. Then becauſe 6 is preſcribed in the Queſtion for the Product of the leaſt Term multiplied into the common difference, let 6 be divided by the ſaid 2 and $\frac{6}{5}$ ſeverally, and the Quotients 3 and 5 ſhall be the two leaſt Terms of two Arithmetical Progreſſions, each of which will ſolve the Queſtion: And therefore

The ſix numbers ſought may be either theſe, 3, 5, 7, 9, 11, 13;
Or theſe, ... 5, $6\frac{1}{5}$, $7\frac{2}{5}$, $8\frac{3}{5}$, $9\frac{4}{5}$, 11.

In each of which Progreſſions, the number of Terms is 6; the ſum of all the Terms is 48; and the common difference multiplied by the leaſt Term produces the number of Terms. Which was preſcribed in the Queſtion.

The End of the Firſt BOOK.

THE ELEMENTS OF THE Algebraical ART.

BOOK II.

CHAP. I.

Concerning the Geneſis or Production of Powers from Roots Binomial, Trinomial, &c.

I. I Shall take it for granted, that the Reader of this Second Book of Algebraical Elements is well exerciſed in the Firſt; and therefore without making any repetition of what has been there explained at large, I ſhall proceed to the handling of new matter in this Myſterious Art. Firſt then, foraſmuch as the Extraction of Roots is undoubtedly the hardeſt Leſſon in Vulgar Arithmetic, and the Reaſon of the Rules delivered in moſt Treatiſes of Arithmetic for extracting of the Square and Cubic Roots is known but to few practical Arithmeticians, I ſhall explain what our learned Divine and famous Mathematician Mr. *William Oughtred*, hath ſuccinctly delivered upon this Subject in the twelfth, thirteenth, and fourteenth Chapters of his Incomparable *Clavis Mathematicæ*; to which end in this and the following ſecond Chapters I ſhall firſt ſhew the Geneſis or Production of Powers from Roots Binomial, Trinomial, *&c.* and then in the third and fourth Chapters their Analyſis, or the Extraction of the Root or Side out of any given Power, whether it be expreſs'd by the Number or Letters.

II. If a Line or Number be divided into any two parts, ſuppoſe a the greater and e the leſſer, theſe connected by the Sign $+$ or $-$ do conſtitute a Binomial Root, as $a+e$ or $a-e$, the latter of which ſome call a Reſidual Root, becauſe it imports a Remainder, *viz.* the difference of the two Names or Parts of the Root. In like manner theſe Compound Quantities $a+b+c$, $a-b-c$; and the like, may be called Trinomial Roots, becauſe each of them conſiſts of three Names or Parts; and $a+b+c+d$ a Quadrinomial Root, that is, a Root conſiſting of four Parts: And ſo of others.

III. From a Root Binomial, Trinomial, *&c.* Algebraical Powers may be produced in like manner as from a ſimple Root, *viz.* by a continued Multiplication of the Root into it ſelf. As for Example: The Binomial Root $a+e$ being multiplied by it ſelf, that is, $a+e$ by $a+e$, produces $aa+2ae+ee$, the Square of $a+e$. Again, if the Square $aa+2ae+ee$ be multiplied by its Root $a+e$, the Product will be $aaa+3aae+3aee+eee$, which is the Cube of the Root $a+e$; and if the ſaid Cube be multiplied by its Root $a+e$, it will produce the fourth Power: and ſo you may proceed to find a fifth, ſixth, or what Power you pleaſe from the Binomial Root $a+e$. But for the greater evidence view the following Operation.

$$\begin{array}{ll}
\text{Binomial Root, .} & a+e \\
 & a+e \\ \cline{2-2}
 & aa+ae \\
 & \quad +ae+ee \\ \cline{2-2}
\text{Square, ...} & aa+2ae+ee. \\
 & a+e \\ \cline{2-2}
 & aaa+2aae+aee \\
 & \quad +\ aae+2aee+eee \\ \cline{2-2}
\text{Cube, ...} & aaa+3aae+3aee+eee \\
 & a+e \\ \cline{2-2}
 & aaaa+3aaae+3aaee+aeee \\
 & \quad +\ aaae+3aaee+3aeee+eeee \\ \cline{2-2}
\text{Biquadrate, . .} & aaaa+4aaae+6aaee+4aeee+eeee.
\end{array}$$

After the ſame manner, if the Reſidual Root $a-e$ be multiplied by it ſelf, the Product will be $aa-2ae+ee$ the Square of $a-e$. Again, if the Square $aa-2ae+ee$ be multiplied by its Root $a-e$, the Product will be $aaa-3aae+3aee-eee$, which is the Cube of the Root $a-e$. And ſo you may proceed to find a fourth, fifth, or what Power you pleaſe from the Reſidual Root $a-e$; view the following Work.

$$\begin{array}{ll}
\text{Reſidual Root . .} & a-e \\
 & a-e \\ \cline{2-2}
 & aa-ae \\
 & \quad -ae+ee \\ \cline{2-2}
\text{Square, ...} & aa-2ae+ee \\
 & a-e \\ \cline{2-2}
 & aaa-2aae+aee \\
 & \quad -aae+2aee-eee \\ \cline{2-2}
\text{Cube, ...} & aaa-3aae+3aee-eee. \\
 & a-e \\ \cline{2-2}
 & aaaa-3aaae+3aaee-aeee \\
 & \quad -aaae+3aaee-3aeee+eeee \\ \cline{2-2}
\text{Biquadrate, : .} & aaaa-4aaae+6aaee-4aeee+eeee.
\end{array}$$

By thoſe two Examples it is manifeſt, that the Powers from the Reſidual Root $a-e$ differ only in the Signs + and — from like Powers formed from the Binomial Root $a+e$; for in every Power of a Reſidual Root, the Signs prefix'd before the Parts or Members of the Power are alternately + and —; *viz.* the greateſt or firſt Member is Affirmative, the ſecond Negative, the third Affirmative, the fourth Negative, and ſo forwards: as you may ſee in the Cube of $a-e$, where aaa the greateſt extreme Member is Affirmative; the next Number in order being $-3aae$ is Negative; the third Member $+3aee$ is Affirmative; and the laſt (to wit, the leaſt) Member $-eee$ is Negative. But in every Power produced from a Binomial Root, whoſe Parts are connected by +, as $a+e$, all the Members of the Power are Affirmative.

IV. If according to the Conſtruction in the laſt preceding *Section* a Scale or Rank of Powers be formed from a Binomial Root, as from $a+e$, the Members of each Power to the tenth incluſive will be ſuch as you ſee in the following Table, where the two laſt Powers are compendiouſly expreſs'd according to *Carteſius* his way.

A Table of Powers produced from the Binominal Root $a+e$.

The Root.	a	e									
(2)	aa	$2ae$	ee								
(3)	aaa	$3aae$	$3aee$	eee							
(4)	$aaaa$	$4aaae$	$6aaee$	$4aeee$	$eeee$						
(5)	$aaaaa$	$5aaaae$	$10aaaee$	$10aaeee$	$5aeeee$	$eeeee$					
(6)	$aaaaaa$	$6aaaaae$	$15aaaaee$	$20aaaeee$	$15aaeeee$	$6aeeeee$	$eeeeee$				
(7)	$aaaaaaa$	$7aaaaaae$	$21aaaaaee$	$35aaaaeee$	$35aaaeeee$	$21aaeeeee$	$7aeeeeee$	$eeeeeee$			
(8)	$aaaaaaaa$	$8aaaaaaae$	$28aaaaaaee$	$56aaaaaeee$	$70aaaaeeee$	$56aaaeeeee$	$28aaeeeeee$	$8aeeeeeee$	$eeeeeeee$		
(9)	a^9	$9a^8e$	$36a^7ee$	$84a^6e^3$	$126a^5e^4$	$126a^4e^5$	$84a^3e^6$	$36aae^7$	$9ae^8$	e^9	
(10)	a^{10}	$10a^9e$	$45a^3ee$	$120a^7e^3$	$210a^6e^4$	$252a^5e^5$	$210a^4e^6$	$120a^3e^7$	$45aae^8$	$10ae^9$	e^{10}

V. By the foregoing Table it is evident, that the Square of $a+e$ conſiſts of $aa+2ae+ee$; which ſhews, that if a Number be divided into any two parts, the Square of that Number ſhall be equal to the Squares of the parts, and to twice the Product made by the Multiplication of the parts one into the other; as if 12 be divided into 10 and 2, which may be ſignified by a and e, then

The Square of 10 is	100	aa
Product of 10 multiplied by 2 is 20, which doubled makes . .	40	$2ae$
The Square of 2 is	4	ee
Which three Numbers, to wit, 100, 40, and 4, added together make the Square of 12, *viz.*	144 $= aa+2ae+ee$.	

In like manner the ſaid Table ſhews, that the Cube or third Power of the Binominal Root $a+e$ conſiſts of the Cubes of the Names or Parts of the Root a and e, together with the triple of the ſolid Product made by the Multiplication of the Square of the greater part a into the leſſer part e, and the triple of the ſolid Product made by the Multiplication of the greater part a into the Square of the leſſer part e. This may be illuſtrated by Numbers thus: Suppoſe 12 to be divided into 10 and 2, which may (as before) be repreſented by a and e; then the Cube of 12 or of $a+e$, will be equal to the Sum of theſe four ſolid Numbers, *viz.*

The Cube of 10 is	1000	aaa
The Square of 10 is 100, which multiplied by 2 produces 200, this tripled makes	600	$3aae$
Again, 10 multiplied by 4 the Square of 2 produces 40, the Triple whereof is	120	$3aee$
The Cube of 2 is	8	eee

Which four Numbers, *viz.* 1000, 600, 120, and 8, added together make the Cube of 12, (or 12×12×12) that is . $1728=aaa+3aae+3aee+eee$

After the ſame manner the reſt of the Powers in the Table might be expreſs'd by Words. Whence 'tis evident, that this literal Method diſcovers many Properties in Powers, which in Numeral Calculations do lie in obſcurity.

VI. Moreover, by a bare Inſpection into the ſaid Table it may be perceived, that the Number prefix'd to every one of the mean Members of every Power produced from the Binomial Root $a+e$, is compoſed of the two Numbers prefix'd to the next ſuperiour and inferiour Members of the next preceding Power. As for example: If you conceive the Line upon which $3aae$ is ſet to be continued forth at length, it will paſs between aa, that is, $1aa$ and $2ae$, in the foregoing ſecond Power (or Square.) Now I ſay that the number 3 prefix'd to aae is the ſum of 1 and 2 the Numbers prefix'd to aa and ae. Likewiſe the number 6 prefix'd to $aaee$, one of the Members of the fourth Power, is compoſed of 3 and 3, the Numbers prefix'd to aae and aee in the third Power. Again, the number 15 prefix'd to $aaaaee$ is the ſum of 5 and 10, the Numbers prefix'd to $aaaae$ and $aaaee$ in the fifth Power. Hence a Table may be made to ſhew what Numbers are to be prefix'd to the mean Numbers of every Power.

A

2	For the Square.
3 . 3	For the Cube.
4 . 6 . 4	For the fourth Power.
5 . 10 . 10 . 5	For the fifth Power.
6 . 15 . 20 . 15 . 6	For the ſixth Power.
7 . 21 . 35 . 35 . 21 . 7	For the ſeventh Power.
8 . 28 . 56 . 70 . 56 . 28 . 8	For the eighth Power.
9 . 36 . 84 . 126 . 126 . 84 . 36 . 9	For the ninth Power.
10 . 45 . 120 . 210 . 252 . 210 . 120 . 45 . 10	For the tenth Power.

B C

In this Table the Numbers from *A* to *B*, and likewiſe from *A* to *C*, do proceed from 2 in an Arithmetical Progreſſion, having 1 (to wit, Unity) for a common difference; and every one of the mean Numbers ſtanding between the ſame Term of each Progreſſion, is compoſed of the two Numbers which ſtand next above each mean Number reſpectively: As 6, which ſtands between 4 and 4, is the Sum of 3 and 3, which ſtand above and on each ſide of 6: likewiſe 10, which is ſet between 5 and 5, is the Sum of 6 and 4 which ſtand above 10; and ſo of the reſt. So that this Table may be eaſily continued further at pleaſure.

VII. Any Power of a Binominal or Reſidual Root expreſs'd by Letters, may without a continued Multiplication of the Root into it ſelf be eaſily formed by the following Method, which is deduced from the Premiſes, *viz.* Suppoſe the fifth Power of the Bi-

Binominal Root $a+e$ be deſired; Firſt, I write all the ſimple Powers of a, deſcending orderly from the fifth Power downwards to the Root a; as $aaaaa$, $aaaa$, aaa, aa, and a, as here you ſee in the firſt Columel: then to all thoſe Powers, except the uppermoſt $aaaaa$, I joyn ſuch ſimple Powers of e, that the Sum of the Indices of both Powers may make 5, *viz.* To $aaaa$ I joyn e; to aaa, ee, to aa, eee; and to a, $eeee$; then I write $eeeee$ underneath; ſo that there are ſix diſtinct Members or Terms, every one of which conſiſts of five Dimenſions, as you ſee in the ſecond Columel. That done, by the Table in the foregoing *Sect.* 6. I find that the Numbers 5, 10, 10, and 5 are to be prefix'd before the mean Members of the fifth Power; and accordingly I ſet 5 before $aaaae$, 10 before $aaaee$, likewiſe 10 before $aaeee$, and 5 before $aeeee$; laſtly, by prefixing $+$, or ſuppoſing it to be prefix'd before every one of the ſaid five Members, the fifth Power of the Binominal Root $a+e$ is compleated, as you ſee in the third Columel, and in every reſpect agrees with the fifth Power in the Table in the foregoing *Sect.* 4. But if the Signs $+$ and $-$ be alternately prefix'd before the Members of the ſaid fifth Power, according to what has been ſaid at the latter end of *Sect.* 3. it will be the fifth Power of the Reſidual Root $a-e$.

(1)	(2)	(3)
$aaaaa$	$aaaaa$	$aaaaa$
$aaaa$	$aaaae$	$5aaaae$
aaa	$aaaee$	$10aaaee$
aa	$aaeee$	$10aaeee$
a	$aeeee$	$5aeeee$
	$eeeee$	$eeeee$

VIII. Laſtly, from a Root conſiſting of three, four, or any number of parts, the Square, Cube, or any higher Power of the Root may be produced by a continued Multiplication of the Root into it ſelf: As the Trinomial Root $a+b+c$ being multiplied by it ſelf, its Square will be found $aa+2ab+2ac+bb+2bc+cc$; and this Square multiplied again by its Root $a+b+c$ produces the Cube of the ſame Root, that is, $aaa+3aab+3aac+3abb+6abc+3acc+bbb+3bbc+3bcc+ccc$. After the ſame manner Powers may be produced from a Root conſiſting of four, or any Number of Parts. And if the Conſtitution of Powers expreſs'd by Letters be ſeriouſly conſidered, it will be ſome help to diſcover whether an Algebraic Quantity conſiſting of more than three Members or Terms be a perfect Power or not, and alſo give ſome light to diſcover its Root.

CHAP. II.

Concerning the Compoſition of Powers in Numbers from a Binominal Root.

Sect. I. *Of the Compoſition of a Square from a Number given for the Side or Root.*

1. SUppoſe the Square of the Root 28 be deſired: Firſt, write down the Root 28 in ſuch manner that there may be ſpace enough to ſet one Figure between 2 and 8, and let a Line be drawn under them; as alſo two downright Lines, the one next after 2, and the other after 8, to the end the Numbers which are to be found out may be orderly placed for Addition: then let the Root 28 be conceived to be divided into theſe two parts 20 and 8, and let a be put for the greater part, and e for the leſſer. Now foraſmuch as the Square of $a+e$ is $aa+2ae+ee$, therefore the Square of 28, or of $20+8$ may be compoſed thus, *viz.* The Square of 20 is 400, (or aa;) the double of 20 is 40, (or $2a$) which multiplied by 8 (or e) produces 320, (that is, $2ae$;) and the Square of 8 is 64 (or ee.) Laſtly, the ſaid three Numbers 400, 320, and 64, being ſet under one ano-

$a=20$
$e=8$

2	8	Root propoſed.
4	00	aa
3	20	$2ae$
	64	ee
7	84	Square required.

another in ſuch order, that Units may ſtand under Units, Tens under Tens, &c. and added together the Sum makes 784, the Square of the Root 28; as may eaſily be proved by multiplying 28 into it ſelf.

2. When the given Number or Root whoſe Square is deſired conſiſts of three or more places, as 47803; firſt, the Square of the two foremoſt Figures towards the left Hand, that is, of 47, muſt be found out in like manner as before in the firſt Example, ſo there will be produced 2209 for the Square of 47, as you ſee in the following Example 2. Secondly, write 47 in a void place, and annex a Cypher to it, ſo it makes 470, this Number muſt now be eſteemed a, and 8 the next following Character of the Root muſt be taken for e; and then according to theſe values of a and e the Numbers ſignified by aa, $2ae$, and ee, being added together make 228484 for the Square of 478, (as you ſee here underneath.) Where obſerve, that to find the Square of 470 (that is, of a) you need only annex two Cyphers to 2209, which was before found for the Square of 47. Thirdly, annex a Cypher to 478 in a void place, and it makes 4780 for a new Value of a, and the next following Character of the Root, to wit 0, is the new Value of e, then according to theſe Values of a and e, the Value of $aa+2ae+ee$ is 22848400, to wit aa only; for $e=0$, and conſequently $2ae+ee=0$: ſo the ſaid 22848400 is found for the Square of 4780. Laſtly, by annexing a Cypher to 4780 it makes 47800 for a new Value of a, and 3 the laſt Figure of the Root is the new Value of e; then according to theſe Values of a and e the Sum of the Numbers ſignified by aa, $2ae$, and ee, makes 2285126809, which is the Square of the ſaid given Root 47803, as may eaſily be proved by multiplying the ſaid Root by it ſelf. Compare the following Example with the precedent Directions.

Example 2. *of* Sect. I.

	4	7	8	0	3	Root propoſed.
$a=40$	16	00				aa
$e=7$	5	60				$2ae$
		49				ee
$a=470$	22	09	00			aa
$e=8$		75	20			$2ae$
			64			ee
$a=4780$	22	84	84	00		aa
$e=0$				00		$2ae$
				00		ee
$a=47800$	22	84	84	00	00	aa
$e=3$			28	68	00	$2ae$
					09	ee
	22	85	12	68	09	Square required.

Sect. II. *Of the Compoſition of a Cube from a Number given for the Side or Root.*

1. Let the Cube of the Root 28 be deſired: Firſt, I write the Root 28 in ſuch manner, that there may be ſpace enough to ſet two Figures between 2 and 8; then having drawn a Line under 28, and downright Lines as before in the Square, I conceive the Root 28 to be divided into 20 and 8, that is, a and e. Now foraſmuch as the Cube of $a+e$ is compoſed of theſe four Members, *viz.* aaa, $3aae$, $3aee$, and eee, (as appears by the Table in *Sect.* 4. *Chap.* 1.) therefore the Cube of 20+8 (that is, of 28) may be compoſed thus, *viz.* Fitſt, the Cube of 20 is 8000, (that is, aaa.) Secondly, the triple of the Square of 20 being mul-

	2	8	Root propoſed.
$a=20$	8	000	aaa
$e=8$	9	600	$3aae$
	3	840	$3aee$
		512	eee
	21	952	Cube deſired.

multiplied by 8 produces 9600, (that is, $3aae$;) thirdly, the triple of 20 being multiplied by the Square of 8 produces 3840, (that is, $3aee$;) fourthly, the Cube of 8 is 512, (that is, eee; laſtly, the ſaid four Numbers 8000,9600,3840,512, being ſet under one another in ſuch order that Units may ſtand under Units, Tens under Tens, &c. and added together make 21952, the Cube of the given Root 28.

2. When the given Number or Root whoſe Cube is deſired conſiſts of three or more places, as 28503; Firſt, the Cube of the two foremoſt Figures, that is, of 28, muſt be found out in like manner as before in *Example* 1. ſo there will be produced 21952. Secondly, write 28 in a void place, and annexing a Cypher to it, it makes 280, this Number muſt now be eſteemed a, and 5 the next following Character of the Root muſt be taken for e; then according to theſe values of a and e the Numbers ſignified by aaa, $3aae$, $3aee$, and eee, being added together make 23149125 for the Cube of 285, (as you ſee in *Example* 2.) where obſerve that to find the Cube of 280, that is, of a, you need only annex three Cyphers to 21952, which was before found for the Cube of 28. Thirdly, annex a Cypher to 285 after it is ſet in a ſpare place, and it makes 2850 for a new value of a, and the next following Character of the Root, to wit, 0, is the new value of e: Then according to theſe values of a and e, the value of $aaa+3aae+3aee+eee$ is 23149125000, that is, aaa only; for $e=0$, and conſequently $3aae+3aee+eee=0$, ſo the ſaid 23149125000 is found for the Cube of 2850. Laſtly, by annexing a Cypher to 2850 it makes 28500 for a new value of a, and 3 the laſt Figure of the Root is the new value of e; then according to theſe values of a and e the Sum of the Numbers ſignified by aaa, $3aae$, $3aee$, and eee, makes 23156436019527, which is the Cube of the given Root 28503; as may eaſily be proved by multiplying the ſaid Root into it ſelf Cubically. Compare the following Example with the precedent Directions.

Example 2. *of* Sect. II.

	2	8	5	0	3	Root propoſed.
$a=20$	8	000				aaa
$e=8$	9	600				$3aae$
	3	840				$3aee$
		512				eee
$a=280$	21	952	000			aaa
$e=5$	1	176	000			$3aae$
		21	000			$3aee$
			125			eee
$a=2850$	23	149	125	000		aaa
$e=0$				000		$3aae$
				000		$3aee$
				000		eee
$a=28500$	23	149	125	000	000	aaa
$e=3$		7	310	250	000	$3aae$
				769	500	$3aee$
					27	eee
	23	156	436	019	527	Cube deſired.

Sect. III. *Of the Compoſition of a Biquadrate, or the fourth Power, from a Number given for the Root.*

1. Let the Root 28 be propoſed, and its Biquadrate or fourth Power deſired. Firſt, I write the Root 28 in ſuch manner that there may be ſpace enough to ſet three Figures between 2 and 8; then having drawn a Line under 28, and downright Lines, as in former Examples, I conceive the Root 28 to be divided into 20 and 8, that is, a and e; now foraſmuch as the Biquadrate, or fourth Power produced from the Binomial Root $a+e$ is $aaaa+4aaae+6aaee+4aeee+eeee$, (as appears by the Table in *Sect.* 4. *Chap.* 1.) therefore the fourth Power of 20+8, (that is, of 28) may be compoſed thus, *viz.* Firſt, the fourth Power of 20 is 160000, (that is $aaaa$;) ſecondly, four times the Cube of 20 being multiplied by 8 produces 256000, (that is, $4aaae$;) thirdly, ſix times the Square of 20 being multiplied by the Square of 8 produces 153600, (that is, $6aaee$,) fourthly, four times 20 multiplied by the Cube of 8 produces 40960, that is, $4aeee$; fifthly, the fourth Power of 8 is 4096, (that is, $eeee$;) laſtly, the Sum of all the ſaid five Numbers, to wit, 160000, 256000, 153600, 40960, and 4096 makes 614656, which is the fourth Power of 28 the Root propoſed; as will eaſily appear by the Multiplication of 28 four times into it ſelf.

	2	8	Root propoſed.
$a=20$	16	0000	$aaaa$
$e=8$	25	6000	$4aaae$
	15	3600	$6aaee$
	4	0960	$4aeee$
		4096	$eeee$
	61	4656	Biquadrate deſired.

2. When the given Number or Root whoſe fourth Power is deſired conſiſts of three Places, as 285; Firſt, the fourth Power of the two foremoſt Figures 28 muſt be found out, in like manner as in Example 1. of this *Sect* ſo there will be produced 614656 for the fourth Power of 28. Secondly, let 28 be ſet in void place, and annex a Cypher to it, ſo it makes 280, which muſt now be eſteemed a, and 5 the next following Character of the Root muſt be taken for e; and then according to theſe values of a and e the Numbers ſignified by $aaaa$, $4aaae$, $6aaee$, $4aeee$, and $eeee$ being added together make 6597500625, which is the fourth Power of the given Root 285 and the work will ſtand as you ſee in the following Example 2. After the ſame manner the work is to be continued when the given Root conſiſts of more than three places, as is manifeſt by the following Example 3.

Example 2. *of* Sect. III.

	2	8	5	Root propoſed.
$a=20$	16	0000		$aaaa$
$e=8$	25	6000		$4aaae$
	15	3600		$6aaee$
	4	0960		$4aeee$
		4096		$eeee$
$a=280$	61	4656	0000	$aaaa$
$e=5$	4	3904	0000	$4aaae$
		1176	0000	$6aaee$
		14	0000	$4aeee$
			625	$eeee$
	65	9750	0625	Biquadrate required.

Example 3. *of* Sect. III.

	2	8	0	5	Root propoſed.
$a=20$	16	0000			*aaaa*
$e=8$	25	6000			4*aaae*
	15	3600			6*aaee*
	4	0960			4*aeee*
		4096			*eeee*
$a=280$	61	4656	0000		*aaaa*
$e=0$			0000		4*aaae*
			0000		6*aaee*
			0000		4*aeee*
			0000		*eeee*
$a=2800$	61	4656	0000	0000	*aaaa*
$e=5$		4390	4000	0000	4*aaae*
		11	7600	0000	6*aaee*
			140	0000	4*aeee*
				625	*eeee*
	61	9058	1740	0625	Biquadrate deſired.

Sect. IV. *Of the Compoſition of the fifth Power from a Number given for its Root.*

1. Let the Root 28 be propoſed, and its fifth Power deſired : Firſt, let the Root 28 be written in ſuch manner, that there may be ſpace enough to ſet 4 Figures between 2 and 8; then having drawn a Line under 28, and downright Lines, as in the Examples of the precedent Section, let 28 be conceived to be divided into 20 and 8, that is, *a* and *e*; now foraſmuch as the fifth Power produced from the Binomial Root $a+e$ is *aaaaa*+5*aaaae*+10*aaaee*+10*aaeee*+5*aeeee*+*eeeee*, (as is manifeſt by the Table in *Sect.* 4. *Chap.* 1.) Therefore the fifth Power of 20+8 (that is, of 28) may be compoſed thus; Firſt, the fifth Power of 20 is 3200000, (that is, *aaaaa*;) ſecondly, five times the fourth Power of 20 being multiplied by 8 produces 6400000, (that is, 5*aaaae*;) thirdly, ten times the Cube of 20 being multiplied by the Square of 8 produces 5120000, that is, 10*aaaee*;) fourthly, ten times the Square of 20 multiplied by the Cube of 8 produces 2048000, (that is, 10*aaeee*;) fifthly, five times 20 multiplied by the fourth Power of 8 produces 409600, (that is, 5*aeee*;) ſixthly, the fifth Power of 8 is 32768, (that is, *eeeee*; laſtly, the Sum of all thoſe ſix Numbers, *viz.* 3200000, 6400000, 5120000, 2048000, 409600, and 32768 makes 17210368, which is the fifth Power of 28 the Root propoſed, as will eaſily appear by multiplying 28 five times into it ſelf.

	2	8	
$a=20$	32	00000	*aaaaa*
$e=8$	64	00000	5*aaaae*
	51	20000	10*aaaee*
	20	48000	10*aaeee*
	4	09600	5*aeeee*
		32768	*eeeee*
	172	10368	

2. When the given number or Root, whoſe fifth Power is deſired, conſiſts of three places, as 285; Firſt, the fifth Power of the two foremoſt Figures 28 muſt be found out in like manner as in Example 1. of this *Sect.* ſo there will be produced 17210368 for the fifth Power of 28. Secondly, let 28 be ſet in a void place, and annex a Cypher to it, ſo it makes 280, which muſt now be eſteemed *a*, and 5 the next following Character of the Root muſt be taken for *e*; then according to theſe values of *a* and *e* the Numbers ſignified by *aaaaa*, 5*aaaae*, 10*aaaee*, 10*aaeee*, 5*aeeee*, and *eeeee*, being added together make 1880287678125, which is the fifth Power of the given Root 285, and the work will ſtand as you ſee in the following Example 2. Nor will the Operation be more difficult (though more laborious) to find the fifth Power of a Number (or Root) conſiſting of four or more places.

Example 2. *of* Sect. IV.

	2	8	5	Root propoſed.
$a=20$	32	00000		*aaaaa*
$e=8$	64	00000		5*aaaae*
	51	20000		10*aaaee*
	20	18000		10*aaeee*
	4	09600		5*aeeee*
		32768		*eeeee*
$a=280$	172	10368	00000	*aaaaa*
$e=5$	15	36640	00000	5*aaaae*
		54880	00000	10*aaaee*
		980	00000	10*aaeee*
		8	75000	5*aeeee*
			3125	*eeeee*
	188	20876	78125	Fifth Power deſired.

By the Precedent Rules and Examples of this Chapter, the Ingenious Reader will eaſily apprehend how to compoſe the ſixth, ſeventh, or any higher Power, from a Root given in Number, and conſidered as a Binomial $a+e$, as before hath been directed. The main Buſineſs conſiſting in a right underſtanding of the Number ſignified by a and e, and in finding out the Numbers anſwering to the Members of the deſired Power of $a+e$, according to the Table in *Sect.* 4. of the precedent *Chap.* 1.

CHAP. III.

Concerning the Reſolution of Powers expreſt by Numbers, or the Extraction of all kinds of Roots out of Powers given in Numbers.

Sect. I. *Of the Extraction of the Square Root out of a Number given.*

1. LEt it be obſerved in general, that the Reſolution of every Power given in Numbers conſiſts in a Regular Subtraction of thoſe Numbers which are ſuppoſed to be added together in the Compoſition of each Power reſpectively, according to the Rules of the laſt preceding Chapter, wherein I preſuppoſe the Reader to be well exerciſed. And for the more ready Extraction of any Root, it will be convenient to have in a readineſs the reſpective Powers of the nine ſingle Figures; as if the Square Root be deſired, then the Squares of 1, 2, 3, 4, 5, 6, 7, 8, 9, will be uſeful, which Roots and Squares are expreſt in the following Tabulet.

ROOTS.	1	2	3	4	5	6	7	8	9
SQUARES.	1	4	9	16	25	36	49	64	81

2. When a whole Number is propoſed, and its Square Root deſired, the Number propoſed muſt be prepared for Extraction, by diſtributing it into parts or members after this manner, *viz.* Firſt, ſet a point over the firſt or Units place of the given Number, then paſſing over the ſecond place ſet another Point over the third; alſo paſſing over the fourth place ſet another Point over the fifth: and in that order if there be more places in the given Number, Points are to be ſet, ſo that between every two Points which ſtand next to one another, there will be one place without any Point over it. As for Example: If the Square Root of 119025 be deſired, I ſet Points as here you ſee, whereby the ſaid Number is diſtributed into 3 Members, to wit 11,90,25. In like manner if the Square Root of 785 be deſired, the

$1\dot{1}9\dot{0}2\dot{5}$

the Points will ſtand as you ſee here, whereby the ſaid 784 is diſtributed into two Members 7 and 84. The Points ſet as aforeſaid ſhew the number of places that will be found in the Root; for if there be two Points, there will be two places in the Root; if three Points, then the Root will conſiſt of three places, &c. The Points alſo ſhew what Member of the Number given belongs to the finding out of every ſingle Character of the Root ſought, as is evident by the Rules in *Sect.* 1. of the precedent *Chap.* 2. Theſe things being premiſed as preparatory to the Extraction of the Square Root, I ſhall proceed to Examples.

$\dot{7}\ 8\ \dot{4}$

Example 1.

3. Let it be required to extract the Square Root of 784. By the Preceding Rule 2. it is evident that the deſired Root conſiſts of two places, *viz.* of ſome number of Tens under 100, and of ſome number of Units under 10; which two numbers (agreeable to the compoſition of a Square in *Sect.* 1. of the ptecedent *Chap.* 2.) may be repreſented by *a* and *e*, ſo that *a* and *e* ſignifie the Root ſought; and conſequently the Square of $a+e$, that is, $aa+2ae+ee$ is equal to the propoſed Number 784. Now to find out the number of Tens, (that is, *a*) in the Root; (after a crooked Line is drawn on the right hand of the given Number, that the Root, like the Quotient in Diviſion, may be ſet next after the ſaid crooked Line, as alſo a downright Line next after each of the Points, as here you ſee;) the firſt work in the Extraction is always to ſubtract the greateſt Square whole Number contained in the firſt Member towards the left hand from the ſaid Member, and to write the Root of the ſaid ſquare Number in the Quotient for the firſt ſingle Figure of the deſired Root: ſo 4 being the greateſt Square contained in the firſt Member 7, I ſubſcribe 4 under 7, and ſet 2 the Root of the ſaid 4 in the Quotient, then after a Line is drawn under 4, I ſubtract 4 from 7, or 400 from 784, and there remains the *Reſolvend* 384, that is, that part of the given Number 784, which is yet to be reſolved. Now obſerve, that the ſaid 2 in the Quotient, in reſpect of the next following unknown Character of the Root, is really 20, which is the Number ſignified by *a* in the Compoſition; and the Square of 20, to wit 400, is *aa*, which being the firſt Number found in the Compoſition, is the firſt Number to be ſubtracted in the Reſolution. Obſerve alſo, that the next ſingle Character of the Root, whither it happen to be a Figure or a Cypher, is called *e*, which is yet unknown.

$\dot{7}$	$8\dot{4}$	(2
4		
3	84	

4. Then I proceed to find the value of *e*, that is, the greateſt ſingle Character with this Condition, that the ſum of the Numbers ſignified by $2ae$ and ee may not excced the *Reſolvend* 384; for from this Number that ſum muſt be ſubtracted. Now becauſe (for the reaſon aforeſaid) *a* is 20, therefore $2a$ is 40, which muſt be eſteemed a *Diviſor*, and ſet under the *Reſolvend*; then I divide the ſaid *Reſolvend* 384 by 40, and find the Quotient 9 for the Number *e*, provided it will anſwer the Condition before mentioned; and therefore I make Tryal (in a waſt Paper) to ſee whether 9 will ſatisfie the ſaid Condition or not in this manner, *viz.* If *e* be 9, and $2a$ 40, then conſequently $2ae$ is 360, and ee is 81; therefore $2ae+ee=441$, this ought to be ſubtracted from the *Reſolvend* 384; but 441 exceeds 384, and therefore cannot be ſubtracted from it, ſo as to leave a real Remainder; whence I conclude, that *e* muſt be leſs than 9: and therefore I make tryal with 8 in like manner as before with 9, *viz.* If $e=8$ and $2a=40$, then conſequently $2ae=320$, and $ee=64$; therefore $2ae+ee=384$, which may be ſubtracted from the *Reſolvend* 384; wherefore I conclude that *e*, that is the Figure which muſt follow 2 in the Quotient) is 8, which I ſet in the Quotienr: then I ſubſcribe 320 and 64 (before found) under the *Reſolvend* 384, (in ſuch order that Units may ſtand under Units, and Tens under Tens) and adding the ſaid 320 and 64 together, the ſum is 384, which ſome Authors call the *Gnomen*, others, the *Ablatitium*) which ſubtracted from the *Reſolvend* 384 leaves 0; ſo the whole Extraction is finiſh'd,

	$\dot{7}$	$8\dot{4}$	(28
Subtract	4	00	*aa*
	3	84	2*Reſolvend*
$a=20$		40	$2a$ *Diviſor*
$e=\ 8$	3	20	$2ae$
		64	ee
Subtract	3	84	*Ablatitium*
		00	

and the Square Root of the given Number 784 is found 28, which is the true Root ſought, for 28 multiplied by 28 produces 784.

NOTE 1.

The firſt Operation in the Extraction of the Square Root is always to ſubtract the greateſt ſquare whole Number, (that is, *aa*) contained in the firſt Member (towards the left hand) of the given number, from the ſaid Member, and to ſet the Root of the ſaid Square in the Quotient, (as has been ſhewn in the third ſtep) which Root is the firſt Figure of the Root ſought. This Work is no more repeated in the whole Extraction, but the work in the fourth ſtep is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

After the firſt Figure of the Root ſought is known, and ſet in the Quotient, let it be written in a void place, and multiplied by 10, (by annexing to the ſaid firſt Figure a Cypher towards the right hand) then is the Product to be taken for the value of *a*, in order to the finding out of the firſt *Diviſor*. Alſo when the firſt and ſecond Characters of the Root are ſet in the Quotient, and there be yet another to come forth, then the Number conſiſting of thoſe two Characters with a Cypher annexed to them, is to be taken for a new value of *a*, in order to the finding out of the ſecond *Diviſor*. Likewiſe, when the firſt, ſecond, and third Characters of the Root are ſet in the Quotient, and there be yet another to come forth, then the number conſiſting of thoſe three Characters with a Cypher annexed to them, is to be taken for a new value of *a*; and ſo forwards, when there be more Characters in the Root. The reaſon of which Work is manifeſt from the Compoſition of Powers in the precedent *Chap.* 2.

But the Letter *e* repreſents every ſingle unknown Figure or Cypher next following that part of the Root which is already diſcovered and ſet in the Quotient. This Note concerning the Eſtimation of *a* and *e* is to be obſerved not only in the Extraction of the Square Root, but of any Root whatever.

NOTE 3.

After the Number ſignified by *a* is found out by Note 2. the *Diviſor*, which ſhews how to begin the Tryal in ſearching out the unknown ſingle Character repreſented by *e*, is conſequently known: for in the Reſolution of every Power produced from the Binomial Root $a+e$, the *Diviſor* conſiſts of ſuch Powers of *a* as are multiplied into the Powers of *e*; and becauſe the Square Root of $a+e$ is $aa+2ae+ee$, therefore in the Extraction of the Square Root the *Diviſor* is $2a$; ſo that when the Number *a* is known, the *Diviſor* $2a$ is conſequently known.

NOTE 4.

When the *Diviſor* is found out by Note 3. as alſo the *Ablatitium*, (that is, the Number to be ſubtracted) which in the Extraction of the Square Root is compos'd of $2ae$ and ee, the two numbers ſignified by $2ae$ and ee muſt each of them be ſet in ſuch order under the preſent *Reſolvend*, (that is, the number remaining to be reſolved) that Units may ſtand under Units, Tens under Tens, &c. to the end that the *Ablatitium* may be rightly compoſed and ſubtracted from the preſent *Reſolvend*.

NOTE 5.

When the *Diviſor* is not contained once in the particular or preſent *Reſolvend*, a Cypher (to wit, 0) muſt be ſet in the Quotient; and then the *Reſolvend* muſt be augmented with the next Member (towards the right hand) of the Power propoſed, for a new particular *Reſolvend*. Alſo a new *Diviſor* muſt be found out by Note 3, and the like is to be done as often as the *Diviſor* is not contained once in the particular *Reſolvend*. The Practice of theſe Notes will be ſhewn in the following Example.

Example 2.

5. If the Square Root of 2285126809 be deſired, it will be found 47803 by the precedent Rules, and the work will ſtand as here you ſee underneath.

	22	85	12	68	09	(47803. *Root.*
Subtract	16					aa
	6	85				*Reſolvend.*
$a = 40$		80				$2a$. *Diviſor.*
$e = 7$	5	60				$2ae$
		49				ee
Subtract	6	09				*Ablatitium.*
		76	12			*Reſolvend.*
$a = 470$		9	40			$2a$. *Diviſor.*
$e = 8$		75	20			$2ae$
			64			ee
Subtract		75	84			*Ablatitium.*
			28	68		*Reſolvend.*
$a = 4780$			95	60		$2a$. *Diviſor.*
$e = 0$			28	68	09	*Reſolvend.*
$a = 47800$			9	56	00	$2a$. *Diviſor.*
$e = 3$			28	68	00	$2ae$
					9	ee
Subtract			28	68	09	*Ablatitium.*
			00	00	00	

Explication of Example 2.

The firſt Figure of the Root is 4, (by the foregoing Note 1.) whoſe Square 16 ſubtracted from 22 the firſt Member towards the Left-hand of the number propoſed leaves 6, to which the ſecond Member 85 being annexed, there ariſes 685 for the next *Reſolvend* : Or to cauſe the ſame Effect, ſuppoſe 0 to be annexed to 4 the firſt Figure of the Root, and it makes 40, (that is, a,) whoſe Square 1600 (or aa) ſubtracted from 2285 the two firſt Members of the Number firſt propoſed, leaves (as before) the *Reſolvend* 685.

Then, the firſt Figure of the Root being found 4, the value of a is 40, (by Note 2.) which doubled gives 80 for a Diviſor to the *Reſolvend* 685 by Note 3.) and then by dividing and making Tryal as is directed in the precedent fourth ſtep, the number e will be found 7 for the ſecond Figure of the Root, and conſequently the numbers ſignified by $2ae$ and ee are 560 and 49 ; theſe being ſet orderly and added together (according to Note 4.) make the *Ablatitium* 609, which ſubtracted from the ſaid *Reſolvend* 685, there remains 76, to which annexing 12 the third Member of the Number firſt propoſed, it makes 7612 for a new *Reſolvend.*

Again, the two formoſt figures of the Root being found 47, the new value of a is 470, (by Note 2.) which doubled gives 940 for a Diviſor to the ſaid *Reſolvend* 7612, (by Note 3.) then by dividing and making Tryal as is directed in the fourth ſtep, the value of e is found 8 for the firſt Figure of the Root ; whence the number ſignified by $2ae$ and ee are 7520 and 64 ; theſe being ſet orderly and added together (according to Note 4.) make the *Ablatitium* 7584, which ſubtracted from the *Reſolvend* 7612 before-mentioned, leaves 28, to which annexing 68 the fourth Member of the Number firſt propoſed, it makes 2868 for a new *Reſolvend.*

Again, the three foremoſt figures of the Root being 478, the value of a is 4780, (by Note 2.) which doubled gives 9560 for a diviſor to the ſaid *Reſolvend* 2868, by Note 3.) then by dividing as aforeſaid the value of e is found 0; therefore, (according to Note 5.) I ſet 0 in the Quotient, and becauſe in this caſe the *Ablatitium* is alſo 0, the *Reſolvend* 2868 from which the ſaid *Ablatitium* ought to be ſubtracted remains the ſame without alteration; therefore by annexing 09 the laſt member of the number firſt propoſed, to the ſaid 2868 it makes 286809 for a new (and the laſt) Reſolvend. Laſtly, by proceeding as before, the laſt Figure of the Root will be found 3; ſo that the Square Root ſought is 47803; for this multiplied by it ſelf produces 2285126809, the number whoſe Square Root was deſired.

The Premiſſes may ſuffice to ſhew a perfect Method of extracting the Square Root of a whole number having an exact Square Root, which I have explain'd at large, that the Reaſon and certainty of the Rules might be apparent. But this Method may be contracted into more practical and compendious Rules, as I have ſhewn in the 32 *Chap.* of Mr. *Wingate*'s common Arithmetic.

6. But when a whole Number has not a Square Root exactly expreſſible by any rational or true Number, then to approach infinitely near the exact Root, firſt, pairs of Cyphers, as 00, 0000, 000000, or 00000000, *&c.* are to be annexed to the Number given; then eſteeming the number given with the Cyphers annexed to be one whole Number, let its Square Root be extracted according to the Precedent (or other practical) Rules; that done, look how many Points were ſet over the Number firſt given, for ſo many of the foremoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt following thoſe Integers expreſs the Fractional part of the Root in Decimal parts. As for Example: If the Square Root of 12 be deſired, I annex ſix Cyphers to 12, thus 12.000000, and then the Square Root of $\dot{1}\dot{2}.\dot{0}0\dot{0}0\dot{0}0$ being extracted, it will be found 3.464, that is, $3\frac{464}{1000}$. But becauſe after the Extraction is finiſhed, there happens to be a Remainder, I conclude that $3\frac{464}{1000}$ is leſs than the true Root, but $3\frac{465}{1000}$ is greater than it. So that by annexing three pairs of Cyphers you will not miſs $\frac{1}{1000}$ part of an Unit of the true Root, and by annexing eight Cyphers you will not want $\frac{1}{10000}$ part: and in that order you may approach as near as you pleaſe, when you cannot obtain the exact Square Root of a whole Number given.

7. The Square Root of a Vulgar Fraction is found out thus, *viz.* Firſt, if the Fraction be not in its leaſt terms, let it be reduced to the leaſt Terms; then extract the Square Root of the Numerator for a new Numerator, and the Square Root of the Denominator for a new Denominator, ſo ſhall this new Fraction be the Square Root of the Fraction propoſed. As for Example: The Square Root of $\frac{9}{16}$ is $\frac{3}{4}$; likewiſe the Square Root of $\frac{1}{4}$ is $\frac{1}{2}$.

But when either the Numerator or Denominator of a vulgar Fraction has not a perfect Square Root, then to find the Square Root of that Fraction very near; firſt reduce the Fraction to a Decimal Fraction, whoſe Numerator may conſiſt of an even number of places, *viz.* of two, four, or ſix places, *&c.* then extract the Square Root that Decimal as if it were a whole Number, and the Root that comes forth ſhall be a Decimal Fraction, expreſſing nearly the Square Root of the Fraction propoſed. As for Example: If the Square Root of $\frac{13}{16}$ be deſired, I firſt reduce it to this Decimal Fraction, .81250000; (for as 16. 13 :: 100000000. 81250000) then by extracting the Square Root of .81250000 as if it were a whole Number, I find .9013, that is, $\frac{9013}{10000}$, which is near the Square Root of $\frac{13}{16}$, for it wants not $\frac{1}{10000}$ part of an Unit of the exact Square Root of $\frac{13}{16}$.

8. Laſtly, if the Square Root of a mixt number be deſired, firſt reduce it to an improper Fraction, and then extract the Square Root of that improper Fraction as before; but if it has not an exact Square Root, then reduce the Fractional part of the mixt number firſt propoſed to a Decimal Fraction of an even number of places, and after this Decimal is annexed to the Integers of this mixt number, extract the Square Root out of the whole, then ſo many Points as were ſet over the Integers, ſo many of the foremoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt expreſs the Fractional part of the Root in Decimal Parts. As for Example: The Square Root of $34\frac{33}{64}$, that is, of $\frac{2209}{64}$, will be found $\frac{47}{8}$ or $5\frac{7}{8}$; and the Square Root of $7\frac{2}{3}$, that is, of 7.666666, *&c.* is 2.708, *&c.* that is: $2\frac{768}{1000}$, *&c.*

Sect.

Sect. II. *Of the Extraction of the Cubic Root out of a Number given.*

1. For the more ready extraction of the Cubic Root of a number given, the following Tabulet will be uſeful, which ſhews at firſt ſight the Cubic Root of any Cubical whole Number leſs than 1000.

ROOTS.	1	2	3	4	5	6	7	8	9
CUBES.	1	8	27	64	125	216	343	512	729

2. When a whole number is propoſed, and its Cubic Root deſired, the Number given muſt be prepared for Extraction, by diſtributing it into parts or members after this manner; *viz.* Firſt, a point is to be ſet over the Units place of the given number; then paſſing over the ſecond and third places towards the left hand, another point is to be ſet over the fourth place; alſo paſſing over the fifth and ſixth places another point is to be ſet over the ſeventh place: and in that order as many points are to be ſet as the number propos'd will admit, and conſequently between every two adjacent points there will be two Places without Points. So if the Cubic Root of 1331 be deſired, after Points are ſet as is above directed, the ſaid 1331 will be diſtributed into 2 Members, to wit, 1, and 331. In like manner if the Cubic Root of 21952 be required, the Points will ſtand as you ſee in the Example, and the ſaid 21952 will be diſtributed into two members 21 and 952; likewiſe this Number 941192 being pointed in the ſame order will be diſtributed into the two members 941 and 192; and this number 23156436019527 into theſe five members, 23, 156, 436, 019, 527. The points ſhew the number of places that will be found in the Root; for ſo many points as there be, ſo many places will the Root conſiſt of; they likewiſe ſhew what member of the Number propos'd belongs to the Extraction of every ſingle Character of the Root ſought.

$\dot{1}33\dot{1}$

$2\dot{1}95\dot{2}$

$94\dot{1}19\dot{2}$

$2\dot{3}15\dot{6}43\dot{6}01\dot{9}52\dot{7}$

3. The given number whoſe Cubic Root is deſired may be conceived to be produced from the Cubical Multiplication of the Binomial Root $a+e$, and then the ſaid number will be compos'd of theſe four members or ſolid numbers, *viz.* aaa, $3aae$, $3aee$, and eee, (as appears by the third Power in the Table in *Sect.* 4. *Chap.* 1.) Now becauſe the Reſolution of the Cubic number, *viz.* the Extraction of the Cubic Root, is deducible from the ſteps of the Compoſition of a Cubic number from its Root, (for ſuch numbers as are added in the Compoſition are to be ſubtracted in the Reſolution,) reſpect muſt be had to *Sect.* 2. *Chap.* 2. of this Book.

Example 1.

4. Let it be required to extract the Cubic Root of 21952. By the precedent ſecond Rule it is evident that the deſired Root conſiſts of two places, *viz.* of ſome number of Tens under 100, and of ſome number of Units under 10, which two Numbers, (agreeable to the Compoſition of a Cube in *Sect.* 2. of the precedent *Chap.* 2.) may be repreſented by a and e, ſo that $a+e$ ſignifies the Root ſought, and conſequently the Cube of $a+e$, that is, $aaa+3aae+3aee+eee$ is equal to the given number 21952. Now to find out the Number of Tens, (that is, a) in the Root, (after a crooked line is drawn on the right hand of the given number, that the Root, like the Quotient in Diviſion may be ſet next after the ſaid crooked Line, as alſo a downright Line next after each of the Points, as here you ſee.) The firſt Work in the Extraction is always to ſubtract the greateſt Cubic whole number contained in the firſt Member towards the Left hand, from the ſaid member, and to write the Root of the ſaid Cube number in the Quotient for the firſt ſingle Figure of the deſired Cubic Root: So 8 being the greateſt Cube contained in the firſt member 21, I ſubſcribe 8 under 21, and ſet 2 the Cubic Root of the ſaid 8 in the Quotient, then after a line is drawn under 8, I ſubtract 8 from 21, or, 8000 from 21952, and there remains the *Reſolvend* 13952, that is, that part of the propoſed number 21952 which is yet to be reſolved. Now obſerve, that the ſaid 2 in the Quotient,

$2\dot{1}$	$95\dot{2}$	(2
8		
13	952	

in reſpect of the next following unknown Character of the Root, is really 20, which is the number ſignified by a in the Compoſition, and the Cube of 20, to wit 8000, is aaa, which being the firſt Number found in the Compoſition, is firſt to be ſubtracted in the Reſolution. Obſerve alſo, that the next ſingle Character of the Root, whether it happen to be a Figure or a Cypher is called e, which is yet unknown.

5. Then I proceed to find the value of e, that is, the greateſt ſingle Character with this condition, that the Sum of the Numbers ſignified by $3aae$, $3aee$, and eee, may not exceed the remaining *Reſolvend* 13952, for from this Number that ſum muſt be ſubtracted. Now becauſe (for the Reaſon aforeſaid) a is 20, therefore $3aa=1200$, and $3a=60$; then ſubſcribing the ſaid 1200 and 60 under the *Reſolvend* 13952, (in ſuch order that Units may ſtand under Units, and Tens under Tens, &c) and adding them together the Sum is 1260, which muſt be eſteemed a *Diviſor*, and ſet under the *Reſolvend*. Then by ſuppoſing I were to divide the ſaid *Reſolvend* 13952 by 1260, I find the Quotient exceeds 9, but e always repreſents a ſingle Figure or a Cypher, and therefore it cannot exceed 9; wherefore I make tryal with 9 (in a void place) to ſee whether it will anſwer the before mentioned Condition, to which e is ſubject, in this manner, *viz.* Foraſmuch as it was before found that $3aa=1200$ and $3a=60$, it will follow, if we ſuppoſe $e=9$, that $3aae=10800$, alſo $3aee=4860$, and $eee=729$; therefore $3aae+3aee+eee=16389$: this ought to be ſubtracted from the *Reſolvend* 13952 but 16389 exceeds 139522 and therefore cannot be really ſubtracted from it; whence I conclude that e muſt be leſs than 9; and therefore I make tryal with 8 in like manner as before with 9, *viz.* having before found that $3aa=1200$. and $3a=60$, it will follow if we ſuppoſe $e=8$, that $3aae=9600$, alſo $3aee=3840$, and $eee=512$; therefore $3aae+3aee+eee=13952$, which may be ſubtracted from the *Reſolvend* 13952; wherefore I conclude that e (that is, the Figure which muſt follow 2 in the Quotient) is 8, which I ſet in the Quotient: then I ſubſcribe the three Numbers before found, to wit, 9600, 3840, and 512, under the *Reſolvend* 13942, (in ſuch order that the Units may ſtand under Units, Tens under Tens, &c.) and adding together the ſaid three Numbers ſo ſubſcribed, their Sum makes 13952, (the *Ablatitium*) which ſubtracted from the *Reſolvend* 13952, leaves 0. So the Extraction is finiſh'd, and 28 is found to be the Cubic Root of the propoſed Number 21952; for 28 multiplied into itſelf cubically, *viz.* 28×28×28 produces 21952.

```
               21|952| (28
Subtract        8|   | aaa
              ---
               13|952  Reſolvend
              ---
a=20            1|200 3aa
                 | 60 3a
              ---
                1|260  Diviſor
              ---
e=8             9|600 3aae
                3|840 3aee
                 |512  eee
              ---
               13|952  Ablatitium.
              ---
                 |000
```

NOTE 1.

The firſt Operation in the Extraction of the Cubic Root is always to ſubtract the greateſt Cubic whole Number, (that is, aaa) contained in the firſt Member (towards the left hand) of the given Number; from the ſaid Member, and to ſet the Root of the ſaid Cube-number in the Quotient; which Root is the firſt Figure of the Root ſought, as hath been ſhewn in the fourth ſtep. This Work is no more repeated in the whole Extraction, but the Work in the fifth ſtep is to be renewed for the finding out of every following Character in the Root.

NOTE 2.

The Number ſignified by a is to be found out by Note 2 in *Sect.*1. of this *Chap.* and then the *Diviſor* for the finding of the unknown ſingle Character repreſented by e is conſequently known: For in the Reſolution of every Power produced from the Binomial Root $a+e$, the *Diviſor* conſiſts of ſuch Powers of a as are multiplied into the Powers of e; and becauſe the Cube of $a+e$ is $aaa+3aae+3aee+eee$, therefore in the Extaction of the Cubic Root the *Diviſor* is compoſed of $3aa$ and $3a$, ſo that when the Number a is known, the *Diviſor* $3aa+3a$ is conſequently known.

NOTE 3.

When the *Diviſor* is found out by the precedent Note 2. as alſo the *Ablatitium*, which in the Extraction of the Cubic Root is compos'd of 3*aae*, 3*aee*, and *eee*; the Numbers ſignified by the ſaid 3*aae*, 3*aee*, and *eee*, muſt each of them be ſet in ſuch order under the particular or preſent *Reſolvend*, that Units may ſtand under Units, Tens under Tens, *&c.* to the end the *Ablatitium* may be rightly compoſed and ſubtracted from the *Reſolvend.*

NOTE. 4.

When the *Diviſor* is not contained once in the particular or preſent *Reſolvend*, a Cypher (to wit, 0) muſt be ſet in the Quotient; and then the *Reſolvend* muſt be augmented with the next Member (towards the right Hand) of the Power propoſed, for a new particular *Reſolvend*. Alſo a new *Diviſor* muſt be found out by Note 2. of this *Sect* and the like is to be done as often as the *Diviſor* is leſs than the *Reſolvend.*

The Practice of theſe Notes will be ſhewn in the following Example.

Example 2.

6. If the Cubic Root of 23156436019527 be deſired, it will be found 28503 by the precedent Rules, and the Work will ſtand as here you ſee underneath.

	23	156	436	019	527	(28503 *Root.*
Subtract	8					*aaa*
	15	156				*Reſolvend.*
a=20	1	200				3*aa*
		60				3*a*
	1	260				*Diviſor.*
e= 8	9	600				3*aae*
	3	840				3*aee*
		512				*eee*
Subtract	13	952				*Ablatitium.*
	1	204	436			*Reſolvend.*
a=280		235	200			3*aa*
			840			3*a*
		236	040			*Diviſor.*
e= 5	1	176	000			3*aae*
		21	000			3*aee*
			125			*eee*
Subtract	1	197	125			*Ablatitium.*
	0	007	311	019		*Reſolvend.*
a=2850		24	367	500		3*aa*
e= 0			8	550		3*a*
		24	376	050		*Diviſor.*
		7	311	019	527	*Reſolvend.*
a=28500		2	436	750	000	3*aa*
				85	500	3*a*
		2	436	835	500	*Diviſor.*
e= 3		7	310	250	000	3*aae*
				769	500	3*aee*
					27	*eee*
Subtract		7	091	019	527	*Ablatitium.*
		0	000	00	000	

Explication of Example 2.

The firſt Figure of the Root is 2 (by Note 1.) whoſe Cube 8 ſubtracted from 23, the firſt Member of the Number propos'd leaves 15, to which the ſecond Member 156 being annexed, there ariſes 15156 for the next *Reſolvend.* Or to cauſe the ſame effect, ſuppoſe 0 to be annexed to 2, the firſt Figure of the Root, and it makes 20, (that is, *a*) whoſe Cube 8000 (or *aaa*) ſubtracted from 23156, the two foremoſt Members of the Number firſt propos'd, leaves (as before) the *Reſolvend* 15156.

Then the firſt Figure of the Root being found 2, the value of *a* is 20, and the *Diviſor* is 1260, (by Note 2.) and then by dividing and making tryal, as is directed in the foregoing fifth ſtep, the Number *e* will be found 8 for the ſecond Figure of the Root, and conſequently the Numbers ſignified by 3*aae*, 3*aee*, and *eee*, are 9600, 3840, and 512; theſe being ſet orderly and added together (according to Note 3.) make the *Ablatitium* 13952, which ſubtracted from the *Reſolvend* 15156 leaves 1204, to which annexing 436, the third Member of the Number firſt propoſed, it makes 1204436 for a new *Reſolvend.* The reſt of the Operation in Example 2. being but a Repetition of what has been directed for finding out the ſecond Figure of the Root, I ſhall leave it to the Learner's Practice.

The precedent Rules and Notes in this *Sect.* 2. for extracting the Cubic Root of a whole Number, having an exact Cubic Root, are expreſs'd at large, that the Reaſon of the Work might be apparent; but this Method may be contracted into more practical and compendious Rules, as I have ſhewn in the 33 *Ch.* of Mr. *Wingate*'s Common Arithmetic.

7. But when a whole Number has not a Cubic Root exactly expreſſible by any rational or true Number, then to approach infinitely near the exact Root, firſt, Ternaries of Cyphers, *viz.* three, or ſix, or nine, or twelve, *&c.* Cyphers are to be annexed to the whole Number given; then eſteeming the Number given with the Cyphers annexed to be one whole Number, let its Cubic Root be extracted by the precedent (or other practical) Rules. That done, look how many Points were ſet over the Number firſt given, for ſo many of the foremoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt following thoſe Integers expreſs the Fractional part of the Root in Decimal Parts. As for Example: If the Cubic Root of 8302348 be deſired, I annex ſix Cyphers to 8302348 thus, 8302348.000000, and then the Cubic Root of 8302348 000000 being extracted, it will be found 202.48, that is, $202\frac{48}{100}$; but becauſe after the Extraction is finiſh'd there happens to be a Remainder, I conclude that $202\frac{48}{100}$ is leſs than the true Cubic Root ſought, but $202\frac{49}{100}$ is greater than it, ſo that by annexing ſix Cyphers you will not miſs $\frac{1}{100}$ part of an Unit of the true Root, and by annexing nine Cyphers you will not want $\frac{1}{1000}$ part; and in that order you may approach as near as you pleaſe when you cannot obtain the exact Cubic Root of a whole Number given.

8. The Cubic Root of a Vulgar Fraction is found out thus, *viz.* firſt, if the Fraction be not in its leaſt Terms, let it be reduced to the leaſt Terms; then extract the Cubic Root of the Numerator for a new Numerator, and the Cubic Root of the Denominator for a new Denominator, ſo ſhall this new Fraction be the Cubic Root of the Fraction propoſed. As for Example: The Cubic Root of $\frac{8}{27}$ is $\frac{2}{3}$, and the Cubic Root of $\frac{1}{8}$ is $\frac{1}{2}$.

9. But when either the Numerator or Denominator of a Vulgar Fraction has not a perfect Cubic Root, then to find the Cubic Root of that Fraction very near, firſt reduce the Fraction to a Decimal Fraction, whoſe Numerator may conſiſt of Ternaries of places, *viz.* either of three, ſix, nine, or twelve, *&c.* places, and then extract the Cubic Root of that Decimal as if it were a whole Number, and the Root that comes forth ſhall be a Decimal Fraction expreſſing nearly the Cubic Root of the Vulgar Fraction propoſed. As for Example: If the Cubic Root of $\frac{2}{3}$ be deſired, I firſt reduce it to this Decimal Fraction, .666666666666, and then by extracting the Cubic Root of the ſaid Decimal as if it were a whole Number, I find .8735, that is, $\frac{8735}{10000}$; which is near the Cubic Root of $\frac{2}{3}$, for it wants not $\frac{1}{10000}$ part of an Unit of the exact Cubic Root of $\frac{2}{3}$.

10. Laſtly, if the Cubic Root of a mixt Number, that is, of a whole Number with a Fraction in its leaſt Terms, be deſired; firſt reduce it to an improper Fraction, and then extract the Cubic Root of that improper Fraction in like manner as before in the eighth ſtep; but if it has not an exact Cubic Root, then reduce the Fractional part of the mixt Number firſt propoſed to a Decimal Fraction, whoſe Numerator may conſiſt of Ternaries of places, and after this Decimal is annexed to the Integers of the mixt Number, extract the Cubic Root out of the whole, then ſo many Points as were ſet over the Integers, ſo many of the foremoſt places in the Quotient are to be taken for the Integers in the Root, and the reſt expreſs the Fractional part of the Root in Decimal parts. As for Example: The

Cubic Root of $12\frac{19}{27}$, that is, of $\frac{343}{27}$, will be found $\frac{7}{3}$ or $2\frac{1}{3}$; and the Cubic Root of $2\frac{3}{8}$, that is, of 3.375000000, *&c.* will be found 1.334, *&c.* that is, $1\frac{334}{1000}$, *&c.*

Seƈt. III. *Of the Extraƈtion of the Biquadratic Root out of a Number given.*

1. The brieſeſt way to extraƈt the Root of a Biquadratic Number, that is, of a Number produced by the Multiplication of ſome Number or Root four times into it ſelf, is firſt to extraƈt the Square Root of the Number propoſed, and then to extraƈt the Square Root of that Root. As for Example: If the Root of the Biquadratic Number, or fourth Power 256 be deſired; firſt, the Square Root of 256 being extraƈted is 16, and then the Square Root of 16 is 4, which is the Root of the fourth Power 256; for 4×4×4×4 produces 256. But my purpoſe being to explain the general Method for the Extraƈtion of all kinds of Roots, I ſhall upon that Foundation ſhew how to extraƈt the Root of a Biquadratic Number.

2. For the more ready Extraƈtion of the Biquadratic Root, the following Tabulet will be uſeful, which ſhews at firſt ſight the Root of any Biquadratic whole Number under 10000.

Roots . . .	1	2	3	4	5	6	7	8	9
Fourth Powers	1	16	81	256	625	1296	2401	4096	6561

3. When a whole Number is propoſed, and it is deſired to extraƈt the Biquadratic Root of that Number, ſet Points over the given Number in this manner, *viz.* firſt, ſet a Point over the Units place, then paſſing over the three next places towards the left Hand ſet another Point over the fifth place, and in that order as many Points are to be ſet as the given Number will admit, that there may be three places between every two adjacent Points. So if the Biquadratic Root of 614656 be deſired, after Points are ſet as is above direƈted, the ſaid 614656 will be diſtributed into two Members, to wit, 61 and 4656. In like manner this Number 6597500625 being pointed in the ſame order will be diſtributed into theſe three Members, 65, 9750, and 0625. The Points ſhew the number of places that will be found in the Root, as alſo what Member of the Number propos'd belongs to the Extraƈtion of every ſingle Charaƈter of the Root ſought.

$$6\dot{1}465\dot{6}$$

$$6\dot{5}975\dot{0}062\dot{5}$$

4. The given Number, whoſe Biquadratic Root is deſired may be conceived to be produced from the Multiplication of the Binomial Root $a+e$ four times into it ſelf, and then the ſaid Number will be compoſed of theſe five Members or Numbers, *viz.* $aaaa$, $4aaae$, $6aaee$, $4aeee$, $eeee$, (as is manifeſt by the fourth Power in the Table in *Seƈt.* 4. *Chap* 1. of this Book.) Now becauſe the Reſolution of a Biquadratic Number, *viz* the Extraƈtion of the Biquadratic Root is deducible from the ſteps of the Compoſition of a Biquadratic Number from its Root, (for ſuch Numbers as are added in the Compoſition are to be ſubtraƈted in the Reſolution) reſpeƈt muſt be had to *Seƈt.* 3. *Chap.* 2. of this Book.

Example.

5. Let it be required to extraƈt the Biquadratic Root of 614656. After the Number given is prepared by Punƈtations as before is direƈted, I ſeek in the Tabulet in the precedent ſecond ſtep of this *Seƈt* 3. for the greateſt Biquadratic whole Number contained in 61, the firſt Member (towards the left Hand) of the Number propoſed, and finding it to be 16, I ſubſcribe 16 under 61, and write 2 the Root of the ſaid fourth Power 16 in the Quotient, for the firſt Figure of the Root ſought; then after a Line is drawn under 16 I ſubtraƈt 16 from 61, or 160000 from 614656, and there remains to be reſolved 454656.

$$\begin{array}{r|l|l} 6\dot{1} & 465\dot{6} & (2 \\ 16 & & \\ \hline 45 & 4656 & \end{array}$$

The *Diviſor* for the finding out of *e*, that is, every Character which is to follow 2, the firſt Figure of the Root, is always in the Extraction of the Biquadratic Root compoſed of theſe Numbers, *viz.* 4*aaa*, 6*aa*, and 4*a*, for theſe are all the Powers of *a* that are drawn into the Powers of *e* in the fourth Power of *a*+*e*; (as is evident by the Table in *Sect.* 4. *Chap.* 1.) and becauſe the firſt Figure of the Root is found 2, and conſequently (by Note 2. in *Sect.* 1. of this *Chap.*) the Number ſignified by *a* is 20, therefore the Sum of the Numbers ſignified by 4*aaa*, 6*aa*, and 4*a*, is 34480, which is the *Diviſor*; then ſuppoſing I were to divide the *Reſolvend* 454656 by the *Diviſor* 34480, I find the Quotient exceeds 9; but in regard *e* always repreſents either a ſingle Figure or a Cypher, it cannot exceed 9: and therefore I make tryal (in a waſt Paper) with 9, to ſee whether it will conſtitute an *Ablatitium* that does not exceed the *Reſolvend* 454656, *viz.* I ſuppoſe *e*=9; then becauſe *a* was before found 20, the *Ablatitium*, which in the Extraction of the Biquadratic Root is always compos'd of 4*aaae*, 6*aaee*, 4*aeee*, and *eeee*, will exceed the *Reſolvend*, from which it ought to be ſubtracted. But if *e*=8, then the *Ablatitium* will be equal to the *Reſolvend*, and conſequently that being ſubtracted from this, there will remain 0, wherefore I ſet 8 in the Quotient, and conclude that the Biquadratic Root of the given Number 614656 is 28; for 28×28×28×28 produces 614656.

	61̇	4656̇	(28. *Root*
Subtract	16		*aaaa*
	45	4656	*Reſolvend.*
a=20	3	2000	4*aaa*
		2400	6*aa*
		80	4*a*
	3	4480	*Diviſor.*
e= 8	25	6000	4*aaae*
	15	3600	6*aaee*
	4	0960	4*aeee*
		4096	*eeec*
Subtract	45	4656	*Ablatitium.*
	0	0000	

Sect. IV. *Of the Extraction of the Root of the fifth Power given in Number.*

1. For the more ready Extraction of the Root of any fifth Power given in Number, this Tabulet will be uſeful, which ſhews at firſt ſight the fifth Powers of every ſingle Figure, and conſequently any fifth Power in Number under 100000 being given, its Root is hereby diſcovered.

Roots.	*5th Powers.*
1	1
2	32
3	243
4	1024
5	3125
6	7776
7	16807
8	32768
9	59049

2. When a whole Number is given for a fifth Power, and its Root deſired, that is, ſuch a Number which being multiplied five times into it ſelf will produce the given Number, it muſt be prepared for Extraction by Punctations in this manner, *viz.* Firſt, let a Point be ſet over the Units place of the given Number, then paſſing over the four next places towards the left Hand, ſet another Point over the ſixth place; and in that order as many Points are to be ſet as the given Number will admit, that there may be four places between every two adjacent Points. So if the Root of the fifth Power 17210368 be deſired, after Points are ſet as is above directed, the ſaid 17210368 will be diſtributed into two Members, to wit, 172 and 10368. In like manner this Number 1880287678125 will be diſtributed into theſe three Members, 188, 02876, and 78125. The Points (as before hath been ſaid) ſhew the number of Places that will be found in the Root, as alſo what Member of the Number given belongs to the Extraction of every ſingle Character of the Root ſought.

172̇10368̇

188̇02876̇78125̇

3. Every Number conſidered as a fifth Power may be conceived to be produced from the Multiplication of the Binomial Root $a+e$ five times into it ſelf, and then the ſaid Number will be compoſed of theſe ſix Members or Numbers, *viz.* $aaaaa$, $5aaaae$, $10aaaee$, $10aaeee$, $5aeeee$, and $eeeee$, (as is manifeſt by the fifth Power in the Table in *Sect.* 4. *Chap.* 1. of this Book.) Now becauſe the Reſolution of the fifth Power, *viz.* the Extraction of $\surd(5)$ out of a given Number, is deducible from the ſteps of the Compoſition of a fifth Power from its Root given in Number; (for ſuch Numbers as are added in the Compoſition are to be ſubtracted in the Reſolution) the Learner muſt be exercis'd in *Sect.* 4. *Chap.* 2. of this Book.

Example.

Let it be required to extract $\surd(5)$ out of 17210368, *viz.* to find a Root or Number, which being multiplied five times into it ſelf will produce 17210368. After the given Number is prepared by Punctations as before is directed, I ſeek in the Tabulet in the firſt ſtep of this Section 4. for the greateſt fifth Power contained in 172 the firſt Member (towards the left Hand) of the given Number, and finding it to be 32, I ſubſcribe 32 under 172, and write 2 the Root of the ſaid fifth Power 32 in the Quotient, for the firſt Figure of the Root ſought; then after having drawn a Line under 32, I ſubtract 32 from 172, or 3200000 from 17210368, and there remains to be reſolved 14010368.

$\dot{1}7\dot{2}$	$10\dot{3}68$	(2
32		
140	10368	

Then to diſcover the *Diviſor*, which ſhews how to begin the tryal in the finding out of e, that is, every Character (whether it be a Figure or Cypher) which is to follow the firſt Figure of the Root, I take ſuch Powers of a as are multiplied into the Powers of e in the fifth Power produced from $a+e$, *viz.* $5aaaa$, $10aaa$, $10aa$, and $5a$; ſo the Sum of theſe four Numbers make the *Diviſor*. And becauſe the firſt Figure of the Root is found 2, and conſequently (by Note 2. in *Sect.* 1. of this *Chap.*) the Number ſignified by a is 20, therefore the Sum of the Numbers ſignified by $5aaa$, $10aaa$, $10aa$, and $5a$ is 884100, which is the *Diviſor*; then ſuppoſing I were to divide the *Reſolvend* 14010368 by the *Diviſor* 884100, I find the Quotient exceeds 9; but in regard e always repreſents a ſingle Figure or Cypher, it cannot exceed 9; therefore I make tryal (in a void place) with 9, to ſee whether it will conſtitute an *Ablatitium* that does not exceed the *Reſolvend* 14010368, *viz.* I ſuppoſe $e=9$, then becauſe a was found 20, the *Ablatitium* $5aaaae+10aaaee+10aaeee+5aeeee$ exceeds the *Reſolvend* from which it ought to be ſubtracted. But if $e=8$, then the *Ablatitium* will be equal to the *Reſolvend*, and conſequently that being ſubtracted from this, there will remain 0, wherefore I ſet 8 in the Quotient; ſo 28 is found to be the $\surd(5)$ of the given Number 17210368, for 28×28×28×28×28 produces 17210368. Compare the following Work with the precedent Rules of *Sect.* 4.

	$\dot{1}7\dot{2}$	$10\dot{3}68$	(28. *Root.*
	32	00000	$aaaaa$
	140	10368	*Reſolvend.*
$a=20$	8	00000	$5aaaa$
		80000	$10aaa$
		4000	$10aa$
		100	$5a$
	8	84100	*Diviſor.*
$e=8$	64	00000	$5aaaae$
	51	20000	$10aaaee$
	20	48000	$10aaeee$
	4	09600	$5aeeee$
		32768	$eeeee$
	140	10368	*Ablatitium.*
	000	00000	

By the precedent Rules and Examples of this *Chap.* the Ingenious Reader will eaſily perceive how to extend this general Method to the Extraction of the Roots of all kinds of Powers in Numbers, *viz.* of the ſixth, ſeventh, eighth, *&c.* Powers; as alſo to find out the Roots infinitely near of ſuch Powers as have not Roots exactly expreſſible by any rational or true Number.

CHAP. IV.

Concerning the Extraction of Roots out of Powers expreſs'd by Letters.

I. IN a Series or Scale of Powers produced from a Root, ſuppoſe from a, as in this Series, $a, aa, aaa, aaaa, aaaaa, a^6, a^7, a^8$, *&c.* thoſe Powers only whoſe Indices are even Numbers are Squares; as $aa, aaaa, a^6, a^8$, *&c.* (whoſe Indices are 2, 4, 6, 8, *&c.*) are Squares. And thoſe Powers only whoſe Indices are diviſible by 3, are Cubes, as $aaa, aaaaaa, a^9$, *&c.* (whoſe Indices are 3, 6, 9, *&c.*) are Cubes. Therefore every Power whoſe Index is a Prime Number greater than 3, as $aaaaa, a^7, a^{11}$, *&c.* (whoſe Indices are 5, 7, 11, *&c.*) is neither a Square nor a Cube. But every Power whoſe Index is diviſible by 6, as a^6, a^{12}, a^{18}, *&c.* is both a Square and a Cube, becauſe the Index is diviſible both by 2 and by 3.

II. If a Simple Quantity be expreſs'd by the ſame Letter repeated an even number of times, the Square Root thereof is eaſily extracted; for the Root muſt be ſuch that its Index may be the half of the Index of the Quantity propoſed: As, $\surd aa$ (that is, the Square Root of aa) is a; for 1, the Index of the Root a is the half of 2, the Index of the Square aa. In like manner $\surd aaaa$ is aa, whoſe Index 2 is the half of 4, the Index of the Square $aaaa$. Again, $\surd aaaaaa$ is aaa, whoſe Index 3 is the half of 6, the Index of the Square a^6.

III. And with the like facility you may extract the Cubic Root of a Simple Quantity, which is expreſs'd by one and the ſame Letter repeated ſuch a Number of times as is diviſible by 3; for the Cubic Root muſt be ſuch that its Index may be $\frac{1}{3}$ of the Index of the Cube propoſed: As $\surd(3)aaa$ (that is, the Cubic Root of the Quantity aaa) is a, whoſe Index 1 is $\frac{1}{3}$ of 3 the Index of aaa. In like manner $\surd(3)\,a^6$ is aa, whoſe Index 2 is $\frac{1}{3}$ of 6 the Index of the Cube a^6.

IV. If the Index of a Simple Power expreſs'd by the ſame Letter be ſome Prime Number greater than 3, as 5, 7, 11, *&c.* then neither $\surd(2)$ nor $\surd(3)$, nor any other Root, except that denoted by ſuch Index or Prime Number can be exactly extracted out of the ſaid Power: ſo no Root can be exactly extracted out of $aaaaa$ or a^5, but $\surd(5)$, which is a; nor any Root out of a^7 but $\surd(7)$, which is alſo a. But when the Root cannot be exactly extracted, the Sign of the Root is to be prefix'd to the Quantity; as to expreſs the Square Root of $aaaaa$ or a^5, I write $\surd aaaaa$ or $\surd a^5$. Likewiſe I expreſs the Cubic Root of a^5 thus, $\surd(3)\,a^5$; and $\surd(4)$ of a^7 thus, $\surd(4)a^7$; and ſo of others.

V. When ſome Power of an unknown Simple Root a is found equal to ſome known Number, and the Index of that unknown Power is not a Prime Number, then the value of the Root a in Number may oftentimes be diſcovered by two or more Extractions, more eaſily than by one ſingle Extraction of a Root out of the ſaid unknown Number. As for Example:

If there be propoſed or found out . . . . $aaaaaa = 729$,

Then to find out the value of a you need not extract the $\surd(6)$ of 729, by the general Method before delivered in *Chap.* 3. but firſt by that Method extract the Square Root of 729, and then by *Sect.* 2. of this *Chap.* the Square Root of $aaaaaa$, ſo thoſe two Roots compared give this Equation, *viz.* . . . . } . $aaa = 27$

Laſtly, by extracting the Cubic Root of each part of the laſt Equation, the value of a the Root ſought is diſcovered, *viz.* . . } . . $a = 3$

Or thus,

Firſt, by extracting the Cubic Root of each part of the Equation tion propoſed, there ariſes . . . . $aa=9$

And then by extracting the Square Root of each part of the laſt Equation, the ſame value of the Root a is found out as before, to wit, . $a=3$

In like manner if . . . . $a^9=19683$

Firſt by extracting the Cubic Root, it gives . . . . $a^3=27$

And again, by extracting the Cubic Root of that Root the Root a is made known, *viz.* . . . . $a=3$

VI. When two or more Squares, Cubes, or other Powers expreſs'd by different Letters, be multiplied one into another, then if the Root of each Power, *viz.* the Square Root if they be Squares, or the Cubic Root if they be Cubes, *&c.* be extracted, the Product made by the Multiplication of theſe Roots one into another, ſhall be a like Root of the Power or Product firſt given. As for Example: $\surd aabb$ is ab, which is the Product of the Square Roots of aa and bb. Likewiſe, $\surd(3)aaabbb$ is ab, which is the Product of the Cubic Roots of aaa and bbb.

Again, $\surd aabbcc$ is abc, which is the Product of the Square Roots of aa, bb, and cc. In like manner, $\surd(3)27aaabbb$ is $3ab$, which is the Product of the Cubic Roots of $27aaa$ and bbb; and $\surd 16aabbcc$ is $4abc$, which is the Product of the Square Roots of $16aa$, bb, and cc. The like is to be underſtood of others.

But if the Square Root of $5aabb$ be deſired, becauſe 5 is not a Square, the ſaid Root is to be expreſs'd either thus, $\surd 5aabb$; or thus, $\surd 5\times ab$; or thus, $ab\surd 5$. In like manner, to denote the Square Root of $aaabbb$ I write $\surd a^3b^3$. And to ſignifie the Cubic Root of $aabb$ I write $\surd(3)aabb$; but the Cubic Root of $3aaabbb$ may be written either thus, $\surd(3)3a^3b^3$; or thus, $\surd(3)3\times ab$; or thus, $ab\surd(3)3$.

Concerning the Extraction of Roots out of Compound Quantities expreſs'd by Letters.

VII. Before the Learner enters upon the Extraction of Roots out of Compound Squares, Cubes, or other Powers expreſs'd by Letters, he ought to be well exercis'd in the eighth and ninth Chapters of my firſt Book of *Algebraical Elements*; as alſo in the foregoing firſt, ſecond, and third Chapters of this Book, and in the precedent Rules of this Chapter; all which well underſtood will render the following Rules and Examples of this Chapter very plain and eaſie.

VIII. *Rules for the Extraction of Square Roots out of Compound Quantities expreſs'd by Letters.*

Rule 1. Set the particular Members of the Compound Algebraic Quantity, whoſe Square Root is required, in ſuch order, that one of the Simple Squares may ſtand outermoſt towards the left Hand; and next after the ſame ſuch other Member or Members, wherein you find the ſame Letter or Letters as are in the ſaid Simple Square. Then the Square Root of the ſaid Simple Square is to be ſet in the Quotient for the firſt Number of the Compound Root ſought, and the Square it ſelf is the firſt Quantity to be ſubtracted from the Compound Quantity propoſed. This is the firſt Work, which is no more to be repeated in the whole Extraction.

Rule 2. Double the Root before ſet in the Quotient for the firſt *Diviſor*; likewiſe to find every following *Diviſor* double every Simple Quantity that ſtands in the Quotient, and take the Sum of the Products for the *Diviſor*.

Rule 3. When the *Diviſor* is found out, divide only the firſt Simple Quantity (towards the left Hand) in the *Reſolvend*, by the firſt Simple Quantity in the *Diviſor*, and ſet that which comes forth next after the Member or Members of the Root ſought that was before found out.

Rule 4. After the firſt Simple Square is ſubtracted (according to *Rule* 1.) then every following *Ablatitium*, that is, the Sum of the Quantities to be ſubtracted from the reſpective *Reſolvend*, muſt be compoſed of theſe two Products, *viz.* the Product made by the Multiplication of the whole *Diviſor* by that particular Quantity which was laſt ſet in the Quotient, and the Square of the ſame Simple Quantity.

The Practice of theſe Rules will be apparent in the following Examples.

Example 1.

Let it be required to extract the Square Root of $aa+2ab+bb$.

Firſt, I exrract the Square Root of aa, and it is a, which I ſet in the Quotient; then multiplying a by it ſelf, I ſet the Product aa under, and ſubtract it from the Qnantity firſt propoſed, and there remains $2ab+bb$. This is the firſt work which anſwers to Rule 1. and is no more to be repeated.

The Square,	$aa+2ab+bb$	$(a+b$ The Root.
Subtract	aa	
Remainder,	$+2ab+bb$	
Diviſor,	$+2a)$	
Subtract	$+2ab+bb$	
Remainder,	0 0	

Secondly; the Diviſor (according to Rule 2.) is $2a$, which I ſet under $2ab$.

Thirdly, I divide $+2ab$ by the Diviſor $+2a$, and the Quorient is $+b$, which I ſet next after a, (the particular Root before found out) according to Rule 3.

Fourthly, I multiply the Diviſor $+2a$ by $+b$, (that was laſt ſet in the Quotient) and the Product is $+2ab$, to which adding $+bb$, (the Square of $+b$) the Sum is $+2ab+bb$, which (according to Rule 4.) I ſet under and ſubtract from the Reſolvend $+2ab+bb$, and there remains 0: So the Extraction being finiſh'd, the Root ſought is found $a+b$; for if it be multiplied by it ſelf it produces $aa+2ab+bb$, the Quantity firſt propoſed.

Note. By what I have ſaid in the eighrh and ninth Chapters of my Firſt Book of *Algebraical Elements*, 'tis eaſie to diſcover at firſt ſight whether a Compound Algebraic Quantity conſiſting of three Terms be a perfect Square or not, and if a Square what its Root is. Nevertheleſs in this firſt Example I have expreſs'd the Work at large according to the four Rules before given, that the like Opertion may the more eaſily be perceived in the following Examples.

Example 2.

If the Square Root of $aa-2ab+2ac-2bc+bb+cc$ be deſired, it will be found $a-b+c$ by the precedent Rules, and the Work ſtands as here you ſee underneath.

The Square,	$aa-2ab+2ac-2bc+bb+cc$	$(a-b+c$ The Root.
Subtract	aa	
Remainder,	$-2ab+2ac-2bc+bb+cc$	
Diviſor,	$+2a)$	
Subtract	$-2ab+bb$	
Remainder,	$+2ac-2bc+cc$	
Diviſor,	$+2a-2b)$	
Subtract	$+2ac-2bc+cc$	
Remainder,	0 0 0	

Example 3.

In like manner the Square Root of $64aabb+32abc-144ab+4cc-36c+81$ will be found $8ab+2c-9$, as is manifeſt by the following Operation.

The Square,	$64aabb+32abc-144ab+4cc-36c+81$	$(8ab+2c-9$
Subtract	$64aabb$	
Remainder,	$+32abc-144ab+4cc-36c+81$	
Diviſor,	$+16ab)$	
Subtract	$+32abc \quad +4cc$	
Remainder,	$-144ab \quad -36c+81$	
Diviſor,	$+16ab \quad +4c)$	
Subtract	$-144ab \quad -36c+81$	
Remainder,	0 0 0	

Example 4.

Again, the Square Root of $dddd+2dddb+3ddbb+2dbbb+bbbb$ will be found $dd+db+bb$; and the Extraction ſtands thus:

The Square,	$d^4+2d^3b+3d^2b^2+2db^3+b^4$	$(dd+db+bb.$
Subtract	d^4	
Remainder,	$+2d^3b+3d^2b^2+2db^3+b^4$	
Diviſor,	$+2d^2)$	
Subtract	$+2d^3b+d^2b^2$	
Remainder,	$+2d^2b^2+2db^3+b^4$	
Diviſor,	$+2d^2\quad+2db)$	
Subtract	$+2d^2b^2+2db^3+b^4$	
Remainder,	$0\quad 0\quad 0$	

IX. *Rules for the Extraction of Cubic Roots out of Compound Quantities expreſs'd by Letters.*

Rule 1. Set the particular Members or Parts of the Compound Algebraic Quantity whoſe Cubic Root is required, in ſuch order, that one of the Simple Cubes may ſtand outermoſt towards the left Hand, and next after the ſame ſuch other Members wherein you find the ſame Letter or Letters as are in the ſaid Simple Cube; then the Cubic Root of the ſaid Simple Cube is to be ſet in the Quotient for the firſt Member of the Root ſought, and the Simple Cube it ſelf is the firſt Quantity to be ſubtracted from the Compound Quantity propoſed. This is the firſt Work, and no more to be repeated in the whole Extraction.

Rule 2. The firſt Diviſor muſt be compoſed of the Triple of the Square of the Root before ſet in the Quotient, (which Triple Square I call the firſt part of the Diviſor) and the Triple of the ſame Root, (which Triple Root I call the latter part of the Diviſor.) Likewiſe every following Diviſor muſt be compoſed of the Triple of the Square of the Sum of all the ſingle Quantities or Parts of the Root already found out and ſet in the Quotient, and of the Triple of the ſame Sum.

Rule 3. When the Diviſor is found out, divide only the firſt Simple Quantity (towards the left Hand) in the *Reſolvend*, by the firſt Simple Quantity in the Diviſor, and ſet that which comes forth in the Quotient next after the Member or Members of the Root ſought before found out.

Rule 4. After the firſt Simple Cube is ſubtracted (according to Rule 1.) then every following *Ablatitium*, that is, the Sum of the Quantities to be ſubtracted from the *Reſolvend*, muſt be compoſed of theſe three Products, *viz.* Firſt, the Product made by the Multiplication of the firſt Part of the Diviſor, (to wit, the Triple Square mentioned in Rule 2.) by the ſimple Quantity laſt ſet in the Quotient. Secondly, the Product made by the Multiplication of the latter part of the Diviſor, (to wit, the Triple Root or Sum mentioned in Rule 2.) by the Square of the ſame ſimple Quantity. And thirdly, the Cube of the ſaid ſimple Quantity laſt ſet in the Quotient.

The Practice of theſe Rules will appear in the following Examples.

Example 1.

Let it be required to extract the Cubic Root out of $aaa+3aae+3aee+eee.$

Firſt, beginning at the left Hand I extract the Cubic Root of aaa, and it is a, which I ſet in the Quotient, then multiplying the ſaid Root a Cubically it makes aaa, which I ſubtract from the Compound Quantity firſt propoſed for Extraction, and there remains to be reſolved $+3aae+3aee+eee.$ This is the firſt Work, which anſwers to Rule 1. and is no more to be repeated in the whole Extraction.

The Cube,	$aaa+3aae+3aee+ee$	$(a+e.$ The Root.
Subtract	aaa	
Remainder,	$+3aae+3aee+eee$	
Diviſor,	$+3aa\quad+3a)$	
Subtract	$+3aae+3aee+eee$	
Remainder,	$0\quad 0\quad 0$	

Secondly, I ſeek a Diviſor thus, *viz.* to $+3aa$, which is the triple of aa the Square of the Root a, I add $+3a$, the triple of the ſaid Root a, and the Sum $3aa+3a$ is the Diviſor, which I ſet underneath the remaining *Reſolvend*, according to Rule 2.

Thirdly, according to Rule 3. I divide $+3aae$ by $+3aa$, and it gives $+e$, which I ſet in the Quotient next after a.

Fourthly, to find out the *Ablatitium* (or Quantity next to be ſubtracted) I make a threefold Multiplication, *viz.* Firſt, I multiply $+3aa$ (the firſt part of the Diviſor) by $+e$ the Root laſt ſet in the Quotient, and the Product is $+3aae$. Secondly, I multiply $+3a$, the latter part of the Diviſor by $+ee$, the Square of the ſaid Root e, and the Product is $+3aee$. Thirdly, I multiply the ſaid Root e Cubically, and the Product is eee. Laſtly, I ſubtract the Sum of the ſaid three Products from the *Reſolvend*, and there remains 0. So the Extraction is finiſh'd, and $a+e$ is the true Cubic Root ſought; for if it be multiplied cubically, it will produce $aaa+3aae+3aee+eee$ firſt propoſed.

Example 2.

In like manner the Cubic Root extracted out of $125aaa+225aae+135aee+27eee$ is $5a+3e$, and the Work ſtand thus:

The Cube.	$125aaa+225aae+135aee+27eee$	$(5a+3e.$ Root.
Subtract	$125aaa$	
Remainder,	$+225aae+135aee+27eee$	
Diviſor,	$+75aa+15a)$	
Subtract	$+225aae+135aee+27eee$	
Remainder,	0 0 0	

Example 3.

So the Cubic Root of $27a^6-54a^5+171a^4-188a^3+285aa-150a+125$ will be found $3aa-2a+5$, and the Operation ſtands thus:

Cube,	$27a^6-54a^5+171a^4-188a^3+285aa-150a+125$	$(3aa-2a+5.$ Root.
Subtract	$27a^6$	
Remainder,	$-54a^5+171a^4-188a^3+285aa-150a+125$	
Diviſor,	$+27a^4+9a^2)$	
Subtract	$-54a^5+36a^4-8a^3$	
Remainder,	$+135a^4-180a^3+285aa-150a+125$	
Diviſor,	$\left\{\begin{array}{l}+27a^4-36a^3+12aa\\ \qquad\qquad\quad+9aa-6a\end{array}\right.$	
Add theſe,	$\left\{\begin{array}{l}+135a^4-180a^3+60aa\\ \qquad\qquad\qquad+225aa-150a\\ \qquad\qquad\qquad\qquad\qquad+125\end{array}\right.$	
Subtract	$+135a^4-180a^3+285aa-150a+125$	
Remainder,	0 0 0 0 0	

If there be occaſion to extract the Root of the fourth, fifth, or other higher Compound Power, the Diviſors and Ablatitious Quantities may be drawn out of the Table in *Sect.* 4. *Chap.* 1. of this Book.

X. *Concerning the Extraction of Roots out of Algebraical Fractions.*

1. Foraſmuch as in the Extraction of Roots out of Fractions, the Root of the Numerator and Denominator being ſeverally extracted gives the Root ſought; therefore if the Square Root of $\frac{aabb}{cc}$ be to be extracted, I write $\frac{ab}{c}$ for the Root ſought; for the Square Root of the Numerator $aabb$ is ab, and the Square Root of the Denominator cc is c.

In like manner if the Square Root of $\frac{aaaa-2aabb+bbbb}{aa+4ab\ \ +4bb}$ be deſired, by extracting the Square Root out of the Numerator and Denominator, there ariſes $\frac{aa-bb}{a+2b}$ for the Root ſought.

And for the ſame Reaſon the Cubic Root of this Fraction $\frac{27a^6-54a^5+171a^4-188a^3+285aa-150a+125}{aaa-9aa+27a-27}$ will be $\frac{3aa-2a+5}{a-3}$, which is found by extracting the Cubic Root out of the Numerator and Denominator of the Fraction propoſed.

2. But if the Root ſought cannot be extracted out of the Numerator and Demoninator, then the Radical Sign √ with the Index of the Power, if it exceed a Square, is to be prefix'd to the Fraction; as to denote the Square Root of $\frac{ccxx}{4bb}-ac$, that is, of $\frac{ccxx-4abbc}{4bb}$, I write $\sqrt{}\frac{ccxx-4abbc}{4bb}$, or (becauſe the Square Root of the Denominator is $2b$) the Square Root of the Quantity propoſed may be expreſs'd thus $\frac{\sqrt{ccxx-4abbc}}{2b}$; likewiſe the Cubic Root of $\frac{a^3b^3}{aa+bb}$ may be deſigned either thus, $\sqrt{}(3)\frac{a^3b^3}{aa+bb}$, or (becauſe the Numerator is a Cube) thus, $\frac{ab}{\sqrt{(3)aa+bb}}$. The like is to be underſtood in expreſſing the irrational Roots of higher Powers.

CHAP. V.

Concerning Geometrical Proportion.

I. THE *Difference* of two Numbers is found out by Subtraction; but the *Ratio, Reaſon*, or *Habitude* of one Number to another is diſcovered by dividing the Antecedent (or firſt Number) by the Conſequent (or ſecond Number;) for the Quotient denominates the *Ratio*, Reaſon, or (as ſome call it) the Proportion which the Antecedent has to the Conſequent. As if 6 be compared to 2, then $\frac{6}{2}$, that is $\frac{3}{1}$, or 3, ſhews that 6 has triple Reaſon to 2, *viz.* 6 contains 2 thrice, or 6 is in proportion to 2 as 3 to 1; but if 2 be compared to 6, then $\frac{2}{6}$ or $\frac{1}{3}$ ſhews, that 2 has ſubtriple Reaſon to 6, *viz.* 2 is $\frac{1}{3}$ part of 6, or 2 is in proportion to 6 as 1 to 3. In like manner if the Quantity a be compared to the Quantity b, then $\frac{a}{b}$ expreſſes the *Ratio* or Reaſon of a to b, and $\frac{b}{a}$ ſhews the Reaſon of b to a.

Note, that the Reaſon of two Numbers or Quantities ought to be expreſs'd by the ſmalleſt Terms or Quantities that can poſſibly be found to expreſs that Reaſon. So the Denominator of the Reaſon of 16 to 12 is $\frac{4}{3}$, where 16 and 12 are firſt reduced to the ſmalleſt Terms 4 and 3, (by dividing the 16 and 12 ſeverally by their greateſt common Diviſor 4) and then dividing the Antecedent 4 by the Conſequent 3, the Quotient $\frac{4}{3}$ expreſſes the Reaſon or Proportion of 16 to 12, *viz.* 16 is to 12 as 4 to 3. In like manner the Reaſon of bb to ba, or of bbb to bba is $\frac{b}{a}$.

II. Quantities which proceed by equal Differences are ſaid to be in a continued Arithmetical Progreſſion, (as has been ſhewn in *Chap.* 17. *Book* 1. of my *Algebraical Elements*;) but Quantities which proceed by equal Reaſons (or Proportions) are ſaid to be in a continued Geometrical Progreſſion or Proportion. So theſe Numbers 2,6,18, 54, 162 are continually proportional, becauſe the Reaſon (or Proportion) of the firſt the ſecond is equal to the Reaſon of the ſecond to the third, alſo of the third to the fourth, and ſo forward; *viz.* $\frac{2}{6}$ (or $\frac{1}{3}$) $=\frac{6}{18}=\frac{18}{54}=\frac{54}{162}$; or backward, $\frac{162}{54}=\frac{54}{18}=\frac{18}{6}=\frac{6}{2}$ (or 3.) In like manner if theſe Quantities a, b, c, d, e, be ſuch, that $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$; or backwards, if $\frac{e}{d}=\frac{d}{c}=\frac{c}{b}=\frac{b}{a}$, then thoſe Quantities are continually proportional; *viz.* as the firſt is in proportion to the ſecond, ſo is the ſecond to the third, the third to the fourth, *&c.*

But if there be four ſuch Quantities, that the Reaſon (or Proportion) of the firſt to the ſecond, is equal to the Reaſon of the third to the fourth; but the Reaſon of the ſecond to the third, is not equal to the Reaſon of the firſt to the ſecond, then thoſe Quantities are ſaid to be in Geometrical Proportion diſcontinued or interrupted; ſuch are theſe four Numbers $2 . 6 :: 12 . 36$; for $\frac{2}{6}$ (or $\frac{1}{3}$) $=\frac{12}{36}$, but $\frac{6}{12}$ (or $\frac{1}{2}$) is not equal to $\frac{2}{6}$ or $\frac{1}{3}$. In like manner, if a, b, c, d, be ſuch Quantities that $\frac{a}{b}=\frac{c}{d}$, but $\frac{b}{c}$ is not equal to $\frac{a}{b}$, (or $\frac{c}{d}$;) then are $a,b,c,d,$ diſcontinual Proportionals.

III. If three Quantities be Proportionals, the Product made by the mutual Multiplication of the Extremes is equal to the Square of the Mean; as,

If there be propoſed . . . . $\left\{\begin{array}{l} 18, 6, 2, \div \\ a, b, c, \div \end{array}\right.$

Then this Equation enſues . . . . $ac=bb=36$

For ſince by ſuppoſition . . . . $a . b :: b . c$

It follows (by *Sect.* 1. and 2.) that . . . . $\left\{\frac{a}{b}=\frac{b}{c}=3\right.$

Whence by multiplying each part by c, ... $\left\{\frac{ac}{b}=b=6\right.$

And by multiplying each part of the laſt Equation by b, it produces . . . . $\left\{ac=bb=36\right.$

Which was to be proved.

IV. If four Quantities be Proportionals, whether they be continual or diſcontinual, the Product made by the mutual Multiplication of the Extremes is equal to the Product of the Means; and conſequently if the Product of the Means be divided by either of the Extremes, the Quotient is the other Extreme. As for Example:

Let four diſcontinual Proportionals be propoſed, . . $\left\{\begin{array}{l} d . c :: b . a \\ 12 . 4 :: 15 . 5 \end{array}\right.$

Then by the foregoing *Sect.* 2. . . . . $\left\{\frac{d}{c}=\frac{b}{a}=3\right.$

And by multiplying each part of that Equation by a this is produced, *viz.* . . . . $\left\{\frac{da}{c}=b=15\right.$

And by multiplying each part of the laſt Equation by c, the firſt part of the Propoſition is manifeſt, *viz.* . . $\left\{da=cb=60\right.$

And by dividing each part by d there ariſes . . . . $\left\{a=\frac{cb}{d}=5\right.$

Which laſt Equation being compared with the four Proportionals firſt propoſed does ſhew, that if three Quantities d, c, b, be given, to find ſuch a fourth as ſhall have the ſame Proportion to b as c has to d, then the Product of the ſecond and third Terms, to wit cb, being divided by the firſt Term d will give the fourth Proportional ſought, which is the very Operation in the Rule of Three Direct.

V. If three Quantities a, b, c be Proportionals, and the firſt and ſecond, to wit a and b be given ſeverally, the third is alſo given; for by *Sect.* 3. of this *Chap.* $ac=bb$, whence by dividing each part by a there ariſe $c=\frac{bb}{a}$ which ſhews, that if the Square of the Mean or ſecond Term be divided by the firſt, the Quotient is the third Proportional; hence a, b, and $\frac{bb}{a}$ are continual Proportionals. In like manner if three Quantities in continual Proportion be given ſeverally, and a fourth Proportional be deſired, the Square of the third Term divided by the ſecond gives the fourth: as if there be given theſe three, $a,\ b,\ \frac{bb}{a}$ ÷; then by dividing the Square of $\frac{bb}{a}$, to wit, $\frac{bbbb}{aa}$ by b, the Quotient $\frac{bbb}{aa}$ ſhall be the fourth continual Proportional: Hence $a,\ b,\ \frac{bb}{a},\ \frac{bbb}{aa}$ are continual Proportionals. Likewiſe if the Square of the fourth continual Proportional be divided by the third, the Quotient will be the fifth; ſo to thoſe four continual Proportionals this fifth will be found, to wit, $\frac{bbbb}{aaa}$; and ſo forwards infinitely. Therefore,

VI. If Numbers, how many ſoever, be continually Proportionals, and the leaſt Term be eſteemed the firſt, that next greater than the leaſt the ſecond, and ſo forwards; then the ſecond Term is produced by the Multiplication of the firſt into the Reaſon of the ſecond Term to the firſt, the third Term is produced by the Multiplication of the firſt into the Square of the ſame Reaſon, the fourth Term is produced by the Multiplication of the firſt into the Cube of the ſame Reeſon; and in like manner every following Term is produced by the Multiplication of the firſt into ſuch a Power of the Reaſon of the ſecond Term to the firſt, as has fewer dimenſions by one than the Number of Terms has Units: as in theſe following ſix continual Proportionals, to wit,

$$a,\ b,\ \frac{bb}{a},\ \frac{bbb}{aa},\ \frac{bbbb}{aaa},\ \frac{bbbbb}{aaaa}\ \div$$
$$2,\ 6,\ 18,\ 54\ ,\ 162\ ,\ 486\ \div$$

Suppoſing a to be the firſt and leaſt Term, the ſecond Term b is equal to the Product of the firſt Term a into $\frac{b}{a}$, to wit, the Reaſon of the ſecond Term to the firſt; alſo the third Term $\frac{bb}{a}$ is produced by the Multiplication of the firſt Term a into the Square of the ſame Reaſon, that is, into $\frac{bb}{aa}$; and the fourth Term $\frac{bbb}{aa}$ is produced by the Multiplication of the firſt Term a into the Cube of the ſame Reaſon, that is, into $\frac{bbb}{aaa}$; and the fifth Term $\frac{bbbb}{aaa}$ is produced by the Multiplication of the firſt Term a into the fourth Power of the ſame Reaſon, that is, into $\frac{bbbb}{aaaa}$: and ſo forwards.

But if the greateſt Term be eſteemed the firſt, that next leſs than the greateſt the ſecond, and ſo downwards; then the ſecond Term is equal to the Quotient that ariſes by dividing the firſt (or greateſt) Term by the Reaſon of the firſt to the ſecond; the third is equal to the Quotient that ariſes by dividing the firſt Term by the Square of the ſame Reaſon; the fourth Term is equal to the Quotient that ariſes by dividing the firſt Term by the Cube of the ſame Reaſon; and in like manner every Term beneath the greateſt is equal to the Quotient that ariſes by dividing the firſt (or greateſt) Term by ſuch a Power of the Reaſon of the greateſt to the greateſt but one (or ſecond) Term, as has fewer Dimenſions by one than the number of Terms: as in theſe following ſix continual Proportionals, to wit,

$$\frac{bbbbb}{aaaa},\ \frac{bbbb}{aaa},\ \frac{bbb}{aa},\ \frac{bb}{a},\ b,\ a,\ \div$$
$$486\quad 162\ ,\ 54\ ,\ 18,\ 6,\ 2\ \div$$

If we ſuppoſe $\frac{bbbbb}{aaaa}$ to be the firſt and greateſt Term, then the ſecond Term $\frac{bbbb}{aaa}$ is equal to the Quotient of the firſt Term $\frac{bbbbb}{aaaa}$ divided by $\frac{b}{a}$, to wit, by the Reaſon of the firſt Term to the ſecond; alſo the third Term $\frac{bbb}{aa}$ is equal to the Quotient of the firſt Term $\frac{bbbbb}{aaaa}$ divided by $\frac{bb}{aa}$, that is, by the Square of the Reaſon $\frac{b}{a}$; and the fourth Term $\frac{bb}{a}$ is equal to the Quotient of the firſt Term $\frac{bbbbb}{aaaa}$ divided by $\frac{bbb}{aaa}$ the Cube of the ſame Reaſon. And ſo of the reſt.

VII. From the laſt preceding Section it follows, that if in a Series or Rank of Numbers which are in continual proportion, the firſt Term, the ſecond Term, and the Number of Terms be given ſeverally, the laſt Term ſhall be alſo given by this Rule, *viz.* firſt, (according to the *Note* in *Sect.* 1. of this *Chap.*) find out the ſmalleſt Numbers that may ſhew the Reaſon of the greater of the two given Terms to the leſs; then eſteeming the ſaid Reaſon as a Root, find ſuch a Power thereof whoſe Index may be equal to the given multitude of Terms leſs by Unity, which Power multiplied by the firſt Term, when the firſt Term is leſs than the ſecond, gives the laſt, to wit, the greateſt Term. But when the firſt Term is greater than the ſecond, then the firſt Term divided by the ſaid Power gives the laſt Term. As if there be given a and b, the firſt and ſecond of ſix Numbers in continual proportion, and that b is greater than a; then the Reaſon of b to a is $\frac{b}{a}$, (by *Sect.* 1. of this *Chap.*) and the fifth Power of $\frac{b}{a}$ is $\frac{bbbbb}{aaaaa}$, this multiplied by the firſt Term a produces $\frac{bbbbb}{aaaa}$, which is the ſixth Proportional ſought, (as is evident by *Sect.* 6.) but if the firſt Term a be greater than the ſecond Term b, then the Reaſon of a to b is $\frac{a}{b}$, whoſe fifth Power is $\frac{aaaaa}{bbbbb}$, by which if you divide the firſt Term a, the Quotient is the ſixth Term $\frac{bbbbb}{aaaa}$.

This Rule may be exemplified by the four following Ranks of Numbers in continual Proportion.

2 , 6 , 18 , 54 , 162 , ~~468~~ 486 ÷

3072 , 768 , 192 , 48 , 12 , 3 ÷

2 , 3 , $\frac{9}{2}$, $\frac{27}{4}$, $\frac{81}{8}$, $\frac{243}{16}$ ÷

$\frac{1024}{81}$, $\frac{256}{27}$, $\frac{64}{9}$, $\frac{16}{3}$, 4 , 3 ÷

VIII. If there be given two Integers expreſſing a Reaſon in the leaſt Terms, and it be deſired to find out a given multitude of continual Proportionals in the ſame Reaſon, and that all the Terms may be Integers; Firſt, to thoſe two Integers, or firſt and ſecond Proportionals given, find out (by *Sect.* 5. or 6. of this *Chap.*) ſo many Proportionals as with thoſe given may make the deſired multitude: then multiply every Term by the Denominator of the laſt Term, ſo ſhall the Products be continual Proportionals in Integers in the ſame Reaſon as the two Terms firſt given. As for Example: If a and b be given, and it be deſired to find three Proportionals in Integers in the Reaſon of a to b, firſt, to a and b I find a third Proportional, which (by *Sect.* 5.) is $\frac{bb}{a}$ then a, b, $\frac{bb}{a}$ being multiplied ſeverally by the Denominator a, the Products aa, ab, bb, are Proportionals expreſs'd by Integers, and in the Reaſon of a to b, as was deſired.

Hence if $a=2$, and $b=3$; then aa, ab, and bb will give 4, 6, and 9, which are continual Proportionals in Integers in the given Reaſon of 2 to 3.

So if four continual Proportionals in the Reaſons of a to b, be deſired; firſt, (by *Sect.* 5. or 6.) theſe will be found continual Proportionals, to wit, a, b, $\frac{bb}{a}$, $\frac{bbb}{aa}$ which multiplied ſeverally by aa, (the Denominator of the laſt Term) will produce aaa, aab, abb, bbb, which are four continual Proportionals in Integers in the given Reaſon of a to b. Hence if $a=2$, and $b=3$, then aaa, aab, abb, and bbb, will give 8, 12, 18, and 27, which are continual Proportionals in Integers in the given Reaſon of 2 to 3.

In like manner theſe five Quantities $aaaa$, $aaab$, $aabb$, $abbb$, and $bbbb$, will be found continual Preportionals in the Reaſon of a to b; ſo that if $a=2$, and $b=3$, then thoſe five Proportionals will give theſe five, to wit, 16, 24, 36, 54, and 81 ÷ in the Reaſon of 2 to 3. After the ſame manner you may proceed infinitely.

IX. If there be Quantities in continual Proportion, how many ſoever, the Product made by the Multiplication of the Extremes is equal to the Product of any two Means equally diſtant from the Extremes; and alſo to the Square of the Mean Term, when the number of Terms is odd. As for Example: If a,b,c,d,e,f, be continual Proportionals, I ſay, the Product of the Extremes a and f, to wit af, is equal to the Product of any two Terms equally diſtant from the Extremes, *viz.* to the Product cd, and to the Product be: For,

1. By ſuppoſition, (and by *Sect.* 1. and 2.) . . . . $\frac{a}{b}=\frac{e}{f}$

2. Therefore by multiplying each part by f, it produces . $\frac{af}{b}=e$

3. And by multiplying each part of the laſt Equation by b, it gives $af=be$

4. Again, by ſuppoſition . . . . $\frac{b}{c}=\frac{d}{e}$

5. Therefore (by multiplying in like manner as before) . . $cd=be$

6. Therefore from the third and fifth Equation (*per* 1. *Axiom.* 1. *Elem. Euclid.* . . . . $\Big\}$ $af=cd=be$

Which was to be proved. And if more continual Proportionals even in multitude were propoſed, the Demonſtration would not be otherwiſe.

But if the multitude of Terms be odd, as in theſe ſeven Quantities which we may ſuppoſe to be continually proportional, $a,b,c,d,e,f,g \div$; then the Product made by the Multiplication of the two Extremes a and g is equal to the Square of the middle Term d, *viz.* $ag=dd$. For,

1. By ſuppoſition (and by *Sect.* 1. and 2.) . . . . $\frac{c}{d}=\frac{d}{e}$

2. Therefore by multiplying each part of that Equation by d, it makes . . . . $\Big\}$ $c=\frac{dd}{e}$

3. And by multiplying each part of the laſt Equation by e, it produces . . . . $\Big\}$ $ce=dd$

4. And by what has been already proved in the firſt part of this Propoſition, ... $\Big\}$ $ce=ag$

5. Therefore from the two laſt Equations (*per* 1. *Ax.* 1 *Elem. Eucl.*) $ag=dd$

Which was to be proved. Therefore the Propoſition is every way manifeſt; but for further Illuſtration:

Let there be propoſed theſe ſix continual Proportionals in Numbers, to wit, . . . $\Big\}$ $2, 6, 18, 54, 162, 486 \div$

Then according to the firſt part of the Propoſition, ... $\Big\}$ $2\times486=6\times162=18\times54=972$

Again, let there be propoſed theſe ſeven continual Proportionals, to wit, ... $\Big\}$ $2, 6, 18, 54, 162, 486, 1458$

Then according to the latter part of the Propoſition, $\Big\}$ $2\times1458=54\times54=2916.$

X. If four Quantities be Proportionals, $a . b :: c . d$, they ſhall be alſo Alternly, and Inverſly, and Compoſedly, and Dividedly, and Converſly, Proportionals, *viz.*

If $\left\{\begin{array}{l} a \;.\; b \;::\; c \;.\; d \\ 6 \;.\; 4 \;::\; 12 \;.\; 8 \end{array}\right\}$

Then Alternly, $\left\{\begin{array}{l} a \;.\; c \;::\; b \;.\; d \\ 6 \;.\; 12 \;::\; 4 \;.\; 8 \end{array}\right\}$ *Per* 16. *Prop.* 5. *Elem. Eucl.*

And Inverſly, $\left\{\begin{array}{l} c \;.\; a \;::\; d \;.\; b \\ 12 \;.\; 6 \;::\; 8 \;.\; 4 \end{array}\right\}$ *Per Cor. of Prop.* 4. *Elem.* 5.

And Compoſedly, $\left\{\begin{array}{l} a+b \;.\; b \;::\; c+d \;.\; d \\ 10 \;.\; 4 \;::\; 20 \;.\; 8 \end{array}\right\}$ *Per* 18. *Prop.* 5. *Elem.*

And Dividedly, $\left\{\begin{array}{l} a-b \;.\; b \;::\; c-d \;.\; d \\ 2 \;.\; 4 \;::\; 4 \;.\; 8 \end{array}\right\}$ *Per* 17. *Prop.* 5. *Elm.*

And Converſly, $\left\{\begin{array}{l} a \;.\; a+b \;::\; c \;.\; c+d \\ 6 \;.\; 10 \;::\; 12 \;.\; 20 \end{array}\right\}$ *Per Cor. of Prop.* 19. *Elem.* 5.

But that the Learner may the better perceive the meaning and uſe of theſe ways of arguing about Proportionals, I ſhall apply ſome of them to the Reſolution of this following

QUEST.

The Difference (b) between the greater extreme and mean of three Quantities continually proportional being given, as alſo the Difference (c) between the mean and the leſſer Extreme, to find the Proportionals; but the firſt Difference muſt be greater than the latter.

RESOLUTION.

1. For the mean Proportional ſought put . . . . a
2. To which adding the given Difference (b) the Sum is the greater Extreme, to wit, ... $a+b$
3. But if from the Mean (a) the given Difference (c) be ſubtracted, the Remainder is the leſſer Extreme, to wit, . . $a-c$
4. Then (according to the Queſtion) theſe three Quantities $a+b$, a, and $a-c$ muſt be in continual proportion, *viz.* $a+b\,.\,a :: a\,.\,a-c$
5. Therefore by Diviſion of Reaſon, ... $b\,.\,a :: c\,.\,a-c$
6. And alternately (or by Permutation) . . . . . . $b\,.\,c :: a\,.\,a-c$
7. And by Diviſion of Reaſon, ... $b-c\,.\,c :: c\,.\,a-c$
8. Wherefore by Converſion of Reaſon, ... $b-c\,.\,b :: c\,.\,a$

Which laſt Analogy if it be expreſs'd by Words gives this

CANON.

As the Difference between the two given Differences is to either of them, ſo is the other to the mean Proportional ſought.

Therefore if $36=b$, and $12=c$, the Canon will diſcover 18 for the mean Proportional ſought, (to wit, a in the Reſolution) which increaſed with 36, and leſſened by 12, gives 54 and 6 for the Extremes. Therefore the three Proportionals ſought are manifeſtly 54, 18, and 6.

Note. If the Analogy in the fourth ſtep of the Reſolution be converted into an Equation, by comparing the Product made by the mutual Multiplication of the Extremes to the Product of the Means, that Equation after due Reduction will give the ſame Canon as above; ſo that the Argumentation in the four laſt ſteps of the Reſolution is not of neceſſity, but only to ſhew how without the help of any Equation the Number ſought may ſometimes be made the fourth Term of an Analogy, whoſe three firſt Terms are known, whence by the Rule of Three the Number ſought is alſo known. Which ways of inferring one Analogy out of another are more proper when the Nature of a Queſtion will admit the ſame, than the common way of proceeding by Equations, eſpecially in the Reſolution of Geometrical Problems, where every ſtep ought to be expreſs'd in the moſt ſimple Terms, to the end the Compoſition of the Problem may the more eaſily be formed by the ſteps of the Reſolution; but in a retrograde or backward Order, as I ſhall hereafter ſhew in the fourth Book of my *Algebraical Elements.*

XI. If Proportionals be multiplied or divided by Proportionals, the Products alſo or Quotients ſhall be Proportionals; as,

If theſe four Proportional Numbers, to wit, ...	a 2	.	b 4	::	ca 3×2	.	$cb.$ 3×4
be multiplied by theſe four Proportional Numbers, ...	d 5	.	f 6	::	gd 7×5	.	$gf.$ 7×6
there will be produced theſe four Proportional Numbers, to wit, . . .	ad 2×5	.	bf 4×6	::	$cgad$ $3\times7\times2\times5$	.	$cgbf$ $3\times7\times4\times6$

whereby the firſt part of the Propoſition is manifeſt.

And if theſe four Proportional Numbers, to wit, ...	ad	.	bf	::	$cgad$	.	$cgbf.$
be divided by theſe four Proportionals, to wit, ...	d	.	f	::	gd	.	$gf.$
the Quotients will be theſe four Proportionals, to wit, ...	a	.	b	::	ca	.	$cb.$

whereby the latter part of the Propoſition is manifeſt.

Hence it may eaſily be proved, that the Squares, Cubes, fourth Powers, fifth Powers, &c. of proportional Numbers ſhall be alſo Proportionals; as,

If . . . . $a \,.\, b :: ca \,.\, cb$

Then their Squares alſo ſhall be proportionals, *viz.* $aa \,.\, bb :: ccaa \,.\, ccbb$

And the Cubes of the firſt four Proportionals ſhall alſo be Proportionals, *viz.* . . . . . . $aaa \,.\, bbb :: cccaaa \,.\, cccbbb$

And ſo of higher Powers.

XII. In every Series or Rank of Quantities continually proportional, all the mean Terms between the firſt and the laſt are both Antecedents and Conſequents of Reaſons; as

If . . . . $a, b, c, d, e, f \div$

That is, ... $a \,.\, b :: b \,.\, c :: c \,.\, d :: d \,.\, e :: e \,.\, f$

It is evident that every Term except the laſt (f) is a Antecedent of a Reaſon, and every Term except the firſt (a) is a Conſequent; wherefore if (s) be put for the ſum of all the Terms in the Series, then $s-f$ ſhall be the ſum of all the Antecedents, and $s-a$ the ſum of all the Conſequents. Therefore,

From the premiſſes (*per* 12 *Prop.* 5. *Elem. Eucl.*) this Analogy ariſes, *viz.* . . . . $a \,.\, b :: s-f \,.\, s-a$

Whence by comparing the Product of the Extremes to the Product of the Means . . . . $as-aa=bs-bf$

Therefore by due Tranſpoſition in that Equation when b is greater than a, ... $bf-aa=bs-as$

And by dividing each part of the laſt Equation by $b-a$, there ariſes . . . . $\frac{bf-aa}{b-a}=s$

But if a exceed b, then there will ariſe . : . $\frac{aa-bf}{a-b}=s$

Which two laſt Equations give a Cannon to find the ſum of all the Terms of a Geometrical Progreſſion, the firſt, ſecond, and laſt Term being ſeverally given.

CANON.

Divide the difference between the ſquare of the firſt Term, and the Product made by the Multiplication of the ſecond Term into the laſt, by the difference of the firſt and ſecond Terms, ſo the Quotient ſhall be the ſum of all the Terms of the Geometrical Progreſſion propoſed.

Examples in Numbers.

Let the Values of theſe . . . . . $a, b, c, d, e, f \div$
be expreſs'd by theſe Numbers, . . 32, 48 72, 108, 162, 243 ÷

Then by the Canon . . . . . $\frac{bf-aa}{b-a}=665$ the ſum of all.

But if the Values of the ſame Proportionals . . . . $a, b, c, d, e, f \div$
be expounded by theſe Numbers . . 243, 162, 108, 72, 48, 32 ÷

Then by the Canon . . . : $\frac{aa-bf}{a-b}=665$ the ſum of all.

XIII. If what has been ſaid in the eight *Sect.* of this *Chap.* be compared with the Table in *Sect.* 4. *Chap.* 1. of this Book, it will be manifeſt, that if we caſt away the Numbers of Multitude which are prefix'd to all the mean Terms or Members belonging to any Compound Power produced from a Binomial Root, ſuppoſe from $a+e$, then all the Members or ſimple Quantities whereof the ſaid Compound Power is compoſed, are in continual Proportion. As for Example: The Members whereof the ſquare of $a+e$ is compoſed are aa, $2ae$, and ee; now if 2 which is prefix'd to ae be caſt away, then aa, ae, and ee are Continual Proportionals, (as is evident by the preceeding eight *Sect.* of this *Chap.*)

Again, it appears by the ſaid Table, that the Members whereof the Cube of $a+e$ is compoſed are aaa, $3aae$, $3aee$, and eee; here if 3 and 3 which are prefix'd to the mean Terms be caſt away, then theſe four Quantities aaa, aae, aee, and eee will be in Continual Proportion.

Likewiſe, foraſmuch as the fourth Power of $a+e$ is compoſed of theſe Members, $aaaa$, $4aaae$, $6aace$, $4aeee$, and $eeee$; by caſting away the Numbers of Multitude 4, 6, and 4, theſe five Quantities $aaaa$, $aaae$, $aace$, $aeee$, and $eeee$, ſhall be continual Proportionals. And ſo of higher Powers infinitely.

XIV. Foraſmuch as by the laſt preceding *Sect.* theſe Quantities are in continual proportion to wit, } . $aa\,,\ ae\,,\ ee \div$

Therefore their ſquareRoots alſo ſhall be in continual proportion, (*per* 22 *Prop.* 6. *Elem. Eucl.*) to wit, } . $a\,,\ \sqrt{ae},\ e \div$

Hence if a mean Proportional between any two given Numbers a and e be deſired, it ſhall be $\sqrt{ae}$; as if $a=12$ and $e=3$, then $ae=36$, and $\sqrt{ae}$ or $\sqrt{36}$, that is, 6, is a mean Proportional between 12 and 3; for as 12 is to 6, ſo is 6 to 3.

Again, foraſmuch as theſe Quantities are in continual Proportion, to wit, ... } $aaa\,,\ aae\,,\ aee\,,\ eeee \div$

Therefore their CubicRoots alſo ſhall be continual Proportionals, (*per* 37. *Prop.* 11. *Elem. Eucl.*) to wit, } $a,\ \sqrt{(3)}aae,\ \sqrt{(3)}aee,\ e \div$

Hence if two mean Proportionals between any two given Numbers (a the greater and e the leſſer) be deſired, then $\sqrt{(3)}\,aae$ ſhall be the greater Mean, and $\sqrt{(3)}aee$ the leſſer; as if $a=54$ and $e=2$, then $aae=5832$, and $\sqrt{(3)}aae=\sqrt{(3)}5832$; therefore $\sqrt{(3)}\,5832$, that is, 18 is the greater Mean ſought; alſo $aee=216$, and therefore $\sqrt{(3)}216$, that is, 6 is the leſſer Mean: ſo that 18 and 6 are the two deſired Mean Proportionals between 54 and 2; for 54, 18, 6, and 2, are in continual proportion. But when one Mean next to either of the Extremes is found out, the other Mean may be found out by *Sect.* 5. of this *Chap.* without extracting any Root.

After the ſame manner by the help of the ſaid Table in *Sect.* 4. *Chap.* 1. of this Book, continued to higher Powers if need be, you may find out as many mean Proportional Numbers as ſhall be deſired between any two given Numbers. As, if you would find five mean proportional Numbers between 1458 (or a) and 2 (or e;) look into the ſaid Table for the ſixth Power, (to wit, a Power whoſe Index exceeds by Unity the number of Means ſought) and you will find $aaaaaa$, $6aaaaae$, $15aaaaee$, $20aaaeee$, $15aaeeee$, $6aeeeee$, and $eeeeee$; then caſting away 6, 15, 20, 15, and 6, which are prefix'd to the mean terms, and extract $\sqrt{(6)}$ out of every one of thoſe ſix Terms after the ſaid Numbers prefix'd are caſt away, there will ariſe a, $\sqrt{(6)}aaaaae$, $\sqrt{(6)}aaaaee$, $\sqrt{(6)}aaaeee$, $\sqrt{(6)}aaeeee$, $\sqrt{(6)}aeeeee$, and $e\div$; now to find the five mean proportional Numbers anſwering to thoſe five proportional Roots expreſs'd by Letters which fall between a and e, it will be convenient to find the ſmalleſt Mean firſt, *viz.* foraſmuch as a was put for 1458, and e for 2; therefore $aeeeee=46656$, and $\sqrt{(6)}aeeeee=\sqrt{(6)}46656$; that is, 6 ſhall be the leaſt Mean ſought: then 2 being the firſt Proportional, or leſſer Extreme, and 6 the ſecond, the third will (by *Sect.* 5. of this *Chap.*) be found 18, the fourth 54, the fifth 162, the ſixth 486, and the ſeventh, to wit, the greater Extreme, was firſt given 1458: ſo that between 2 and 1458 five mean Proportionals are found out, as was deſired; and the ſeven continual Proportionals are theſe, to wit, 2, 6, 18, 54, 162, 486, and 1458.

Many other admirable Properties adherent to Numbers in Geometrical Proportion continued, are deducible from the ſaid Table of Powers in *Sect.* 4. *Chap* 1. of this Book, as will partly appear by the Theorems in the following ſixth Chapter, which I find diſperſed in ſeveral Algebraical Treatiſes.

CHAP.

CHAP. VI.

Various Theorems about Quantities in Continual Proportion.

Theorem 1.

IF three Numbers be Proportionals, the Solid Number made by the Continual Multiplication of all the three is equal to the Cube of the Mean.

Let three Proportionals be expoſed in Integers according to *Sect.* 8. or 13. of the preceding *Chap.* 5.

$$\left.\begin{matrix} aa\,, & ae\,, & ee \div \\ 9\,, & 6\,, & 4 \div \end{matrix}\right.$$

Thence it is evident, that $aaaeee$, the Product made by the Multiplication of all the three Proportionals one into another, is equal to the Cube of the Mean ae, as is affirmed by the Theorem.

Theorem 2.

If three Numbers be Proportionals, the Product made by the Multiplication of the Square of the firſt by the third, is equal to the Product of the Square of the ſecond by the firſt.

As in theſe three, ... $\left\}\begin{matrix} aa\,, & ae\,, & ee \div \\ 9\,, & 6\,, & 4 \div \end{matrix}\right.$

It is evident that $aaaa \times ee = aaee + aa = aaaaee$.

Theorem 3.

If three Numbers be Proportionals, the Square of the Sum of the Extremes is equal to both the Squares of the Extremes, together with twice the Square of the Mean.

As in theſe three, ... $\left\}\begin{matrix} aa\,, & ae\,, & ee \div \\ 9\,, & 6\,, & 4 \div \end{matrix}\right.$

The Square of $aa+ee$ is $aaaa+2aaee+eeee$, which is manifeſtly equal to the Squares of aa and ee, together with twice the Square of ae.

Theorem 4.

If three Numbers be Proportionals, the Product of the leſſer Extreme multiplied by the difference of the Extremes, is equal to the difference of the Squares of the mean and leſſer Extreme.

As in theſe three, ... $\left\}\begin{matrix} aa\,, & ae\,, & ee \div \\ 9\,, & 6\,, & 4 \div \end{matrix}\right.$

It is evident that $ee \times aa - ee = aaee - eeee$

Theorem 5.

If three Numbers be Proportionals, the Product of the greater Extreme multiplied by the difference of the Extremes, is equal to the difference of the Squares of the greater Extreme and the Mean.

As in theſe three . . . . $\left\}\begin{matrix} aa\,, & ae\,, & ee \div \\ 9\,, & 6\,, & 4 \div \end{matrix}\right.$

It is evident that $aa \times aa - ee = aaaa - aaee$.

Theorem 6.

If three Numbers be Proportionals, the difference of the Squares of the Extremes is equal to the Square of the difference of the Extremes, together with twice the difference of the Squares of the mean and leſſer Extreme.

As in theſe three, ... $\left\}\begin{matrix} aa\,, & ae\,, & ee \div \\ 9\,, & 6\,, & 4 \div \end{matrix}\right.$

1. The difference of the Squares of the Extremes is $aaaa - eeee$
2. The ſquare of $aa - ee$ (the difference of the Extremes) is . . . . $\}\, aaaa - 2aaee + eeee$
3. The double of the difference of the Squares of the mean and leſſer Extreme is . . . . . . $\}\, +2aaee - 2eeee$

Now the Sum of the two later of thoſe three Quantities is manifeſtly equal to the firſt, as the Theorem affirms.

Theorem 7.

If three Numbers be Proportionals, the difference of the Squares of the greater Extreme and the Mean is equal to the Square of the difference of the Extremes, and to the difference of the Squares of the Mean and the leſſer Extreme

As in theſe three, ... $aa\ ,\ ae\ ,\ ce\ \div$ / $9\ ,\ 6\ ,\ 4$

1. The difference of the Squares of the greater Extreme and the Mean is . . . . $aaaa-aaee$
2. The Square of $aa-ee$ (the difference of the Extremes) is . . . . $aaaa-2aaee+eeee$
3. The difference of the Squares of the Mean and leſſer Extreme is . . . . $+aaee-eeee$

Now the Sum of the two latter of thoſe three Quantities is manifeſtly equal to the firſt, as the Theorem affirms.

Theorem 8.

If three Numbers be Proportionals, then as the firſt is to the third, ſo is the Square of the firſt to the Square of the ſecond; and ſo is the Square of the ſecond to the Square of the third.

As in theſe three, ... $aa\ ;\ ae\ ,\ ee\ \div$ / $9\ ,\ 6\ ,\ 4\ \div$

1. It is evident that . . . . $aa\ .\ ee :: aa\ .\ ee$
2. Therefore by drawing *aa* as a common Factor into the two latter Terms of that Analogy, this ariſes, $aa\ .\ ee :: aaaa\ .\ aaee$
3. And by drawing *ee* as a common Factor into the two latter Terms of the firſt Analogy, this ariſes, ... $aa\ .\ ee :: aaee\ .\ eeee$

By which two laſt Analogies the truth of the Theorem is manifeſt.

Theorem 9.

If three Numbers be Proportionals, then as the firſt is to the ſecond, (or as the ſecond is to the third) ſo is the difference of the firſt and ſecond, to the difference of the ſecond and third.

As in theſe three, ... $aa\ ,\ ae\ ,\ ee\ \div$ / $9\ ,\ 6\ ,\ 4$

1. It is evident (as before hath been ſhewn in *Theorem* 4.) that, ... $ee\times\overline{aa-ee}=aaee-eeee$
2. And by Multiplication it will appear that, . . . $\overline{ae+ee}\times\overline{ae-ee}=aaee-eeee$
3. Therefore from the two laſt Equations *(per* 1 *Ax.* 1 *Elem. Eucl.)* . . . . $ee\times\overline{aa-ee}=\overline{ae+ee}\times\overline{ae-ee}$
4. Therefore by reſolving the laſt Equation into Proportionals, ... $aa-ee\ .\ ae-ee :: ae+ee\ .\ ee$
5. Therefore by diviſion of Reaſon, ... $aa-ae\ .\ ae-ee :: ae\ .\ ee$

Which was to be Demonſtrated.

Theorem 10.

If four Numbers be continually proportional, the Sum of the Means is a mean Proportional between the ſum of the firſt and ſecond, and the ſum of the third and fourth.

Let four continual Proportionals be expos'd in Integers, to wit, ... $aaa\ ,\ aae\ ,\ aee\ ,\ eee\ \div$ / $8\ ,\ 4\ ,\ 2\ ,\ 1\ \div$

Then according to the import of the Theorem, it muſt be proved that theſe three Quantities are Proportionals, *viz.*

$$aaa+aae\ .\ aae+aee\ .\ aee+eee\ \div$$

But that they are Proportionals it will be evident by Multiplication, for the Product of the Extremes is equal to the Square of the Mean: therefore the Truth of the Theorem is manifeſt.

Theorem 11.

If four Numbers be continual Proportionals, the Sum of all is to the Sum of the Means, as the Sum of the firſt and third to the ſecond.

As in theſe four, ... } aaa, aae, aee, eee ∺
8, 4, 2, 1 ∺

1. The Sum of all four is . . . . $aaa+aae+aee+eee$
2. The Sum of the Mean is . . . . $+aae+aee$
3. The Sum of the firſt and third is . . . . $aaa\ +aee$
4. And the ſecond is . . . . $+aae$

I ſay, thoſe four Quantities are Proportionals in ſuch order as they are above written; for it will appear by Multiplication, that the Product of the Extremes is equal to the Product of the Means: therefore the Theorem is manifeſt.

Theorem 12.

If four Numbers be in continual Proportion, the Sum of all is to the Sum of the Means, as the Sum of the Squares of the Means is to the Product of the Means or Extremes.

As in theſe four, ... } aaa, aae, aee, eee ∺
8, 4, 2, 1 ∺

1. The Sum of all is . . . . $a^3+a^2e+ae^2+e^3$
2. The Sum of the Mean is . . . . $+a^2e+ae^2$
3. The Sum of the Squares of the Means is . . . . $+a^4e^2+a^2e^4$
4. The Product of the Means or Extremes is . . . . $+a^3e^3$

I ſay, thoſe four Quantities are Proportionals, in ſuch order as they are above written; for it will appear by Multiplication, that the Product of the Extremes is equal to the Product of the Means: therefore the Theorem is manifeſt.

Theorem 13.

If four Numbers be continual Proportionals, the Sum of the Squares of the Mean is a mean Proportional between the Sum of the Squares of the firſt and ſecond, and the Sum of the Squares of the third and fourth.

As in theſe four, ... } aaa, aae, aee, eee ∺
8, 4, 2, 1 ∺

1. The ſum of the Squares of the firſt and ſecond is . . . . } $a^6+a^4e^2$
2. The Sum of the Squares of the Means is . . . . . . $a^4e^2+a^2e^4$
3. The Sum of the Squares of the third and fourth is . . . . } $a^2e^4+e^6$

I ſay, thoſe three Quantities are Proportionals in ſuch order as they are above written; for it will appear by Multiplication that the Square of the Mean (or ſecond Quantity) is equal to the Product of the Extremes: therefore the Theorem is manifeſt.

Theorem 14.

If four Numbers be continual Proportionals the Square of the Sum of the Means is equal to the Square of their difference, together with four times the Product of the Extremes or Means.

As in theſe four, ... } aaa, aae, aee, eee ∺
8, 4, 2, 1 ∺

1. The Square of a^2e+ae^2 (the ſum of the Means) is . . . . } $a^4e^2+2a^3e^3+a^2e^4$
2. The Square of a^2e-ae^2 (the difference of the Means is . . . . } $a^4e^2-2a^3e^3+a^2e^4$
3. The Quadruple of the Product of the Extremes or Means is . . . . } . $+4a^3e^3$

Now it is Evident that the firſt of thoſe three Quantities is equal to the Sum of the ſecond and third: therefore the Theorem is manifeſt.

Theorem 15.

If four Numbers be continual Proportionals, the Sum of their Squares ſhall be to the Sum of the Products of the firſt into the ſecond, and the third into the fourth; as the ſum of all the four Proportionals to the ſum of the Means.

As in theſe four, ... $\left.\begin{matrix} aaa, & aae, & aee, & eee \div \\ 8, & 4, & 2, & 1 \div \end{matrix}\right.$

1. The ſum of the Squares of the four Proportionals is . . . . $a^6 + a^4e^2 + a^2e^4 + e^6$
2. The ſum of the Products of the firſt into the ſecond, and the third into the fourth is . . . . . . $a^5e + ae^5$
3. The ſum of all the four Proportionals is . . . . . $a^3 + a^2e + ae^2 + e^3$
4. The ſum of the Means is . . . . $a^2e + ae^2$

I ſay, thoſe four Quantities are Proportionals in ſuch order as they are above ſeated; for it will appear by Multiplication, that the Product of the Extremes is equal to the Product of the Means. Therefore the Theorem is manifeſt.

Theorem 16.

If from the ſquare of the ſum of four Numbers in continual proportion the ſum of their ſquares be ſubtracted, and from half the Remainder there be alſo ſubtracted the ſquare of the ſum of the two Means, this latter Remainder ſhall be the ſum of the Products of the firſt Proportional into the ſecond, and of the third into the fourth, and ſhall be to the ſum of the ſquares of thoſe four Proportionals, as the ſum of the two Means is to the ſum of all the Proportionals.

As in theſe four, ... $\left.\begin{matrix} aaa, & aae, & aee, & eee \div \\ 8, & 4, & 2, & 1 \div \end{matrix}\right.$

1. The ſquare of the ſum of the four Proportionals will by Multiplication be found
$$a^6 + 2a^5e + 3a^4e^2 + 4a^3e^3 + 3a^2e^4 + 2ae^5 + e^6.$$
2. The Sum of the ſquares of the four Proportionals is
$$a^6 \qquad + a^4e^2 \qquad + a^2e^4 \qquad + e^6.$$
3. Which Sum of the ſquares being ſubtracted from the ſaid ſquare of the ſum, the half of the Remainder will be
$$+ a^5e + a^4e^2 + 2a^3e^3 + a^2e^4 + ae^5$$
4. The ſquare of the ſum of the two Means, to wit, of $a^2e + ae^2$ is
$$+ a^4e^2 + 2a^3e^3 + a^2e^4.$$
5. Which laſt mentioned ſquare being ſubtracted from the half Remainder in the third ſtep, there will remain the ſum of the Products of the firſt Proportional into the ſecond, and of the third into the fourth, to wit,
$$+ a^5e + ae^5$$
6. Now according to the import and meaning of the Theorem it remains to prove, that the Remainder in the laſt ſtep is to the ſum of the ſquares in the ſecond ſtep, as the ſum of the two mean Proportionals is to the ſum of all four, *viz.* that

$$\text{Theſe four Quantities are Proportionals,}\left\{\begin{array}{l} + a^5e + ae^5 \\ + a^6 + a^4e^2 + a^2e^4 + e^6 \;:: \\ + a^2e + ae^2. \\ + a^3 + a^2e + ae^2 + e^3. \end{array}\right.$$

7. But that they are Proportionals will be evident by Multiplication; for the Product of the Extremes is equal to the Product of the Means, each Product being
$$a^8e + a^7e^2 + a^6e^3 + a^5e^4 + a^4e^5 + a^3e^6 + a^2e^7 + ae^8.$$

Therefore the Theorem is manifeſt.

Theorem 17.

If four Numbers be Continual Proportionals, the ſum of all their Squares ſhall be to the ſum of the ſquares of the Means, as the ſum of the Products of the firſt into the ſecond, and the third into the fourth, to the Product of the Means or Extremes.

This is inferr'd from *Theorem* 12. and 15. by exchange of equal Reaſons.

Theorem 18.

If four Numbers be Continual Proportionals, the ſum of the ſquares of the Extremes ſhall be to the ſum of the ſquares of the Means; as the Exceſs whereby the ſum of the Products of the firſt into the ſecond, and third into the fourth, exceeds the Product of the Means, is to the Product of the Means or Extremes.

This is inferr'd from *Theorem* 17. by Diviſion of Reaſon.

Theorem 19.

If four Numbers be Continual Proportionals, the ſum of the firſt and third ſhall be to the ſecond; as the ſum of the Squares of the Means is to the Product of the Means or Extremes.

This is deduced from *Theorem* 11. and 12. by exchange of equal Reaſons.

Theorem 20.

If four Numbers be continual Proportionals, the ſum of all their Squares ſhall be to the ſum of the Products of the firſt into the ſecond, and the third into the fourth; as the ſum of the firſt and third is to the ſecond.

This is deduced from *Theorem* 17. and 19. by exchange of equal Reaſons.

Theorem 21.

If four Numbers be continual Proportionals, the ſum of the Cubes of the Means is equal to the Product made by the Multiplication of the ſum of the Extremes into the Product of the Means or Extremes.

As in theſe four, ... $\left\}\begin{matrix} aaa, & aae, & aee, & eee \div\div \\ 8, & 4, & 2, & 1 \div\div \end{matrix}\right.$

1. The Sum of the Cubes of the Means is . $a^6e^3+a^3e^6$
2. The ſum of the Extremes is . . . . a^3+e^3
3. The Product of the Means or Extremes is . a^3e^3

Now it is evident, that the firſt of thoſe three Quantities is equal to the Product of the ſecond Quantity multiplied by the third, as affirmed by the Theorem.

Theorem 22.

If four Numbers be continual Proportionals, the Cube of the ſum of the Extremes is equal to the Cubes of the Extremes, together with the triple ſum of the Cubes of the Means.

As in theſe four, ... $\left\}\begin{matrix} aaa, & aae, & aee, & eee \div\div \\ 8, & 4, & 2, & 1 \div\div \end{matrix}\right.$

1. The Cube of a^3+e^3 (the ſum of the Extremes) is . . . . $a^9+3a^6e^3+3a^3e^6+e^9$
2. The Cubes of the Extremes is . . . . a^9+e^9
3. The triple ſum of the Cubes of the Means is . $3a^6e^3+3a^3e^6$

Now it is manifeſt, that the firſt of thoſe three Quantities is equal to the ſum of the other two, as the Theorem affirms.

Theorem 23.

If four Numbers be continual Proportionals, the difference of the Cubes of the Extremes is equal to the triple of the difference of the Cubes of the Means, together with the Cube of the difference of the Extremes.

As in theſe four, ... $\left\}\begin{matrix} aaa, & aae, & aee, & eee \div\div \\ 8, & 4, & 2, & 1 \div\div \end{matrix}\right.$

1. The difference of the Cubes of the Extremes is a^9-e^9
2. The Triple of the difference of the Cubes of the Means is . . . . $3a^6e^3-3a^3e^6$
3. The Cube of a^3-e^3 (the difference of the Extremes) is . . . . $a^9-3a^6e^3+3a^3e^6-e^9$

Now it is manifeſt, that the firſt of thoſe three Quantities is equal to the ſum of the other two; which was to be prov'd.

Theorem 24.

If four Numbers be Continual Proportionals, the Cube of the Sum of the firſt and ſecond is equal to the Product made by the Multiplication of the ſquare of the firſt by the Aggregate of the ſum of the Extremes and the triple ſum of the Means.

As in theſe four, ... } aaa, aae, aee, eee ÷ 8, 4, 2, 1 ÷

1. The Cube of the ſum of the firſt and ſecond to wit, of a^3+aae is . . . . . } $a^9+3a^8e+3a^7e^2+a^6e^3$
2. The Square of the firſt is . . . . a^6
3. The Aggregate of the Extremes and the triple of the ſum of the Means is . . . } $a^3+e^3+3a^2e+3ae^2$

Now it is evident that the firſt of thoſe three Quantities is equal to the Product made by the Multiplication of the third by the ſecond; which was to be proved.

Theorem 25.

If four Numbers be continual Proportionals, the Cube of the ſum of the Means is equal to the Product made by the Multiplication of the Product of the Extremes or Means into the Aggregate of the Extremes and the triple ſum of the Means.

As in theſe four, ... } aaa, aae, aee, eee ÷ 8, 4, 2, 1 ÷

1. The Cube of the ſum of the Means, to wit, of a^2e+ae^2 is . . . . } $a^6e^3+3a^5e^4+3a^4e^5+a^3e^6$
2. The Product of the Extremes or Means is . a^3e^3
3. The Aggregate of the Extremes and the triple ſum of the Means is . . . . . } $a^3+e^3+3a^2e+3ae^2$

Now it is evident that the firſt of thoſe three Quantities is equal to the Product of the two latter; which was to be proved.

Theorem 26.

If four Numbers be continual Proportionals, the Product made by the Multiplication of the ſum of the Extremes by the Sum of the Squares of the Extremes, is equal to the Cubes of the four Proportionals.

As in theſe four, ... } aaa, aae, aee, eee ÷ 8, 4, 2, 1 ÷

1. The ſum of the Extremes is . . . . . . a^3+e^3
2. The ſum of the ſquares of the Extremes is . a^6+e^6
3. The Product of theſe two ſums is . . . . $a^9+a^6e^3+a^3e^6+e^9$
4. The ſum of the Cubes of the four Propornals is . . . . } $a^9+a^6e^3+a^3e^6+e^9$

But the Product in the third ſtep is manifeſtly equal to the ſum in the fourth; as the Theorem affirms.

Theorem. 27.

If five Number be continual Proportionals, the Product of the Mean (or third Proportional) into the ſum of the Extremes, is equal to the Squares of the ſecond and fourth.

As in theſe five, ... } $aaaa$, $aaae$, $aaee$, $aeee$, $eeee$ 16, 8, 4, 2, 1

1. The Product of the Mean into the Sum of the Extremes is . . . . } $a^6e^2+a^2e^6$
2. And the ſum of the Squares of the ſecond and fourth is alſo . . . . } $a^6e^2+a^2e^6$

Therefore the Theorem is manifeſt.

Theore

Theorem 28.

If five Numbers be continual Proportionals, the ſum of the firſt, third, and fifth, ſhall be to the third; as the ſum of the Squares of the ſecond, third, and fourth, is to the ſquare of the third.

As in theſe five, ... $\left.\begin{array}{l} aaaa,\ aaae;\ aaee,\ aeee,\ eeee \\ 16;\ 8,\ 4,\ 2,\ 1 \end{array}\right.$

1. The ſum of the firſt, third, and fifth is . . . . $a^4+a^2e^2+e^4$
2. The third is . . . . a^2e^2 ::
3. The ſum of the Squares of the ſecond, third, and fourth is . . . . $a^6e^2+a^4e^4+a^2e^6$
4. The ſquare of the third is . . . . a^4e^4.

I ſay, thoſe four Quantities are Proportionals, in ſuch order as they are above ſeated; for it will appear by Multiplication, that the Product of the Extremes is equal to the Product of the Means; each Product being $a^8e^4+a^6e^6+a^4e^8$: Therefore the Theorem is manifeſt.

Theorem 29.

If five Numbers be continual Proportionals, the ſum of the Extremes more by the double of the Mean, the ſum of the ſecond and fourth, and the Mean, are alſo continual Proportionals.

As in theſe five, ... $\left.\begin{array}{l} aaaa,\ aaae,\ aaee,\ aeee,\ ecee \\ 16,\ 8,\ 4,\ 2,\ 1 \end{array}\right.$

1. The ſum of the Extremes more by the double of the Mean is . . . . $a^4+e^4+2a^2e^2$
2. The ſum of the ſecond and fourth is . . . . . . a^3e+ae^3
3. The Mean is . . . . a^2e^2

I ſay, thoſe three Quantities are Proportionals; for it will be evident by Multiplication that the Product of the firſt and third is equal to the ſquare of the ſecond: therefore the Theorem is manifeſt.

Theorem 30.

If five Numbers be continual Proportionals, the Sum of the Extremes is to the Mean; as the difference of the Squares of the Extremes, to the difference of the Squares of the ſecond and fourth.

As in theſe five, ... $\left.\begin{array}{l} aaaa,\ aaae,\ aaee,\ aeee,\ eeee \\ 16,\ 8,\ 4,\ 2,\ 1 \end{array}\right.$

1. The ſum of the Extremes is . . . . a^4+e^4.
2. The Means is . . . . a^2e^2 ::
3. The difference of the Squares of the Extremes is . . a^8-e^8.
4. The difference of theSquares of the ſecond and fourth is . . . . $a^6e^2a^2-a^2e^6$.

I ſay, thoſe four Quantities are Proportionals in ſuch order as they are above placed; for it will be evident by Multiplication, that the Product of the Extremes is equal to the Product of the Means, each Product being $a^{10}e^2-a^2e^{10}$: Therefore the Theorem is manifeſt.

Theorem 31.

If five Numbers be continual Proportionals, the ſum of the Squares of the ſecond and fourth ſhall be to the ſquare of the Mean, as the difference of the Squares of the Extremes to the difference of the Squares of the ſecond and fourth.

As in theſe five, ... $\left.\begin{array}{l} aaaa,\ aaae,\ aaee,\ aeee,\ eeee \\ 16,\ 8,\ 4,\ 2,\ 1 \end{array}\right.$

1. The ſum of the Squares of the ſecond and fourth is . $a^6e^2+a^2e^6$.
2. The Square of the Mean is . . . . a^4e^4 ::
3. The difference of the Squares of the Extremes is . a^8-c^8.
4. The difference of the Squares of the ſecond and fourth is . . . . $a^6e^2-ae^6$:

I ſay, thoſe four Quantities are Proportionals in ſuch order as they are above ſeated; for it will be evident by Multiplication, that the Product of the Extremes is equal to the Product of the Means; therefore the Theorem is manifeſt.

Theorem 32.

If five Numbers be continual Proportionals, the Sum of the Exrremes ſhall be to the Mean, as the Sum of the Squares of the ſecond and fourth is to the Square of the Mean. This is evident from the two laſt preceding Theorems by exchange of equal Reaſons.

Theorem 33.

If five Numbers be continual Proportionals, the Sum of the Squares of the ſecond and fourth ſhall be equal to the Product made by the Multiplication of the third into the Sum of the firſt and fifth.

As in theſe five, ... $\left\}\begin{matrix} aaaa, & aaae, & aaee, & aeee, & eeee \\ 16, & 8, & 4. & 2, & 1 \end{matrix}\right.$

1. The Sum of the Squares of the ſecond and foutth is . . . . $a^6e^2+a^2e^6$
2. The Mean or third is . . . . a^2e^2
3. The Sum of the firſt and fifth is . . . a^4+e^4

But the Product of the ſecond and third of thoſe three Quantities above written is equal to the firſt; therefore the Theorem is manifeſt.

CHAP. VII.

Queſtions about Quantities in Continual Proportion reſolved by Literal Algebra.

QUEST. 1.

THE Sum (b) of three Proportional Quantities being given, as alſo (c) the Sum of their Squares, to find the Proportionals.

RESOLUTION.

1. For the Mean Proportional ſought put . . . . a
2. Then ſubtracting the ſaid Mean from (b) the given Sum of all the three Proportionals, there will remain the Sum of the Extremes, to wit, ... $b-a$
3. Therefore the Square of the Sum of the Extremes is . $bb-2ba+aa$
4. From which Square if there be ſubtracted the double of the Square of the Mean, to wit, ... $2aa$
5. There will remain (as is manifeſt by *Th.* 3. of the preceding *Chap.* 6.) the Sum of the Squares of the Extremes, to wit, $bb-2ba-aa$
6. To which Sum of the Squares of the Extremes if you add (aa) the Square of the Mean, the Aggregate ſhall be the ſum of the Squares of the three Proportionals ſought, to wit, $bb-2ba$
7. Which ſum in the laſt ſtep muſt be equal to (c) the given ſum of the Squares; hence this Equation, *viz.* . . $bb-2ba=c$
8. Which Equation after due Reduction gives . . . . $\frac{bb-c}{2b}=a$

And the laſt Equation in words is this

CANON.

From the Square of the given ſum of the three Proportionals ſought ſubtract the given ſum of their Squares; then divide the Remainder by the double of the ſum of the three Proportionals, and the Quotient is the mean Proportional.

Therefore if 14 be given for the ſum of the three Numbers in continual proportion, and 84 for the ſum of their Squares, the mean Proportional will be found 4 by the ſaid Canon. Then the Mean being given 4, as alſo 10 the ſum of the Extremes, the Ex-

Extremes will be found 2 and 8, (by the Canon of *Queſt.* 4. *Chap.* 16. of my Firſt Book of *Algebraical Elements*,) and therefore the three Proportionals ſought are 2, 4, and 8.

QUEST. 2.

The Sum (b) of three proportional Quantities being given, as alſo (c) the Sum of the Squares of the Extremes, to find the Proportionals.

RESOLUTION.

1. For the mean Proportional ſought put . . . . . . a
2. Then ſubtracting the ſaid Mean from (b) the given Sum of all the three Proportionals, there will remain the Sum of the Extremes, to wit, ... $b-a$
3. Therefore the Square of the Sum of the Extremes is . $bb-2ba+aa$
4. From which ſquare if you ſubtract the double of the Square of the Mean, to wit, ... $2aa$
5. There will remain (as is manifeſt by the third Theorem of the preceding ſixth *Chap.*) the Sum of the Squares of the Extremes, to wit, ... $bb-2ba-aa$
6. Which Sum of the Squares of the Extremes muſt be equal to the given Sum (c,) hence this Equation, *viz.* $bb-2ba-aa=c$
7. From which Equation after due Reduction this will ariſe $bb-c=aa+2ba$
8. Therefore by reſolving the laſt Equation, (according to the Canon in *Sect.* 6. *Chap.* 1. of my Firſt Book of *Algebraical Elements*;) the value of (a) the mean Proportional will be made known, *viz.* . . . . $\sqrt{:2bb-c:}-b=a$

Which laſt Equation in words is this

CANON.

From the double of the Square of the given Sum of all the three Proportionals ſought ſubtract the given Sum of the Squares of the Extremes; then from the ſquare Root of the Remainder ſubtract the Sum of the three Proportionals, ſo ſhall this laſt Remainder be the mean Proportional ſought.

Therefore if 14 be given for the Sum of three Continual Proportionals, and 68 for the Sum of the Squares of the Extremes, the mean Proportional will be found 4 by the ſaid Canon Then the Mean being given 4, as alſo 10 the Sum of the Extremes, the Extremes will be found 2 and 8, (by the Canon of *Queſt.* 4. *Chap.* 15. of my Firſt Book of *Algebraical Elements*;) and therefore the three Proportionals ſought are 2, 8, and 4.

QUEST. 3.

The difference (b) of the Extremes of three proportional Quantities being given, as alſo (c) the Sum of the Squares of the three Proportionals; to find the Proportionals.

RESOLUTION.

1. For the Sum of the Extremes, (to wit, of the firſt and third Proportionals ſought) put . . . . a
2. Then foraſmuch as the difference of the Extremes is given (b,) and their Sum is aſſumed to be (a,) therefore (by the Theorem in *Queſt.* 1. *Chap.* 14. of my Firſt Book of *Algebraical Elements)* the greater Extreme ſhall be . $\frac{1}{2}a+\frac{1}{2}b$
3. And by the ſame Theorem the leſſer Extreme is . . . $\frac{1}{2}a-\frac{1}{2}b$
4. Then the Product made by the Multiplication of the Extremes in the ſecond and third ſteps will give the Square of the Mean, to wit, ... $\frac{1}{4}aa-\frac{1}{4}bb$
5. And from the ſecond ſtep the Square of the greater Extreme is . . . . $\frac{1}{4}aa+\frac{1}{2}ab+\frac{1}{4}bb$
6. And from the third ſtep the Square of the leſſer Extreme is $\frac{1}{4}aa-\frac{1}{2}ab+\frac{1}{4}bb$
7. Therefore from the fourth, fifth, and ſixth ſteps the Sum of the Squares of all the three Proportionals is . $\frac{3}{4}aa+\frac{1}{4}bb$

8. Which ſum in the laſt ſtep muſt be equal to (*c*) the ſum of the Squares given in the Queſtion, hence this Equation ariſes, to wit, ... $\frac{3}{4}aa+\frac{1}{4}bb=c$
9. Which Equation after due Reduction will give . . . $aa=\frac{4c-bb}{3}$
10. Therefore by extracting the ſquare Root out of each part of the laſt Equation the ſum of the extreme Proportionals is diſcovered, to wit, ... $a=\sqrt{\frac{4c-bb}{3}}$

Which laſt Equation gives this

CANON.

From four times the given ſum of the ſquares of the three Proportionals ſought, ſubtract the ſquare of the given difference of the Extremes; then the ſquare Root of one third part of that Remainder ſhall be the ſum of the extreme Proportionals.

Then half the ſum of the Extremes increaſed with half their difference gives the greater Extreme, and half the ſaid ſum leſſened by half the ſaid difference leaves the leſſer Extreme.

Laſtly, the ſquare Root of the Product made by the mutual Multiplication of the Extreme is the mean Proportional.

Therefore if 16 be given for the difference of the Extremes of three Proportionals, and 364 for the ſum of the ſquares of all the three Proportionals, the Proportionals are alſo given ſeverally, to wit. 2, 6, 18 ∻

QUEST. 4.

One Extreme (*b*) of three Proportional Quantities being given, as alſo (*c*) the ſum of the ſquares of the other Extreme and the Mean, to find out that other Extreme and Mean

RESOLUTION.

1. For the extreme Proportional ſought put . . . . . a
2. Which multiplied by the given Extreme (*b*) produces the ſquare of the Mean, to wit, ... ba
3. But from the firſt ſtep the ſquare of the extreme Proportional ſought is . . . . aa
4. Therefore from the ſecond and third ſteps the ſum of the ſquares of the two Proportionals ſought is . . . $aa+ba$
5. Which ſum in the laſt ſtep muſt be equal to (*c*) the ſum given in the Queſtion; hence this Equation ariſes, *viz.* $aa+ba=c$
6. Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. of my Firſt Book of *Algebraic Elements*, will diſcover the extreme Proportional ſought, to wit, . . $a=\sqrt{:c+\frac{1}{4}bb:}-\frac{1}{2}b$

The laſt Equation in words is this

CANON.

To the given ſum add the ſquare of half the extreme Proportional given, and out of this ſum extract the ſquare Root; then this ſquare Root leſſened by half the given Extreme will give the other Extreme.

Therefore if 18 be given for one of the Extremes of three Proportionals, and 40 for the ſum of the ſquares of the other two Proportionals, the Canon will diſcover 2 for the Extreme ſought. Laſtly, the ſquare Root of the Product of the Extremes, to wit, 6 is the Mean ſought; therefore the three Proportionals are 18, 6, and 2.

QUEST. 5.

The difference (*b*) between the Extremes of three proportional Quantities being given, as alſo the Proportion which the difference of the ſquares of the Extremes has to the ſum of the ſquares of all the three Proportionals, ſuppoſe the difference be to the ſum as (*r*) to (*s*;) to find the Proportionals. But (*r*) muſt be leſs than (*s*.)

RESOLUTION.

1. For the ſum of the Extremes put . . . . a
2. Then foraſmuch as their difference is given . . . . b
3. Therefore the difference of the ſquares of the Extremes ſhall be *ba*; (for the Product of the Multiplication of the ſum of any two Numbers into their difference is equal to the difference of their ſquares.) ba

4. Then from the firſt and ſecond ſteps (by the *Theorem* of *Queſt.* 1. *Chap.* 14. of my Firſt Book of *Algebraical Elements*) the greater Extreme ſhall be . . . . $\frac{1}{2}a+\frac{1}{2}b$

5. And (by the ſame *Theorem*) the leſſer Extreme ſhall be $\frac{1}{2}a-\frac{1}{2}b$

6. Therefore from the fourth ſtep the ſquare of the greater Extreme is . . . . $\frac{1}{4}aa+\frac{1}{4}bb+\frac{1}{2}ba$

7. And from the fifth ſtep the ſquare of the leſſer Extreme is $\frac{1}{4}aa+\frac{1}{4}bb-\frac{1}{2}ba$

8. And becauſe the Product made by the mutual Multiplication of the Extremes is equal to the Square of the Mean, therefore the Extremes in the fourth and fifth ſteps being multiplied one by the other, will give the Square of the Mean, to wit, ... $\frac{1}{4}aa-\frac{1}{4}bb$

9. Therefore by adding together the Squares in the three laſt ſteps, the Sum of the ſquares of the three Proportionals ſhall be . . . . $\frac{3}{4}aa+\frac{1}{4}bb$

10. Then according to the Queſtion as r is to s, ſo muſt the difference in the third ſtep be to the ſum in the ninth ſtep; hence this Analogy ariſes, *viz.*

$$r\ .\ s\ ::\ ba\ .\ \tfrac{3}{4}aa-\tfrac{1}{4}bb$$

11. Whence by comparing the Product made by the mutual Multiplication of the Extremes to the Product of the Means this Equation comes forth, *viz.*

$$sba=\tfrac{3}{4}raa+\tfrac{1}{4}rbb.$$

12. From which Equation after due Reduction there will ariſe $\frac{4sb}{3r}a-aa=\frac{bb}{3}$

13. Therefore (*per Canon in Sect.* 10. *Chap.* 15. *Book* 1.) the two Roots or Values of a in the laſt Equation are theſe, to wit,

$$a=\frac{2sb+\sqrt{:4ssbb-3rrbb:}}{3r}\text{ the greater; } a=\frac{2sb-\sqrt{:4ssbb-3rrbb:}}{3r}\text{ the leſſer.}$$

14. But the greater of thoſe two Values of (a) is the deſired ſum of the extreme Proportionals ſought; for if we ſhould ſuppoſe the leſſer Value to be the ſum of the Extremes, it ought to exceed (b) the difference of the Extremes: but from that ſuppoſition it will follow that (r) is greater than (s,) and conſequently that the difference of the ſquares of the Extremes is greater than the ſum of the ſquares of all the three Proportionals, which is impoſſible. Now to prove the ſaid Conſequence,

15. Suppoſe . . . . $\frac{2sb-\sqrt{:4ssbb-3rrbb:}}{3r}\sqsubset b.$

16. Then by multiplying each part by $3r$, it follows that . . . . $2sb-\sqrt{:4ssbb-3rrbb:}\sqsubset 3rb.$

17. And by adding $\sqrt{:4ssbb-3rrbb:}$ to each part in the ſixteenth ſtep; . . $2sb\sqsubset 3rb+\sqrt{:4ssbb-3rrbb:}$

18. And by ſubtracting $3rb$ from each part in the ſeventeenth ſtep, ... $2sb-3rb\sqsubset\sqrt{:4ssbb-3rrbb:}$

19. And by ſquaring each part in the eighteenth ſtep, ... $4ssbb-12srbb+9rrbb\sqsubset 4ssbb-3rrbb.$

20. And by adding $3rrbb$ to each part in the nineteenth ſtep, ... $4ssbb-12srbb+12rrbb\sqsubset 4ssbb.$

21. And by adding $12srbb$ to each part in the twentieth ſtep, ... $4ssbb+12rrbb\sqsubset 4ssbb+12srbb.$

22. And by ſubtracting $4ssbb$ from each part in the twenty firſt ſtep, ... $12rrbb\sqsubset 12srbb.$

23. Wherefore by dividing each part in the twenty ſecond ſtep by $12rbb$, . . . $r\sqsubset s.$

24. Thus from a ſuppoſition that the leſſer Value of (a) in the thirteenth ſtep is greater than (b) the given difference of the Extremes, it follows by juſt conſequence that (r) is greater than (s,) which is impoſſible; for in regard the difference of the ſquares of the Extremes is leſs than the ſum of the Squares of all 3 Proportionals, and that according to the Queſtion the ſaid difference is to the ſaid ſum as (r) to (s,) therefore (r) is leſs than (s.) And becauſe the ſeries of Inferences drawn from the ſaid ſuppoſition ends in an Impoſſibility, therefore that which was ſuppoſed cannot be true, *viz.* The leſſer

value of *(a)* is not greater than *(b)* the given difference of the Extremes, and conſequently it cannot be equal to the Sum of the Extremes, which was to be proved.

But by the like Argumentation it may be proved, that the greater value of *(a)* in the thirteenth ſtep exceeds *(b)* the given difference of the Extremes; and if it be expreſs'd by words, it will give the following Canon to find out the Sum of the extreme Proportionals ſought; whence by the help of the given difference of the Extremes, the Extremes are ſeverally given.

CANON.

From four times the ſquare of the latter or greater Term *(s)* of the given Reaſon ſubtract thrice the ſquare of the firſt Term *(r,)* and multiply the Remainder by the ſquare of the given difference of the extreme Proportionals ſought; then add the ſquare Root of that Product to the double of the Product made by the Multiplication of the latter Term *(s)* into the difference of the Extremes, and divide the Sum of that Addition by the triple of the firſt Term *(r;)* ſo ſhall the Quotient be the Sum of the extreme Proportionals. Laſtly, half the Sum of the Extremes increaſed with half their difference gives the greater Extreme, but the ſaid half Sum leſſened by the ſaid half difference leaves the leſſer Extreme.

As for Example: If 6 be given for the difference of the Extremes of three Continual Proportionals, and the difference of the ſquares of the Extremes has ſuch proportion to the Sum of the Squares of all the three Proportionals as 5 to 7, then by the Canon the three Proportionals will be found 2, 4, and 8.

Again, if $2\frac{1}{4}$ be given for the difference of the Extremes, and the difference of the Squares of the Extremes be to the Sum of the Squares of all the three Proportionals, as 123 to 427, the Proportionals will be found 4, 5, and $6\frac{1}{4}$.

QUEST. 6.

The Sum *(b)* of the Extremes, and the Sum *(c)* of the Means of four Quantities in Continual Proportion being given, to find out the Proportionals; but *(b)* muſt exceed *(c.)*

RESOLUTION.

1. For one of the Means put . . . . a
2. Then by ſubtracting the Mean from *(c)* the given Sum of the Means, the Remainder is the other Mean, to wit, $\Big\}\ c - a$
3. And by dividing the Square of the latter Mean by the former, the Quotient gives one of the Extremes, to wit, $\Big\}\ \frac{cc - 2ca + aa}{a}$
4. In like manner the ſquare of the firſt Mean *(a)* being divided by the other Means *(c—a)* gives the other Extreme, to wit, $\Big\}\ \frac{aa}{c - a}$
5. Therefore from the third and fourth ſteps the Sum of the two Extremes is . . . . $\Big\}\ \frac{ccc - 3cca - 3caa}{ca - aa}$
6. Which Sum muſt be equal to *(b)* the given Sum of the Extremes; hence this Equation ariſes, to wit, ... $\Big\}\ \frac{ccc - 3cca - 3caa}{ca - aa} = b$
7. From which Equation after due Reduction this ariſes, to wit, ... $\Big\}\ \frac{ccc}{3c + b} = ca - aa$
8. Wherefore by reſolving the laſt Equation by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of *(a,)* to wit, the mean Proportionals ſought will be made known, *viz.*

$$a = \tfrac{1}{2}c + \sqrt{:\frac{cc}{4} - \frac{ccc}{3c + b}}: \text{ the greater Mean:}$$

$$a = \tfrac{1}{2}c - \sqrt{:\frac{cc}{4} - \frac{ccc}{3c + b}}: \text{ the leſſer Mean.}$$

Which values of *(a)* give this

CANON.

Divide the Cube of the Sum of the Means by the Aggregate of the triple Sum of the Means and the Sum of the Extremes; ſubtract the Quotient from the ſquare of half the ſum of the Means, and extract the ſquare Root of the Remainder; then the ſaid ſquare Root being added to and ſubtracted from half the ſum of the Means, the Sum and Remainder ſhall be the Means ſought.

Then the Square of the leſſer Mean being divided by the greater will give the leſſer Extreme, and the Square of the greater Mean divided by the leſſer gives the greater Extreme.

Therefore if 18 be given for the ſum of the Extremes, and 12 for the ſum of the Means of four continual Proportionals, the Proportionals are given ſeverally by the ſaid Canon, to wit, 2, 4, 8, and 16.

QUEST. 7.

The difference (b) of the Extremes, and the difference (c) of the Means of four Quantities continually proportional being given, to find out the four Proportionals.

RESOLUTION.

1. For the leſſer mean Proportional put . . . . a
2. Which added to (c) the given difference of the Means gives the greater Mean, to wit, ... $c+a$
3. Then the Square of the ſaid greater Mean being divided by the leſſer, gives for the greater Extreme . . . . $\frac{cc+2ca+aa}{a}$
4. Likewiſe by dividing (aa) the Square of the leſſer Mean by the greater, there ariſes for the leſſer Extreme . . $\frac{aa}{c+a}$
5. Therefore the difference of the two Extremes in the third and fourth ſteps is . . . . $\frac{ccc+3cca+3caa}{ca+aa}$
6. Which difference muſt be equal to (b) the given difference of the Extremes; hence this Equation ariſes, *viz.* $\frac{ccc+3cca+3caa}{ca+aa}=b$
7. From which Equation after due Reduction this ariſes, to wit, ... $\frac{ccc}{b-3c}=ca+aa$
8. Wherefore by reſolving the laſt Equation by the Canon in *Sect. 6. Ch.* 15. *Book* 1. the value of (a,) to wit, the leſſer mean Proportional ſought will be made known, *viz.*

$$a=\sqrt{:\frac{cc}{4}-\frac{ccc}{b-3c}:}-\tfrac{1}{2}c.$$

Which Equation in words is this

CANON.

Divide the Cube of the given difference of the Means by the exceſs of the given difference of the Extremes above the triple of the difference of the Means; add the Quotient to the Square of half the difference of the Means; then from the ſquare Root of that ſum ſubtract half the difference of the Means, ſo ſhall this Remainder be the leſſer Mean.

Then to the leſſer Mean add the difference of the Means, and the ſum is the greater.

Laſtly, the Square of the greater Mean divided by the leſſer gives the greater Extreme, and the Square of the leſſer Mean divided by the greater gives the leſſer Extreme.

Therefore if 52 be given for the difference of the Extremes of 4 continual Proportionals, and 12 for the difference of the Means, the Proportionals will be found 2, 6, 18, 54.

QUEST. 8.

The ſum (b) of four Quantities in continual proportion being given, as alſo (c) the ſum of their ſquares, to find the Proportionals.

RESOLUTION.

1. For the ſum of the Means put . . . . a
2. Which ſubtracted from (b) the given ſum of all the four Proportionals. leaves the ſum of the Extremes, to wit, . $b-a$
3. The ſquare of (b) the given ſum of all the four Proportionals is . . . . bb
4. Now (according to *Theor.* 16. of the preceding *Chap.* 6.) from the ſaid ſquare (bb) I ſubtract (c) the given ſum of the ſquares of the four Proportionals, and from the half of the Remainder I alſo ſubtract (aa) the ſquare of the ſum of the Means, ſo this Quantity remains, to wit, . $\tfrac{1}{2}bb-\tfrac{1}{2}c-aa$
5. Which Remainder, to wit, $\tfrac{1}{2}bb-\tfrac{1}{2}c-aa$ (by the ſaid *Theor.* 16.) ſhall be to the given ſum of the ſquares of the four Proportionals, as the ſum of the Means is to the ſum of all the four Proportionals, hence this Analogy ariſes, *viz.*

$$\tfrac{1}{2}bb-\tfrac{1}{2}c-aa\ .\ c\ ::\ a\ .\ b$$

6. Which Analogy, by comparing the Product made by the mutual Multiplication of the Extremes to the Product of the Means, will be converted into this Equation, *viz.*

$$\tfrac{1}{2}bbb-\tfrac{1}{2}bc-baa=ca$$

7. Whence after due Reduction this Equation ariſes, to wit,

$$\tfrac{1}{2}bb-\tfrac{1}{2}c=aa+\frac{c}{b}a$$

Which Equation being reſolved (*per Canon in Sect. 6. Chap.* 15. *Book* 1.) gives this following.

CANON.

From the ſquare of the given ſum of the four Proportionals ſubtract the given ſum of their Squares, and to the half of the Remainder add the ſquare of half the Quotient that ariſes by dividing the ſum of the Squares of the four Proportionals by the ſum of the four Proportionals. Then extract the ſquare Root of the ſum of that Addition, and from the ſaid ſquare Root ſubtract half the Quotient aforeſaid, ſo ſhall the Remainder be the ſum of the two deſired mean Proportionals.

Then the ſum of the Means of four continual Proportionals being given, as alſo the ſum of the Extremes, the Proportionals ſhall be given ſeverally by the Canon of the preceding *Queſt.* 6. of this *Chap.*

So if 30 be given for the ſum of four Proportionals, and 340 for the ſum of their Squares; firſt, by the Canon above expreſs'd the ſum of the Means will be found 12, which ſubtracted from 30 the given ſum of the four Proportionals, leaves 18 for the ſum of the Extremes; then the ſum of the Means being given 12, and the ſum of the Extremes 18, the four Proportionals (by the Canon of the preceding ſixth Queſtion) will be found 2, 4, 8, 16.

QUEST. 9.

The ſum (b) of four Quantities in continual proportion being given, as alſo (c) the ſum of the ſquares of the Means; to find the Proportionals.

RESOLUTION.

1. For the ſum of the Means put . . . . a

Then becauſe (by *Theorem* 12. of the preceding *Chap.* 6.) the ſum of the four Quantities continually proportional is to the ſum of the Means, as the ſum of the Squares of the Means is to the Product made by the mutual Multiplica- of the Means or Extremes, ſay by the Rule of Three,

If . . . b . a :: c . $\frac{ca}{b}$

Whence the Product of the Means or Extremes is found } $\frac{ca}{b}$

3. And becauſe if from the ſquare of the ſum of the Means there be ſubtracted the ſum of the ſquares of the Means, there will remain the double Product of the Means or Extremes; therefore if from (aa) you ſubtract (c) the half of the Remainder ſhall be the Product of the Means or Extremes, to wit, } $\tfrac{1}{2}aa-\tfrac{1}{2}c$

4. Which Product, to wit, $\tfrac{1}{2}aa-\tfrac{1}{2}c$ muſt be equal to $\frac{ca}{b}$ the Product in the ſecond ſtep; hence this Equation ariſes, to wit, ... } $\tfrac{1}{2}aa-\tfrac{1}{2}c=\frac{ca}{b}$

5. From which Equation after due Reduction there ariſes } $aa-\frac{2c}{b}a=c$

Which laſt Equation being reſolved (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) gives this following

CANON.

To the given ſum of the Squares of the Means add the Square of the Quotient that ariſes by dividing the ſaid ſum by the given ſum of the four Proportionals, and out of the ſum made by that Addition extract the ſquare Root; then this ſquare Root added to the aforeſaid Quotient gives the Sum of the Mean Proportionals ſought.

Then the Sum of the Means being given, as alſo the Sum of the Extremes, (for the Sum of the Means found out being ſubtracted from the given ſum of all the four Proportionals leaves the Sum of the Extremes) the four Proportionals will be diſcovered by the Canon of the ſixth Queſtion of this Chapter.

Therefore if 30 be given for the ſum of four Continual Proportionals, and 80 for the ſum of the Squares of the Means, the four Proportionals are alſo ſeverally given; to wit, 2,4,8,16, by the Canon above expreſs'd.

QUEST. 10.

The ſum (b) of four Quantities continually proportional being given, as alſo (c) the ſum of the ſquares of the Extremes, to find out the Proportionals.

RESOLUTION.

1. For the ſum of the Means put . . . . a

2. Which ſubtracted from (b) the given ſum of the four Proportionals leaves the ſum of the Extremes, to wit, . . $b-a$

3. Therefore the ſquare of the ſum of the Extremes is . $bb-2ba+aa$

4. From which Square if (c) the given ſum of the ſquares of the Extremes be ſubtracted, there will remain the double Product made by the mutual Multiplication of the Extremes or Means; therefore the Product of the Means is $\frac{bb-2ba+aa-c}{2}$

5. And becauſe if from aa the ſquare of the ſum of the Means there be ſubtracted $bb-2ba+aa-c$, the double Product of the Means, there will remain the ſum of the ſquares of the Means; therefore the ſum of the ſquares of the Means is $2ba-bb+c$

6. And becauſe by *Theor.* 12. in the preceding *Chap.* 6 the ſum of the ſquares of the Means is to the Product of the Means, as the ſum of all the 4 Proportionals is to the ſum of the Means; therefore from the premiſes this following Analogy ariſes, *viz.*

$$2ba-bb+c \;.\; \frac{bb-2ba+aa-c}{2} :: b \;.\; a$$

7. From which Analogy by comparing the Product of the Extremes to the Product of the Means, this Equation ariſes, *viz.*

$$2baa-bba+ca=\frac{bbb-2bba+baa-bc}{2}$$

8. Which Equation after due Reduction gives this following Equation, *viz.*

$$aa+\frac{2c}{3b}a=\frac{bb-c}{3}$$

Whence *(per Canon in Sect. 6. Chap. 15. Book 1.)* there ariſes this following

CANON.

Divide the given ſum of the ſquares of the Extremes by the triple of the given ſum of all the four Proportionals, and to the ſquare of the Quotient add one third part of the exceſs of the ſquare of the ſum of the four Proportionals above the Sum of the ſquares of the Extremes; then from the ſquare Root of the Sum made by that Addition ſubtract the Quotient firſt found out; ſo ſhall the Remainder be the deſired ſum of the mean Proportionals.

Then the ſum of the Means being given, as alſo the ſum of the Extremes, (for the ſum of the Means being ſubtracted from the given ſum of the four Proportionals leaves the ſum of the Extremes) the four Proportionals will be diſcovered by the Canon of the ſixth Queſtion of this Chapter.

Therefore if 80 be given for the ſum of four continual Proportionals, and 2920 for the ſum of the Squares of the Extremes, the 4 Proportionals will be found 2,6,18,54.

QUEST. 11.

The ſum (b) of the ſquares of the Extremes of 4 Quantities in continual proportion being given, as alſo (c) the ſum of the ſquares of the Means, to find out the Proportionals.

RESOLUTION.

1. Add the two given ſums into one that you may have the ſum of the ſquares of the four Proportionals ſought, for which laſt mentioned Sum put . . . . d

2. Then for the ſum of the Squares of the firſt and ſecond Proportionals put . . . . a

3. Therefore the ſum of the Squares of the third and fourth Proportionals is . . . . $d-a$

4. Then becauſe (by *Theorem* 13. of the preceding *Chap.* 6) the ſum of the Squares of the two Means is a mean Proportional between the Sum of the Squares of the firſt and ſecond, and the ſum of the Squares of the third and fourth, this Analogy is manifeſt, *viz.* . . . . $a . c :: c . d-a$

5. Therefore by comparing the Product made by the Multiplication of the Extremes of that Analogy to the Product of the Means this Equation ariſes, *viz.* . . . . . $da-aa=cc$

6. Which Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 16. *Book* 1. gives this following

CANON.

Add the given Sum of the Squares of the Extremes to the given Sum of the Squares of the Means, and reſerve half of the ſum. From the ſquare of this half ſum ſubtract the Square of the ſum of the Squares of the Means, and extract the ſquare Root of the Remainder; add this ſquare Root to the half ſum before reſerved, and alſo ſubtract it from the ſame half ſum, ſo the ſum ſhall be the ſum of the Squares of the firſt and ſecond Proportionals, and the Remainder ſhall be the ſum of the Squares of the third and fourth.

Then (according to *Theor.* 3. of the preceding *Chap.* 6.) add ſeverally the ſum of the Squares of the firſt and ſecond Proportionals, and the ſum of the Squares of the third and fourth to the ſum of the Squares of the Means, and out of each ſum extract the ſquare Root; ſo ſhall one of theſe Roots be the ſum of the firſt and third Proportionals, and the other ſhall be the ſum of the ſecond and fourth. Which two laſt mentioned ſums being added together give the ſum of the four Proportionals ſought.

Laſtly, the ſum of four Proportionals being given, as alſo the ſum of the Squares of the Means, the Proportionals ſhall be given ſeverally by the ninth Queſtion of this *Chap.*

Therefore if 260 be given for the ſum of the Squares of the Extremes of four continual Proportionals, and 80 for the ſum of the Squares of the Means, the Proportionals will be found 16,8,4,2.

QUEST. 12.

The ſum (*b*) of the Extremes of four Quantities in continual Proportion being given, as alſo (*c*) the ſum of the Cubes of the Means; to find out the Proportionals.

RESOLUTION.

1. For one of the Extreme Proportionals, put . . . . . a

2. Then the other Extreme (by ſubtracting (*a*) from (*b*) the given ſum of the Extremes) ſhall be . . . . $b-a$

3. Therefore the Product made by the mutual Multiplication of the Extremes is . . . . $ba-aa$

4. And becauſe (*per Theorem* 21. of the preceding *Chap.* 6) the Product made by the Multiplication of the Means or Extremes into the ſum of the Extremes, is equal to the ſum of the Cubes of the Means; therefore if you multiply $ba-aa$ by b, this Product ſhall be equal to (*c*) the given ſum of the Cubes of the Means; hence ariſes this Equation, *viz.* . . $bba-baa=c$

5. And by dividing every Term of that Equation by (*b*,) there ariſes . . . . $ba-aa=\frac{c}{b}$

Which laſt Equation being reſolved (by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) gives this following

CANON.

6. From the Square of half the given ſum of the Extremes ſubtract the Quotient that ariſes by dividing the given ſum of the Cubes of the Means by the ſum of the Extremes, and extract the ſquare Root of the Remainder, then half the ſum of the Extremes being increaſed, and alſo leſſened by the ſaid ſquare Root, gives the Extremes ſeverally. Then you may find out the Means by a new Work thus;

7. Let the greater Extreme found out as above be . . . . f.

8. And the leſſer Exteme . . . . g

9. Then for the greater Mean put . . . . a

10. Therefore by dividing (*aa*) the ſquare of the greater Mean by the greater Extreme (*f*,) the Quotient ſhall be the leſſer Mean, to wit, ... $\frac{aa}{f}$

11. But the ſquare of the leſſer Mean is equal to the Product of the leſſer Extreme multiplied by the greater Mean; therefore from the three laſt preceding ſteps this Equation ariſes, *viz.* $\frac{aaaa}{ff}=ga$

12. Which Equation after due Reduction gives . . . . . $aaa=ffg$

13. Therefore by extracting the Cubic Root out of each part of the laſt Equation the greater Mean is made known, *viz.* $a=\sqrt{(3)}ffg$

Which laſt Equation, together with that in the tenth ſtep, will give this

CANON.

14. Multiply the ſquare of the greater Extreme by the leſſer, then the Cubic Root of the Product ſhall be the greater Mean. Laſtly, the Square of the greater Mean divided by the greater Extreme gives the leſſer Mean.

Therefore if 18 be given for the ſum of the Extremes of four Numbers in continual proportion, and 576 for the ſum of the Cubes of the Means, then by the firſt Canon of this Queſtion the Extremes will be found 16 and 2. And laſtly, by the latter Canon the Means will be found 8 and 4. Wherefore the four continual Proportionals ſought are 16, 8, 4, 2.

QUEST. 13.

The ſum (b) of the Cubes of the Extremes of four Quantities in continual proportion being given, as alſo (c) the ſum of the Cubes of the Means, to find the four Proportionals.

RESOLUTION.

1. For the ſum of the Extremes put . . . . a
2. Therefore the Cube of that ſum is . . . . aaa
3. Then becauſe by *Theor.* 22. of the preceding *Chap.* 6. if four Quantities be continually proportional, the ſum of the Cubes of the Extremes more by the triple of the Cubes of the Means is equal to the Cube of the ſum of the Extremes; therefore if to (b) you add $3c$, it gives the Cube of the ſum of the Extremes, which Cube muſt be equal to aaa; hence this Equation $b+3c=aaa$
4. Therefore by extracting the Cubic Root out of each part of that Equation, the ſum of the Extremes is made known, *viz.* $\sqrt{(3)}\overline{:b+3c:}=a$

Which laſt Equation in words is this following

CANON.

Add the triple of the given ſum of the Cubes of the Means to the given ſum of the Cubes of the Extremes, and out of the ſum made by that Addition extract the Cubic Root, which ſhall be the ſum of the Extremes ſought.

Then the ſum of the Extremes being given, as alſo the ſum of the Cubes of the Means, the four Proportionals ſhall be given ſeverally by the Canon of the preceding twelfth Queſtion. As for Example, if 157472 be given for the ſum of the Cubes of the Extremes of four Numbers in continual proportion, and 6048 for the ſum of the Cubes of the Mean; firſt, by the Canon of this Queſtion the ſum of the Extremes will be found 56, and then by the Canon of the preceding twelfth Queſtion, the four Proportionals will be found 2, 6, 18, 54.

QUEST. 14.

The ſum of the Extremes (b) of five Quantities in continual Proportion being given, as alſo (c) the ſum of the three Means; to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle Term of all the five, put . . . . a
2. Then ſubtract that middle Term (a) from (c) the given ſum of the three Means, and there will remain the ſum of the ſecond and fourth, *viz.* . . . . $c-a$
3. And becauſe by *Theorem* 29. of the preceding *Chap.* 6. the ſum of the Extremes of five continual Proportionals, together with the double of the Mean, the ſum of the ſecond and fourth, and the Mean, are alſo in continual proportion; therefore this Analogy is manifeſt, *viz.* . . . . $b+2a\,.\,c-a::c-a\,.\,a$

4. From which Analogy, by comparing the Product made by the Multiplication of the Extremes to the Product of the Means, this Equation is produced, *viz.* . . . . . $ba+2aa=cc-2ca+aa$

5. Which Equation after due Reduction gives . . . . . $aa+ba+2ca=cc$

Laſtly, by reſolving the laſt Equation according to the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. there will ariſe this following

CANON.

Add the ſum of the Extremes to the double of the ſum of the three Means, and take the half of the ſum made by ſuch Addition; then to the Square of the ſaid half ſum add the ſquare of the ſum of the three Means, and out of this ſum extract the ſquare Root; from which Root ſubtract the half ſum firſt taken, and the Remainder ſhall be the middle (or third) Proportional of the five ſought.

Then by ſubtracting the ſaid third Proportional from the ſum of the three Means, the Remainder is the ſum of the ſecond and fourth; by which ſum and the third Proportional, the ſecond and fourth ſhall be given ſeverally, (by the Canon of *Queſt.* 4. *Chap.* 16. *Book* 1.) Then the ſquare of the ſecond Proportional being divided by the third gives the firſt, and the Square of the fourth being divided by the third gives the fifth.

Therefore if 34 be given for the ſum of the firſt and fifth of five continual Proportionals, and 28 for the ſum of the three Means, the five Proportionals ſhall be given ſeverally, *viz.* 2, 4, 8, 16, 32 ÷.

QUEST. 15.

The Sum (b) of the firſt, third, and fifth of five Quantities in continual proportion being given, as alſo (c) the ſum of the ſecond and fourth, to find the five Proportionals.

RESOLUTION.

1. For the third Proportional, that is, the middle Term of the 5, put a
2. Then ſubtract that middle Term (a) from the given ſum (b,) and the Remainder is the ſum of the firſt and fifth, *viz.* . . . $b-a$
3. And becauſe (by *Theorem* 27. of the preceding *Chap.* 6.) the Product made by the Multiplication of the third or middle Term of five continual Proportionals into the ſum of the firſt and fifth, is equal to the Squares of the ſecond and fourth; therefore (from the firſt and ſecond ſteps) the ſum of the Squares of the ſecond and fourth Proportionals is . . . . $ba-aa$
4. The ſquare of the third Proportional (a) is equal to the Product of the ſecond multiplied into the fourth, therefore the double of that Product is . . . . $2aa$
5. Therefore from the two laſt ſteps the Aggregate of the Squares and the double Product of the ſecond and fourth Proportional is $aa+ba$
6. But the Aggregate of the Squares and the double Product of the ſecond and fourth Proportional is equal to the Square of their ſum, therefore the Aggregate in the fifth ſtep muſt be equal to the Square of the given ſum (c) *viz.* . . . . $aa+ba=cc$

Which Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. will give this following

CANON.

Add the Square of half the given ſum of the firſt, third, and fifth Proportionals to the Square of the given ſum of the ſecond and fourth, then from the ſquare Root of the ſum made by that Addition ſubtract the ſaid half ſum, and the Remainder ſhall be the third Proportional.

Then by ſubtracting the ſaid third Proportional from the given ſum of the firſt, third, and fifth, the Remainder is the ſum of the firſt and fifth; by which ſum and the third (or mean) Proportional, the firſt and fifth (to wit, the Extremes) ſhall be given ſeverally by the Canon of *Queſt.* 4. *Chap.* 16. *Book* 1. Then the third Proportional being multiplied into the firſt and fifth ſeverally, and the ſquare Root being extracted out of each Product, theſe Roots ſhall be the ſecond and fourth Proportionals.

Therefore if 42 be given for the ſum of the firſt, third, and fifth of five Numbers in continual proportion, and 20 for the ſum of the ſecond and fourth, the five Proportionals will be found theſe, to wit, 2,4,8,16,32.

QUEST. 16.

The third Proportional (b) of five Quantities in continual proportion being given, as alſo (c) the ſum of the other four, to find out the five Proportionals.

RESOLUTION.

1. For the ſum of the ſecond and fourth Proportional put . . . a
2. Then ſubtract that Sum (a) from (c) the given ſum of the firſt, ſecond, fourth, and fifth Proportionals, and there will remain the ſum of the firſt and fifth, to wit, ... $c-a$
3. The ſquare of the third (that is, of the mean) Proportional (b) is equal to the Product of the ſecond multiplied into the fourth, therefore the double of that Product is, ... $2bb$
4. Which double Product ($2bb$) ſubtracted from (aa) the Square of the ſum of the ſecond and fourth Proportionals, leaves for the ſum of the Squares of the ſecond and fourth, ... $aa-2bb$
5. And becauſe (by *Theor.* 33. of the preceding *Chap. 6.*) the ſum of the Squares of the ſecond and fourth of 5 continual Proportionals is equal to the Product of the third (or mean) multiplied by the ſum of the firſt and fifth, therefore if ($aa-2bb$) the ſum of the Squares of the ſecond and fourth be divided by the mean (b) the Quotient ſhall be the ſum of the firſt and fifth, *viz.* $\frac{aa-2bb}{b}$
6. Which Sum found out in the laſt ſtep muſt be equal to the ſum of the firſt and fifth Proportionals found out in the ſecond ſtep; hence this Equation ariſes, *viz.* . . . . $\frac{aa-2bb}{b}=c-a$
7. Which Equation after due Reduction gives . . . . . . $aa+ba=2bb+bc$

Wherefore by reſolving the laſt Equation (according to the Canon in *Sect.* 6. *Chap.* 15. *Book.* 1.) there will come forth this following

CANON.

To the ſquare of the half of the given third (or mean) Proportional add the double of the ſquares of the ſaid Mean, as alſo the Product of the ſaid Mean multiplied into the given ſum of the other four Proportionals, and out of the ſum of that Addition extract the ſquare Root; this Root leſſened by half the given Mean, gives the ſum of the ſecond and fourth Proportionals.

Then from the given ſum of the firſt, ſecond, fourth, and fifth Proportionals ſubtract the ſum of the ſecond and fourth (found out as above) and the Remainder is the ſum of the firſt and fifth; by which ſum and the third (or mean) Proportional, the ſaid firſt and fifth ſhall be given ſeverally by the Canon of *Queſt* 4. *Chap.* 16. *Book* 1.

Laſtly, the ſquare Roots of the Product of the firſt multiplied into the third, and of the Product of the third into the fifth, ſhall be the ſecond and fourth Proportionals.

Therefore if 8 be given for the third of five Numbers in continual proportion, and 54 for the ſum of the other four, the five Proportionals will be found theſe, to wit, 2,4,8,16,32.

QUEST. 17.

The ſum (b) of the Extremes of five Quantities in continual Proportion being given, as alſo (c) the ſum of the Squares of three Means; to find the five Proportionals.

RESOLUTION.

1. For the Mean (or third Proportional put . . . . a
2. Then (by *Theor.* 33. of the preceding *Chap. 6.*) the Mean (a) multiplied by (b) the given ſum of the Extremes, produces the ſum of the Squares of the ſecond and fourth Proportionals, *viz.* ba
3. Therefore if to (aa) the ſquare of the Mean you add (ba) the ſum of the Squares of the ſecond and fourth, there will come forth the ſum of the Squares of the ſecond, third, and fourth Proportionals, *viz.* . . . . $aa+ba$
4. Which ſum found out in the laſt ſtep muſt be equal to the given ſum (c;) hence this Equation ariſes, *viz.* . . . . . . $aa+ba=c$

Wherefore by reſolving that Equation (according to the Canon in *Sect. 6. Chap.* 15. *Book* 1.) there will ariſe this following

CANON.

Add the ſquare of half the given ſum of the Extremes to the given ſum of the Squares of the three Means, and out of the ſum of that Addition extract the ſquare Root; this Root leſſened by half the ſum of the Extremes will give the Mean (or third) Proportional.

Then the mean (or third) Proportional being given, and the ſum of the Extremes, (*viz.* of the firſt and fifth) the ſaid Extremes ſhall be given ſeverally by the Canon of *Queſt.* 4. *Chap.* 16. *Book* 1.

Laſtly, the ſquare Roots of the Products of the firſt into the third, and of the third into the fifth ſhall be the ſecond and fourth Proportionals.

Therefore if 34 be given for the ſum of the Extremes of five Numbers in continual proportion, and 336 for the ſum of the Squares of the three Means, the five Proportionals ſhall be alſo given, to wit, 2, 4, 8, 16, 32.

QUEST. 18.

The Sum (b) of the Extremes of five Quantities in continual Proportion being given, as alſo (c) the Sum of the Squares of the ſecond and fourth, to find the 5 Proportionals.

RESOLUTION.

1. For the mean Proportional put . . . . a
2. Then (by *Theorem* 33. of the preceding *Chap.* 6.) the Mean (a) multiplied by (b) the ſum of the Extremes, produces the ſum of the Squares of the ſecond and fourth, *viz.* . . ba
3. Which ſum muſt be equal to the given ſum (c,) therefore $ba=c$
4. Wherefore by dividing each part of that Equation by (b,) the mean Proportional will be made known, *viz.* . . . . $a=\frac{c}{b}$

Which laſt Equation in words is this following

CANON.

Divide the given ſum of the Squares of the ſecond and fourth Proportionals by the given ſum of the firſt and fifth, ſo ſhall the Quotient be the mean or third Proportional.

Then the mean (or third) Proportional being given, as alſo the ſum of the firſt and fifth, theſe ſhall be given ſeverally by the Canon of *Queſt.* 4. *Chap.* 16. *Book* 1.

Laſtly, the ſquare Roots of the Products of the firſt into the third, and of the third into the fifth, ſhall be the ſecond and fourth Proportionals.

Therefore if 34 be given for the ſum of the Extremes of five Numbers in continual proportion, and 272 for the ſum of the Squares of the ſecond and fourth, the Proportionals will be diſcovered ſeverally, *viz.* 2, 4, 8, 16, 32.

QUEST. 19.

A Vintner having a Veſſel full of Wine containing 16 (or b) Gallons, draws out 4 (or c) Gallons, and then pours into the Veſſel as much Water as he drew out Wine; then out of that mixt Quantity of Wine and Water he draws out the ſame number of Gallons as before, and pours in the ſame quantity of Water. Again, he makes a third draught of the ſame quantity as at firſt. The queſtion is, to find how much pure Wine remained in the Veſſel after the third draught.

RESOLUTION.

1. The Number of Gallons of Wine in the Veſſel at firſt was b
2. Out of which Quantity (c) Gallons being drawn, there remained of pure Wine in the Veſſel . . . . $b-c$
3. To which remaining quantity of pure Wine (c) Gallons of Water being added, the Veſſel is again full, and contains (b) Gallons of Wine and Water together; out of which drawing again (c) Gallons, we muſt ſeek how much pure Wine was in this ſecond draught, ſaying by the Rule of Three,

$$\begin{array}{lcccc} & \textit{mixt} & \textit{Wine} & \textit{mixt} & \textit{Wine} \\ \text{If} & b\ . & b-c & ::\ c\ . & \left(\frac{bc-cc}{b}\right. \end{array}$$

Whence it is found, that the quantity of pure Wine in the ſecond draught was . . . . $\frac{bb-cc}{b}$

4. Which Quantity $\frac{bc-cc}{b}$ being ſubtracted from $b-c$, the Quantity of pure Wine in the Veſſel before the ſecond draught was made, there remains for the Quantity of pure Wine in the Veſſel after the ſecond draught . . . . . . $\frac{bb-2bc+cc}{b}$

5. To which remaining Quantity of pure Wine add (c) Gallons of Water, ſo the Veſſel is again full, and contains (b) Gallons of Wine and Water together; out of which drawing again (c) Gallons, we muſt ſeek how much pure Wine was in this third draught, ſaying,

$$\text{As } \overset{mixt}{b} \;.\; \overset{Wine}{\frac{bb-2bc+cc}{b}} \;::\; \overset{mixt}{c} \;.\; \text{to a fourth Pro-}$$

portional or Quantity of pure Wine in the third draught, which will be found . . . . $\frac{bbc-2bcc+ccc}{bb}$

6. Then by ſubtracting the ſaid fourth Proportional or Quantity of pure Wine in the third draught, from $\frac{bb-2bc+cc}{b}$ the Quantity of pure Wine in the Veſſel when the third draught was made, there remains for the deſired Quantity of pure Wine in the Veſſel after the third draught . . . . . . $\frac{bbb-3bbc+3bcc-ccc}{bb}$

Which Quantity laſt found out is the Anſwer of the Queſtion; and if it be reſolved into Numbers it gives $6\frac{3}{4}$ for the number of Gallons of pure Wine that remained in the Veſſel after the third draught. Moreover, if the firſt, ſecond, fourth, and ſixth ſteps of the Reſolution be well examined and compared with *Sect.* 2, 5, and 6 *Chap.* 5. of this ſecond Book, it will be manifeſt that the Quantity of pure Wine in the Veſſel at firſt, and the ſeveral Quantities of Wine remaining in the Veſſel after each draught are in continual Proportion:

$$\textit{Viz.}\left\{\begin{array}{ccccccc} b & . & b-c & . & \frac{bb-2bc+cc}{b} & . & \frac{bbb-3bbc+3bcc-ccc}{bb} \quad \div \\ 16 & . & 12 & . & 9 & . & 6\frac{3}{4} \quad \div \end{array}\right.$$

Of which continual Proportionals the firſt is the given Quantity of Wine in the Veſſel at firſt; the ſecond is the Exceſs of the ſame Quantity above the given Quantity drawn out at each draught; and then the fourth continual Proportional is the Quantity of pure Wine remaining in the Veſſel when three draughts have been made, according to the import of the Queſtion; but the fifth continual Proportional when four draughts, the ſixth when five draughts, the ſeventh when ſix draughts, ſhall be the remaining Quantity of pure Wine ſought by the Queſtion. Laſtly, the firſt and the ſecond Terms of a rank of Numbers in continual proportion being given, any of the following Terms ſhall be given by the Rule in *Sect.* 5. and 6 *Chap.* 5. of this ſecond Book.

QUEST. 20.

A Vintner having a Veſſel full of Wine containing 16 (or b) Gallons, draws out a certain quantity, and then pours into the Veſſel as much Water as he drew out Wine. Again, out of that mixt quantity of Wine and Water he draws out the ſame quantity as before, and pours in the ſame quantity of Water. Then he makes a third draught of the ſame quantity as at firſt, and after this third draught there remained $6\frac{3}{4}$ (or d) Gallons of pure Wine. The Queſtion is, to find what quantity of pure Wine was drawn out at the firſt draught, or what quantity of Wine and Water together at the ſecond or third draught, (for the three draughts were equal quantities.)

RESOLUTION.

1. For the Number of Gallons of Wine in the Veſſel at firſt was b
2. For the Number of Gallons of Wine drawn out at the firſt draught put . . . . a
3. Then the quantity of Wine remaining in the Veſſel after the firſt draught was . . . . $b-a$
4. By proſecuting the ſearch as in the preceding nineteenth Queſtion, ſaving that (a) is to

be uſed here inſtead of (*c*) there, you will find this Quantity, *viz.* $\frac{bbb-3bba+2baa-aaa}{bb}$ to be the Number of Gallons of pure Wine remaining in the Veſſel after the third draught, and therefore it muſt be equal to the given Quantity $6\frac{3}{4}$ (or d;) hence ariſes this Equation, *viz.*

$$\frac{bbb-2bba+3baa-aaa}{bb}=d.$$

5. Therefore by multiplying each part of that Equation by the Denominator bb, there will come forth this Equation in Integers, *viz.*
$$bbb-3bba+3baa-aaa=bbd.$$
6. And by extracting the Cubic Root out of each part of the laſt Equation, there ariſes $$b-a=\sqrt{(3)}bbd.$$
7. Wherefore from the laſt Equation after due Tranſpoſition the Value of (a) will be made known, *viz.* $$a=b-\sqrt{(c)}bbd=4.$$

Whence it is manifeſt, that four Gallons were drawn out at every one of the three draughts. But if the Reſolution had been wrought out at large, as in the preceding nineteenth Queſtion, then it would appear, that if between (b) and (d,) *viz.* the quantity of Wine firſt given, and the quantity of Wine remaining after the laſt draught, there be found the greater of two mean Proportionals when three draughts are propoſed, or the greateſt of three Means when four draughts, and ſo forwards; then the Mean ſo found out being ſubtracted from the greater Extreme (b) leaves the Quantity drawn out at each draught. The manner of finding out mean proportional Numbers between any two Numbers given for Extremes, has already been ſhewn in *Sect.* 14. *Chap.* 5. of this ſecond Book.

If the Reader deſires more variety of Queſtions about Quantities in continual Proportion, he may conſult the *Algebra* of *Jac. de Billy*, intituled *Nova Geometriæ Clavis*, and the firſt Part of our Learned Dr. *Wallis* his Mathematical Works.

CHAP. VIII.

The manner of finding out all the Aliquot Parts *both of Numbers and Algebraical Quantities, as alſo the ſmalleſt Numbers that ſhall have given Multitudes of* Aliquot Parts.

I. IN the Reſolution of knotty Queſtions about Quantity, there is oftentimes great uſe of finding out all the *Aliquot Parts*, or juſt Diviſors, as well of Numbers, as of Quantities repreſented by Letters; and therefore in this Chapter I ſhall ſhew how that Work may be done; as alſo how to find out the leaſt Number that ſhall have a given Multitude of *Aliquot Parts*, according to the Method of *Fran. van Schooten*, in *Sect.* 2, 3, and 4. of his *Miſcellanies*, and in his *Principia Matheſ. Univerſal.*

II. A *Prime* or Incompoſit Number is that which can only be meaſured or divided by it ſelf or by Unity, and leave no Remainder; as 2,3,5,7,11,13, *&c.* are *Prime* Numbers.

III. A *Compoſit Number* is that which may be divided by ſome Number leſs than the Compoſit it ſelf, but greater than Unity; as 4,6,8,9,10, *&c.* are Compoſits.

IV. *Juſt Diviſors* are ſuch Numbers or Quantities as will divide a given Number or Quantity, and leave no Remainder; every one of which Diviſors, except that which is equal to the given Quantity is called an *Aliquot Part*, becauſe if it be taken *Aliquoties*, that is, certain times, it will preciſely conſtitute the given Quantity: As if 6 be a Number propoſed, its juſt Diviſors are 1,2,3, and 6; but the Aliquot Parts of 6 are only 1,2, and 3; for 6 cannot be a part of 6, but it may be a Diviſor to it ſelf, that is, 6 may be divided by 6, and the Quotient is Unity. Hence it is manifeſt, that the juſt Diviſors of a Number are more in multitude by one than the Number of its Aliquot Parts.

V. The Aliquot Parts of a whole Number may be found out in this manner, *viz.* Firſt, if the Number propoſed be even, divide it by 2, and reſerve the Diviſor. Again, if the Quotient be even divide it by 2, and reſerve the Diviſor; and continue the Diviſion of every following Quotient by 2, until the Quotient be an odd number. But if either the number firſt propoſed, or the Quotient reſulting from ſuch Diviſion by 2 be odd, divide it by 3, if it will give an Integer Quotient, and continue the Diviſion by 3 in like manner as before by 2, ſo long as the Quotient is an Integer without any Fraction; likewiſe when the Diviſion by 3 ceaſeth, divide by 5,7,11,13,17,19,*&c.* that is, by every prime Number,until you find a Quotient leſs than the Diviſor; and if no ſuch Diviſor will give an Integer Quotient before the Quotient is leſs than the Diviſor,you may conclude the number firſt propoſed to be Incompoſit, (*viz* ſuch as has no Diviſor but it ſelf or Unity) and that laſt Diviſor to be greater than the ſquare Root of the propoſed Number. Then by the help of the prime Diviſors to the given Number, all the reſt may be found out by the Operation directed in the following Examples.

Example 1.

Suppoſe it be deſired to find out all the Aliquot Parts and Diviſors of 360; firſt, I divide 360 by 2, and the Quotient is 180; this divided by 2 gives 90, which divided by 2 gives 45; this being an odd Number the Diviſion by 2 ceaſes. Then I divide the ſaid 45 by 3, and the Quotient is 15; this divided by 3 gives the Quotient 5, and ſo the Diviſion by 3 ceaſes; then I divide 5 by it ſelf, and the Quotient is Unity. Now by the help of thoſe Diviſors or prime Numbers, which (as may eaſily be proved) are ſuch, that if they be continually multiplied will produce the given number 360, all the reſt of the juſt Diviſors of the ſaid 360 may be found out thus.

360	180	90	45	15	5	1
2	2	2	3	3	5	

Firſt, I ſet every one of the ſaid prime Diviſors 2,2,2,3,3, and 5, at the head of a Columel, as you ſee in this Table; then I multiply the firſt Diviſor 2 by the ſecond Diviſor 2, and ſet the Product 4 under 2 in the ſecond Columel. Again, I multiply the ſaid 4 by 2, (which ſtands at the head of the third Columel) and ſet the Product 8 under 2 in the third Columel. Then I multiply every one of the Numbers in the firſt, ſecond,and third Columels,by 3, which ſtands at the head of the fourth Columel, and write the Products under 3 in the ſaid fourth Columel; except ſuch Products which happen to be the ſame with any of thoſe before written, (for one and the ſame Product muſt not be written twice;) ſo multiplying 2, 4, and 8, by 3, I ſet the Products 6, 12, and 24 under 3 in the fourth Columel. Again, I multiply every one of the Numbers in the firſt, ſecond, third, and fourth Columels by 3, (which ſtands at the top of the fifth Columel) and ſet the Products under the ſaid 3; except (as before) ſuch Products which happen to be the ſame with any of thoſe before written in any of the precedent Columels: ſo the Products written under 3 in the fifth Columel are 9, 18, 36, and 72. Laſtly, I multiply every one of the Numbers in the firſt, ſecond, third, fourth, and fifth Columels by 5, (which ſtands at the head of the laſt Columel) and write the ſeveral Products (except as before excepted) under the ſaid 5. So at length all the juſt Diviſors to the given Number 360 are found theſe, to wit, 1,2,3,4,5,6,7,8,9,10,12,15, 18,20,24,30,36,40,45,60,72,90,120, 180, and 360; every one of which Diviſors (except the greateſt, which is always equal to the Number firſt propoſed) is an Aliquot part of 360, which (as you ſee) hath 23 Aliquot parts and 24 Diviſors.

2	2	2	3	3	5
	4	8	6	9	10
			12	18	20
			24	36	40
				72	15
					30
					60
					120
					45
					90
					180
					360

Example 2.

Again, if it be required to find out all the Aliquot Parts and Diviſors of 2310, the Operation will be like that in Example 1. For, firſt the prime Diviſors will be found theſe, to wit, 2, 3, 5, 7, 11; then after the ſaid prime Diviſors are ſet at the heads of ſo many Columels, as you ſee in the Table in the Margin, the reſt of the Diviſors will be found out by Multiplication according to the foregoing directions; which in ſum amounts to this, *viz.* each prime Diviſor ſtanding at the head of every Columel following the firſt, is to be multiplied by every one of the Numbers in the foregoing Columels, (except ſuch which make the ſame Products as were before produced) and the Products are to be ſet under each prime Diviſor reſpectively by which they were produced. So all the Diviſors to the given Number 2310 are diſcovered to be theſe, to wit, 1, 2, 3, 5, 6, 7, 10, 11, 14, 15, 21, 22, &c. as you ſee in this Table; every one of which Diviſors, except the greateſt, to wit, 2310, (which is the ſame with the Number propoſed) is an Aliquot part of the ſaid 2310, which has 31 Aliquot parts, but 32 Diviſors.

2310	1155	385	77	11	1
2	3	5	7	11	

2	3	5	7	11
	6	10	14	22
		15	21	33
		30	42	66
			35	55
			70	110
			105	165
			210	330
				77
				154
				231
				462
				385
				770
				1155
				2310

Upon the ſame foundation the Diviſors of Quantities expreſt by Letters may be found out, as will appear by the following Examples. But this work requires that the Analyſt be well exercis'd in the Rules of Algebraical Multiplication, Diviſion, and the Extraction of Roots; for the finding out of the Primitive or Incompoſit Diviſors, when the given Quantity is compos'd of many large Members connexed by different Signs, is oftentimes both difficult and laborious.

Example 3.

Let it be required to find out all the Diviſors and Aliquot parts of this Quantity $aaabbc$. Firſt, I divide the ſaid $aaabbc$ by a, and the Quotient is $aabbc$, which divided by a gives $abbc$, this divided by a gives bbc; and ſo the Diviſion by a ceaſes. Then I divide bbc by b, and the Quotient is bc; this divided by b gives c, which being a Primitive or Incompoſit Quantity I divide by it ſelf, and the Quotient is 1. So all the primitive Diviſors of the propoſed Quantity $aaabbc$ are found a, a, a, b, b, and c; which are manifeſtly ſuch as being multiplied continually will produce the given Quantity $aaabbc$.

$aaabbc$	$aabbc$	$abbc$	bbc	bc	c	1
a	a	a	b	b	c	

Now out of thoſe Diviſors, after they are ſet at the heads of ſo many Columels as you ſee in this Table, I ſearch out the reſt of the Diviſors by Algebraical Multiplication, in like manner as in Example 1. So all the different Diviſors to the given Quantity $aaabbc$ are found theſe, to wit, 1, a, aa, aaa, b, ab, aab, $aaab$, bb, abb, $aabb$, $aaabb$, c, ac, aac, $aaac$, bc, abc, $aabc$, $aaabc$, bbc, $abbc$, $aabbc$, $aaabbc$; every one of which Diviſors, except the laſt and greateſt is an Aliquot part of the given Quantity $aaabbc$, which has 23 parts, and 24 Diviſors.

a	a	a	b	b	c
	aa	aaa	ab	bb	ac
			aab	abb	aac
			$aaab$	$aabb$	$aaac$
				$aaabb$	bc
					abc
					$aabc$
					$aaabc$
					bbc
					$abbc$
					$aabbc$
					$aaabbc$

Note, That this third Example differs not from Example 1. ſaving that Algebraical Diviſion and Multiplication is uſed here inſtead of vulgar Diviſion and Multiplication in Numbers there.

Example

Example 4.

After the ſame manner 31 Aliquot parts and 32 Diviſors will be found to this Quantity $abcde$, viz. 1. a, b, ab, c, ac, bc, abc, d, ad, bd, &c. as you ſee them expreſt in the following Table.

	$abcde$	$bcde$	cde	de	e	1
Primitive Diviſors,	a	b	c	d	e	

a	b	c	d	e
	ab	ac	ad	ae
		bc	bd	be
		abc	abd	abe
			cd	ce
			acd	ace
			bcd	bce
			$abcd$	$abce$
				de
				ade
				bde
				$abde$
				cde
				$acde$
				$bcde$
				$abcde$

Compare this Example with the precedent Example 2.

Example 5.

Again, to find all the Diviſors of this compound Quantity $aaabc-abbbc$, firſt, I ſearch out all its prime Diviſors thus, *viz.* I divide the ſaid Compound Quantity by a, and the Quotient is $aabc-bbbc$; this divided by b gives $aac-bbc$, which divided by c gives the Quotient $aa-bb$; this divided by $a-b$ gives the Quotient $a+b$, which being a primitive Quantity I divide it by itſelf, and the Quotient is 1. So the prime Diviſors are found $a, b, c, a-b$, and $a+b$, which are to be reſerved.

$aaabc-abbbc$	$aabc-bbbc$	$aac-bbc$	$aa-bb$	$a+b$	1
a	b	c	$a-b$	$a+b$	

Then (as in the foregoing Examples) I ſet the ſaid primitive Diviſors at the heads of ſo many Columels, and from thoſe Diviſors (according to the directions in Example 1.) I find out all the reſt by Multiplication; ſo at length it appears that $aaabc-abbbc$ the compound Quantity propoſed has 31 Aliquot parts and 32 Diviſors, wit, 1, a, b, ab, c, ac, bc, abc, $a-b$, $aa-ab$, $ab-bb$, &c. as you ſee them expreſt in the following Table.

a	b	c	$a-b$	$a+b$
	ab	ac	$aa-ab$	$aa+ab$
		bc	$ab-bb$	$ab+bb$
		abc	$aab-abb$	$aab+abb$
			$ac-bc$	$ac+bc$
			$aac-abc$	$aac+abc$
			$abc-bbc$	$abc+bbc$
			$aabc-abbc$	$aabc+abbc$
				$aa-bb$
				$aaa-abb$
				$aab-bbb$
				$aaab-abbb$
				$aac-bbc$
				$aaac-abbc$
				$aabc-bbbc$
				$aaabc-abbbc$

Example 6.

Again, to find out all the Diviſors of this Quantity $aaabbc-2aabbbc+abbbbc$; firſt, (as before) I ſearch out all the primitive Diviſors, *viz.* I divide the Quantity propoſed by a, and the Quotient is $aabbc-2abbbc+bbbbc$, which divided by b gives the Quotient $aabc-2abbc+bbbc$; this divided again by b gives $aac-2abc+bbc$, which divided by c gives $aa-2ab+bb$. This laſt Quotient being a Square whoſe ſide is either $a-b$ or $b-a$, according as a is greater or leſs than b, I ſhall ſuppoſe a to be greater than b; and then dividing the ſaid Square $aa-2ab+bb$ by its ſide $a-b$ the Quotient is alſo $a-b$. And laſtly, by dividing $a-b$ by it ſelf (becauſe 'tis a Primitive Quantity) the Quotient is 1. Thus the primitive Diviſors of the Quantity propoſed are found a, b, b, c, $a-b$ and $a-b$. Then every one of them being ſet at the head of a Columel, and Multiplication made according to the Operation in the precedent Examples, the reſt of the deſired Diviſors to the Quantity $aaabbc-2aabbbc+abbbbc$ will be found out; and at length all the Diviſors to the ſaid Quantity are diſcovered to be theſe, *viz.* 1. a, b, ab, bb, abb, c, ac, bc, abc, bbc, $abbc$, $a-b$, $aa-ab$, $ab-bb$, &c. as you ſee them expreſt in the following Table.

a	b	b	c	$a-b$	$a-b$
	ab	bb	ac	$aa-ab$	$aa-2ab+bb$
		abb	bc	$ab-bb$	$aaa-2aab+abb$
			abc	$aab-abb$	$aab-2abb+bbb$
			bbc	$abb-bbb$	$aaab-2aabb+abbb$
			$abbc$	$aabb-abbb$	$aabb-2abbb+bbbb$
				$ac-bc$	$aaabb-2aabbb+abbbb$
				$aac-abc$	$aac-2abc+bbc$
				$abc-bbc$	$aaac-2aabc+abbc$
				$aabc-abbc$	$aabc-2abbc+bbbc$
				$abbc-bbbc$	$aaabc-2aabbc+abbbc$
				$aabbc-abbbc$	$aabbc-2abbbc+bbbc$
					$aabbbc-2aabbbc+abbbbc$

Example 7.

In like manner, if it be deſired to find out all the Diviſors of this Quantity $aaaaaa+2aaaacc+aacccc$, that is, $a^6+2a^4cc+aac^4$; I divide it firſt by a, and the Quotient is $a^5+2a^3cc+ac^4$, this divided again by a gives $a^4+2aacc+c^4$. Now 'tis evident that this laſt Quotient cannot be divided by a or by c, or the like quantity, but becauſe (by *Sect.* 4. *Chap.* 8. *Book* 1.) the ſaid $a^4+2aacc+c^4$ is a Square, whoſe Root is $aa+cc$, I divide the Square by its Root $aa+cc$, and the Quotient is alſo the ſame Root ſaid $aa+cc$, which being a primitive Quantity I divide it by it ſelf, and the Quotient is 1. So the Diviſors to be reſerved are a, a, $aa+cc$ and $aa+cc$.

$$\begin{array}{c|c|c|c|c}
a^6+2a^4cc+aac^4 & a^5+2a^3cc+ac^4 & a^4+2aacc+c^4 & aa+cc & 1 \\
\hline
a & a & aa+cc & aa-cc &
\end{array}$$

Then after thoſe Diviſors are ſet at the heads of ſo many Columels, (as you ſee in the following Table) I proceed to find out the reſt of the Diviſors by Multiplication according to the directions in Example 1. *viz.* I multiply each primitive Diviſor ſtanding at the head of every Columel following the firſt by every one of the Quantities in the preceding Columels, and ſet the Products under the reſpective primitive Diviſor, with this caution, that one and the ſame Product be not written down twice. So at length I find all the different Diviſors to be theſe, *viz.* 1, a, aa, $aa+cc$, a^3+acc, $a+aacc$, $a^4+2aacc+c^4$, $a^5+2a^3cc+ac^4$, and $a^6+2a^4cc+aac^4$; all which Diviſors except the laſt are Aliquot parts of the propoſed Quantity $a^6+2a^4cc+aae^4$.

a		$aa+cc$	$aa+cc$
	aa	a^3+acc	$a^4+2aacc+c^4$
		a^4+aacc	$a^5+2a^3cc+ac^4$
			$a^6+2a^4cc+aac^4$

VI. By this skill of finding out all the Diviſors of Quantities we may reduce two or more given quantities, when they are not prime between themſelves, to others in the ſame Reaſon (or Proportion) with thoſe given, and in the ſmalleſt Terms. As to reduce thoſe three Quantities $aaa-abb$, $aab-bbb$, and $aaa+aab-abb-bbb$, to the ſmalleſt quantities in the ſame proportion with thoſe propoſed; firſt, I ſeek (by the Method before delivered) all the different Diviſors to every one of thoſe three given quantities, ſo I find the Diviſors of the firſt quantity $aaa-abb$ to be theſe, 1, a, $a+b$, $a-b$, $aa+ab$, $aa-ab$, $aa-bb$, $aaa-abb$; and the Diviſors of the ſecond quantity $aab-bbb$ to be theſe, *viz.* 1, b, $a-b$, $ab-bb$, $a+b$, $ab+bb$, $aa+bb$, and $aab-bbb$; alſo the Diviſors of the third quantity, $aaa+aab-abb-bbb$, to be theſe, to wit, 1, $a-b$, $a+b$, $aa-bb$, $aa+2ab+bb$, and $aaa+aab-abb-bbb$. Now becauſe among thoſe three companies of Diviſors theſe three $a-b$, $a+b$, and $aa-bb$ are found in each company, we may by the help of any one of thoſe three Diviſors reduce the given quantities to others more ſimple, and in the ſame proportion with thoſe given. But to find out the ſmalleſt Terms I divide the propoſed quantities $aaa-abb$, $aab-bbb$, and $aaa+aab-abb-bbb$, ſeverally by $aa-bb$, to wit, ſuch of the ſaid three Diviſors which has moſt Dimenſions, and there ariſe a, b, and $a+b$; which three quantities are the ſmalleſt Terms that can be found in the ſame proportion with the three quantities firſt propoſed.

Note, The Quantities propos'd to be reduced are ſaid to be Prime the one to the other, when they have no common Diviſor beſides 1, (to wit, Unity) in which caſe the quantities propoſed are already in their ſmalleſt Terms.

VII. The finding out of Diviſors may be very fitly be applied to the reducing of Fractions to their ſmalleſt Terms; as to abbreviate this Fraction.

$$\frac{aaa+aab-abb-bbb}{aaa-abb}$$

Firſt, the Diviſors of the Numerator (by the precedent Method) are found 1, $a-b$, $a+b$, $aa-bb$, $aa+2ab+bb$; and $aaa+aab-abb-bbb$. Likewiſe the Diviſors of the Denominator are 1, $a+b$, $a-b$; $aa+ab$, $aa-ab$, $aa-bb$, and $aaa-abb$. Then becauſe among thoſe Diviſors theſe three, to wit, $a+b$, $a-b$, and $aa-bb$, are common both to the Numerator and Denominator, I divide the Numerator and Denominator ſeverally by $aa-bb$, to wit, that common Diviſor which has moſt Dimenſions; ſo there ariſes $a+b$ for a new Numerator, and a for a new Denominaror, which gives this Fraction $\frac{a+b}{a}$ (or $1+\frac{b}{a}$) equal to that propoſed, and in the ſmalleſt Terms, as was deſired.

In like manner to abbreviate $\frac{aaa-abb}{aa+2ab+bb}$, becauſe the greateſt Diviſor common to the Numerator and Denominator is $a+b$, I divide the Numerator and Denominator ſeverally by $a+b$, and there ariſes $\frac{aa-ab}{a+b}$; which is equal to the Fraction propoſed, and in the ſmalleſt Terms.

VIII. *Obſervations upon the Examples in the foregoing* Sect. V.

Firſt, When two, three or four of the foremoſt Letters (towards the left hand) of a ſimple quantity are equal to one another, (*viz.* expreſt by one and the ſame letter) then mark well how many equal letters ſtand foremoſt together, for ſo many Aliquot parts they will give. As in Example 3. in *Sect.* 5. where the quantity propoſed is $aaabbc$, the three firſt letters a, a, a, (that is, aaa) give three Aliquot parts, to wit, 1, a, aa; but four Diviſors, 1, a, aa, aaa. In like manner, if four equal letters ſtand foremoſt together, as a, a, a, a, or $aaaa$, they will afford theſe four parts, 1, a, aa, aaa; but five Diviſors, to wit, 1, a, aa, aaa, $aaaa$. The like property enſues, when five or more equal letters ſtand foremoſt together.

Hence it is evident, that every Power has ſo many Aliquot Parts as there be Dimenſions in the Power; as the Square aa, whoſe Index (or number of Dimenſions) is 2, has two parts, to wit, 1 and a; likewiſe the Cube aaa, or a^3, has three parts; the fourth Power $aaaa$, or a^4, has four parts; and ſo forwards.

Secondly, It is evident from all the precedent Examples in *Sect.* 5 that when among the primitive Diviſors (which are ſet at the tops of the Columels) a following Diviſor differs from the next precedent primitive Diviſor, then the multitude of Diviſors in the Columel of the ſaid following Diviſor is more by 1 than the multitude of all the different Diviſors in the precedent Columels. As in Example 3. in *Sect.* 5. where the quantity propoſed is *aaabbc*, the letter (or primitive Diviſor) *b*, which follows and is different from the next foregoing primitive Diviſor *a*, gives four Diviſors, to wit, *b*, *ab*, *aab*, and *aaab*; which are more in multitude by 1 than all the foregoing different Diviſors *a*, *aa*, & *aaa*.

Again, in Example 4. *Sect.* 5. where the quantity propoſed is *abcde*, the Diviſors *b* and *ab* in the ſecond Columel are more in number by 1 than *a* in the firſt. Likewiſe the Diviſor *c*, *ac*, *bc*, and *abc*, in the third Columel, are more in multitude by 1 than *a*, *b*, and *ab*, to wit, all the Diviſors in the firſt and ſecond Columels. Alſo *d*, *ad*, *bd*, *abd*, *cd*, *acd*, *bcd*, and *abcd* in the fourth Columel, are more in multitude by 1 than all the Diviſors in the firſt, ſecond, and third Columels, and ſo forward. The reaſon is manifeſt, for every primitive Diviſor which ſtands at the top of a following Columel, is multiplied into all the different Diviſors ſeverally in all the foregoing Columels; & therefore if that multiplying primitive Diviſor be added to the number of thoſe Products, the total multitude muſt neceſſarily be more by 1 than the multitude of different Diviſors in all the foregoing Columels.

Thirdly, It is alſo evident, that when the ſaid primitive Diviſors are all different, than the numbers which expreſs the multitude of Diviſors in every Columel are in continual proportion increaſing from Unity in a duple Reaſon. As in the fourth example in *Sect.* 5. where the primitive Diviſors *a*, *b*, *c*, *d*, *e*, are all different, there is one Diviſor in the firſt Columel, two in the ſecond, four in the third, eight in the fourth, and ſixteen in the fifth, which numbers of multitude, to wit, 1, 2, 4, 8, and 16, are manifeſtly in duple proportion. Therefore when all the primitive Diviſors of a quantity propoſed are different or unlike, then if ſo many of the foremoſt Terms of the ſaid continual Proportionals 1, 2, 4, 8, 16, *&c.* be added together, as there be primitive Diviſors, (to wit, thoſe Incompoſit quantities, which being continually multiplied will produce the quantity propoſed) the ſum ſhall be the number of Aliquot parts contained in that quantity, and the number of Diviſors ſhall be more by 1 than that ſum.

As for Example, if the number of Aliquot parts in the quantity *ab* be deſired, I add 1 and 2 together, (to wit, the two firſt Terms of the ſaid Geometrical Progreſſion 1, 2, 4, 8, 16, *&c.*) and the ſum 3 ſhews that *ab* contains three Aliquot parts, and four (that is, 3+1) Diviſors. Likewiſe if there be propoſed the quantity *abc*, (which conſiſts of three different letters) the ſum of 1, 2, 4, (to wit, of the three firſt Terms of the ſaid Geometrical Progreſſion) is 7; which ſhews, that *abc* contains ſeven parts, but eight (or 7+1) Diviſors. Again, if *abcd* (which conſiſts of four different letters) be propoſed, the ſum of 1, 2, 4, 8, (the four foremoſt Terms of the ſaid Progreſſion) is 15; which ſhews that the quantity *abcd* contains fifteen Aliquot parts, and ſixteen (or 15+1) Diviſors, and ſo forward. But becauſe the ſaid Proportionals proceed in a duple reaſon from Unity, the ſum of any number of Terms may be found out by this brief Rule, *viz.* the third Term (or Proportional) leſſened by Unity (the firſt Term) gives the ſum of the firſt and ſecond Terms. Likewiſe the fourth Term leſſened by 1 gives the ſum of the firſt, ſecond, and third Terms; and the fifth Term leſſened by 1 gives the ſum of the firſt, ſecond, third, and fourth Terms, and ſo forward infinitely. All which may be further illuſtrated by the ten quantities, and their reſpective multitudes of Aliquot parts, expreſt in the following Table.

Quantities given.	Multitude of Parts	Sums of Terms in continual Proportion, proceeding from 1 in duple Reaſon.
a	has 1=	1
ab	. . . 3=	1+2
abc	. . . 7=	1+2+4
abcd	. . 15=	1+2+4+8
abcde	. . 31=	1+2+4+8+16
abcdef	. . 63=	1+2+4+8+16+32
abcdefg	. . 127=	1+2+4+8+16+32+64
abcdefgh	. . 255=	1+2+4+8+16+32+64+128
abcdefghi	. . 511=	1+2+4+8+16+32+64+128+256
abcdefghik	. 1023=	1+2+4+8+16+32+64+128+256+512

Fourthly, When two, three, or more equal Letters in a ſimple quantity ſtand together, and follow ſome different foregoing letter or letters, then as many Aliquot parts as the firſt of thoſe following equal letters produces, (according to *Obſervat.* 2.) ſo many parts every one of the reſt of the ſaid following letters will produce. As in Example 3. in *Sect.* 5. where this quantity *aaabbc* is propoſed, the three firſt letters *a*, *a*, *a* (or *aaa*) gives three parts (by *Obſervat.* 1.) And the firſt following letter *b*, in regard it differs from the next preceding letter *a*, gives four parts (by *Obſervat.* 2.) Now I ſay, the ſecond *b* ſhall alſo give four parts, and if there had been a third *b*, or a fourth *b*, *&c.* every one of them would give four parts, to wit, as many as the firſt *b* produced.

In like manner, if this quantity *abbbbb* or ab^5 be propoſed, the firſt letter *a* gives one part; then (by *Obſervat.* 2.) the next following letter *b* (in regard it differs from *a*) gives two parts. Now I ſay, every *b* following the firſt *b* will alſo give two parts, and ſo *bbbbb* will give ten, (to wit, five times two) parts, which added to one part noted for *a* makes 11 parts. Whence I conclude, that the quantity *abbbbb* contains 11 Aliquot parts and 12 Diviſors. All which may be produced particularly by the Rule in the foregoing *Sect.* 5.

Again, if this quantity *abcddd* be propoſed, firſt, (by *Obſervat.* 3.) *abc* will give ſeven parts, and (by *Obſervat.* 2.) the next following letter *d* gives eight parts; therefore (by this fourth *Obſervat.*) every *d* following the firſt *d* gives alſo eight parts, and conſequently *ddd* gives 24 parts, which added to the ſeven parts before noted for *abc*, makes 31 parts. So that the Quantity *abcddd* has 31 Aliquot parts, and 32 Diviſors; and the ſame number of Parts and Diviſors will be found in the Number produced by the continual Multiplication of theſe five prime Numbers 2, 3, 5, 7, 7, 7.

Fifthly, From what has been ſaid in the precedent Obſervations 'tis eaſie to diſcover how many Aliquot parts are contained in any ſimple Quantity deſign'd by letters, without producing the particular parts. As if *aaabbc* be propoſed, firſt, three parts are to be noted for *aaa* (according to *Obſervat.* 1.) and eight parts more for *bb* (by *Obſervat.* 4.) which eight parts added to the three parts before noted make eleven parts; then for *c* twelve parts are to be noted, (to wit, 11 + 1, according to *Obſervat.* 2.) which added to the ſaid 11 parts makes 23 parts. Whence I conclude, that the quantity *aaabbc* has 23 Aliquot parts and 24 Diviſors, which are particularly expreſt in Example 3. *Sect.* 5.

In like manner we may diſcover, that this quantity *aaaaabbbbcccdd*, or $a^5b^4c^3d^2$ has 359 Aliquot parts, and 360 Diviſors. For firſt, I note 5 parts for a^5 (according to *Obſervat.* 1.) then (by *Obſervat.* 4.) *bbbb* or b^4 gives 24 parts, which added to the 5 parts before noted makes 29 parts. And becauſe one ſingle *c* gives 30 parts, to wit, 29 + 1 (by *Obſervat.* 2.) *ccc* or c^3 will give 90, to wit, three times 30 parts (by *Obſervat.* 4.) which added to 29 parts before noted, makes 119 parts. Laſtly, becauſe the letter *d* is written twice, and one ſingle *d* gives 120, to wit, 119 + 1 parts, (by *Obſervat.* 2) *dd* will give 240 parts (by *Obſervat.* 4.) which added to 119 parts before noted, makes 359 parts, which is the multitude of Aliquot parts the propoſed quantity has, but its number of Diviſors is 360.

And with the like facility we may diſcover the multitude of Parts and Diviſors of a given number, after its primitive Diviſors are found out. As for Example, to find how many Parts and Diviſors 15876000 has, I ſearch out by Diviſion (in like manner as in the Examples in *Sect.* 5.) all the primitive Diviſors, which being continually multiplied will produce the ſaid given Number, and find them to be theſe, to wit, 2, 2, 2, 2, 2, 3, 3, 3, 3, 5, 5, 5, 7, 7, which may be noted by $a^5b^4c^3dd$; but this quantity (as before has been ſhewn) has 359 Aliquot parts and 360 Diviſors, and therefore the ſaid 15876000 has the ſame Number of Parts and Diviſors, which may be particularly found out by the Method in the precedent Examples in *Sect.* 5.

Sixthly, If a quantity be compoſed of different Letters or Powers, and Unity be added ſeverally to the Indices of thoſe Powers, that is, to the numbers expreſſing how oft each Letter is found in that quantity, then the Numbers reſulting by thoſe Additions being multiplied one into the other continually, will produce a Number greater by Unity than the number of Aliquot parts that quantity has. As for Example, if *aaaabbb* or a^4b^3 be propoſed, I add 1 to 4 and 3 ſeverally, (becauſe the Indices of *aaaa* and *bbb* are 4 and 3) and it makes 5 and 4; theſe multiplied one into the other make 20, which is greater by 1 than 19, the number of Aliquot parts that the propoſed quantity a^4b^3 has. The reaſon of this Property is not difficult to be conceived; for ſince (by *Obſervat.* 1.)

aaaa hath four parts, that is, five parts wanting one part; and *bbb* following *aaaa* has thrice five parts (by *Obſervat.* 4.) therefore the whole Quantity *aaaabbb* (or a^4b^3) has 4×5 parts wanting one part, *viz.* 19 parts; which numbers 4 and 5 exceed 3 and 4 the Indices of *bbb* and *aaaa* ſeverally by Unity.

Again, if *aaaabbbcc* be propoſed, the Indices of *aaaa*, *bbb*, and *cc*, are 4, 3, and 2, which increaſed ſeverally by 1 make 5, 4, and 3; theſe multiplied continually produce 60, which is greater by Unity than 59, the number of Aliquot parts which the propoſed Quantity *aaaabbbcc* has. For ſince (for the Reaſon in the laſt preceding Example) *aaaabbb* has 4×5 parts wanting one part, and *cc* following *aaaabbb* has (by *Obſervat.* 4.) $2\times4\times5$ parts, the propoſed Quantity *aaaabbbcc* has conſequently $3\times4\times5$ parts wanting one part, that is, 59 parts; which Numbers 3, 4, and 5 do ſeverally exceed the Indices of *cc*, *bbb*, and *aaaa*, by Unity.

Seventhly, From the preceding *Obſervat.* 6. it follows, that if a Compoſit Number be reſolved into any two or more of ſuch of its Factors, the leaſt of which exceeds Unity; and if from every one of thoſe Factors Unity be ſubtracted, the Remainders ſhall be Indices of ſo many ſeveral Powers expreſſible by different Letters, that being joyned together (that is, multiplied one into the other) will give a Quantity having a number of Aliquot parts leſs by Unity than the Compoſit Number propoſed. As for example, if 20 be propoſed; foraſmuch as 5 and 4 multiplied one by the other produce 20, I ſubtract 1 from 5 and 4 ſeverally; ſo the Remainders 4 and 3 do ſhew, that if the fourth Power of ſome Quantity *a*, as *aaaa*, be multiplied into the third Power of ſome other Quantity *b*, as into *bbb*, the Quantity produced, to wit, *aaaabbb* has 19 Aliquot parts, which 19 is leſs by Unity than 20 the Number propoſed. Again, becauſe the Product of 10 into 2 does alſo make 20, I ſubtract 1 from 10 and 2 ſeverally, ſo the Remainders 9 and 1 do ſhew, that if the ninth Power of ſome Quantity *a*, as a^9, be multiplied by ſome other different Quantity *b*, the Quantity produced, to wit, a^9b, has alſo 19 Aliquot parts. Hence it is manifeſt, that often times many Quantities may be found out, every one of which ſhall have a given multitude of Aliquot parts, as will appear in the next following Section.

IX. *The manner of finding out all ſuch Quantities as ſhall have a given Multitude of Aliquot Parts.*

If the multitude of Aliquot parts deſired be any of the Numbers of the ſecond Columel of the Table in *Obſervat.* 3. *Sect.* 8. the Quantity there ſtanding on the left hand of that number, and on the ſame Line with it, has the number of parts deſired. As if it be deſired to find a Quantity that has 63 Aliquot parts, that Table ſhews that *abcdef* has 63 parts; and therefore if ſix prime Numbers, ſuppoſe 2, 3, 5, 7, 11, 13, be taken for the values of thoſe ſix Letters *a*, *b*, *c*, *d*, *e*, *f*, the Product made by the continual Multiplication of the ſaid prime Numbers, to wit, 30030, ſhall have 63 Aliquot parts, and 64 Diviſors.

But without reſpect to that Table, by the help of the Obſervations in the foregoing *Sect.* 8. many Quantities for the moſt part, and always one Quantity may eaſily be found out, that ſhall have a given Multitude of Aliquot parts, as will be made manifeſt by the following Examples.

Example 1.

Let it be required to find out all ſuch ſimple Quantities expreſſible by Letters, that may every one of them have 15 Aliquot parts and 16 Diviſors.

1. To the ſaid 15 I add 1 and it make 16, this I divide by 2 and the Quotient is 8, which divided by it ſelf gives 1; then from each of the Diviſors 2 and 8 (the Product of whoſe Multiplication makes the firſt Dividend 16) I ſubtract 1; ſo the Remainders 1 and 7 do ſhew, that if ſome letter, as *a*, be written once, and next after it another different letter *b* ſeven times, the Quantity ſo Compoſed, to wit, *abbbbbbb* (or ab^7) ſhall have 15 Aliquot parts, and 16 Diviſors, as was deſired.

$$\begin{array}{c|c|c} 16 & 8 & 1 \\ \hline 2 & 8 & \end{array}$$

2. Again, I divide the ſaid 16 (to wit, $15+1$) by 2, and the Quotient is 8; this divided again by 2 gives 4, which divided again by 2 gives 2, which divided by it ſelf gives 1; then from every one of the Diviſors 2, 2, 2, 2, I ſubtract 1; ſo the Remainders 1, 1, 1, 1 do ſhew, that if four different ſingle Letters be ſet together, as *abcd*, this Quantity ſhall have 15 Parts and 16 Diviſors, as before.

$$\begin{array}{ccccc} 16 & 8 & 4 & 2 & 1 \\ \hline 2 & 2 & 2 & 2 & \end{array}$$

3. Again, I divide 16 by 2, and the Quotient is 8; this divided by 2 gives 4, which divided by itſelf gives 1; then from every one of the Diviſors 2,2,4, I ſubtract 1, and the Remainders 1, 1, and 3 do ſhew, that if two different Letters *a* and *b* be joined together, and next after them a third different from each of them (as *c*) be written thrice, the Quantity ſo compoſed, to wit, *abccc*, ſhall have 15 Aliquot Parts, and 16 Diviſors, as before.

$$\begin{array}{c|c|c|c} 16 & 8 & 4 & 1 \\ \hline 2 & 2 & 4 & \end{array}$$

4. Again, I divide 16 by 4, and the Quotient is 4; this divided by it ſelf gives 1: then from each of the Diviſors 4 and 4 I ſubtract 1, and the Remainders 3 and 3 do ſhew, that if ſome Letter *a* be written thrice, as *aaa*, and next after the ſame another Letter different from *a* (as *b*) be likewiſe written thrice, the Quantity ſo compoſed, to wit, *aaabbb*, or a^3b^3, ſhall have 15 Aliquot Parts and 16 Diviſors, as before.

$$\begin{array}{c|c|c} 16 & 4 & 1 \\ \hline 4 & 4 & \end{array}$$

5. Laſtly, I divide 16 by it ſelf and the Quotient is 1; then from 16 I ſubtract 1, and the Remainder 15 ſhews, that if ſome Letter *a* be written 15 times, as *aaaaaaaaaaaaaaa*, or a^{15}, this Quantity ſhall have 15 Parts and 16 Diviſors, as before.

$$\begin{array}{c|c} 16 & 1 \\ \hline 16 & \end{array}$$

Hence becauſe 16 cannot be divided by any other ways than thoſe five before expreſs'd, we may conclude that the five Quantities found out, and thoſe only, to wit, ab^7, $abcd$, abc^3, a^3b^3, and a^{15}, have each of them 15 Aliquot Parts and 16 Diviſors. All which Operations do clearly reſult from *Obſervat.* 6. and 7. in the precedent *Sect.* 8.

Example 2.

Let it be required to find out all ſuch Quantities expreſſible by Letters, which may every one of them have 23 Aliquot Parts and 24 Diviſors.

Firſt, as before I add 1 to 23, and it makes 24; this may be divided by its Factors in a ſevenfold manner before the Quotient be Unity, as here you ſee.

$$\begin{array}{c|c|c|c|c} 24 & 8 & 4 & 2 & 1 \\ \hline 3 & 2 & 2 & 2 & \end{array} \cdot \begin{array}{c|c|c|c} 24 & 6 & 2 & 1 \\ \hline 4 & 3 & 2 & \end{array} \cdot \begin{array}{c|c|c|c} 24 & 4 & 2 & 1 \\ \hline 6 & 2 & 2 & \end{array}$$

$$\begin{array}{c|c|c} 24 & 4 & 1 \\ \hline 6 & 4 & \end{array} \cdot \begin{array}{c|c|c} 24 & 3 & 1 \\ \hline 8 & 3 & \end{array} \cdot \begin{array}{c|c|c} 24 & 2 & 1 \\ \hline 12 & 2 & \end{array} \cdot \begin{array}{c|c} 24 & 1 \\ \hline 24 & \end{array}$$

Whence I conclude that ſeven different Quantities may be produced, every one of which ſhall have 23 Aliquot Parts and 24 Diviſors; now to find out the ſaid Quantities I ſubtract 1 (to wit, Unity) from every one of the Diviſors of the foregoing ſevenfold Diviſion, ſo the Diviſors 3,2,2,2, of the firſt Diviſion being ſeverally leſſened by Unity give 2,1,1,1; whence according to the precedent directions in Example 1 of this *Sect.* 9. this Quantity may be compoſed, to wit, *aabcd*; and by proceeding in like manner with the reſt of the Diviſors ſeven different Quantities, every one of which has 23 Aliquot Parts and 24 Diviſors, are diſcovered, and may be expreſs'd either

$$\text{Thus, } \left\{ \begin{array}{l} aabcd \\ aaabbc \\ aaaaabc \\ aaaaabbb \\ aaaaaaabb \\ aaaaaaaaaaab \\ aaaaaaaaaaaaaaaaaaaaaaa \end{array} \right\} \text{Or thus, } \left\{ \begin{array}{l} a^2bcd \\ a^3b^2c \\ a^5bc \\ a^5b^3 \\ a^7b^2 \\ a^{11}b \\ a^{23} \end{array} \right.$$

Example 3.

Let it be required to find out a Quantity which has 42 Aliquot Parts.

Firſt, as before I add 1 to 42 and it makes 43, which being a prime Number (that is, ſuch as cannot be divided by any Number but by itſelf or Unity) does ſhew, that there is only one Quantity can be found that has 42 Aliquot Parts, *viz.* ſome Letter (as *a*) being written 42 times one after another, or a ſingle *a* with its Index 42, as a^{42}, does expreſs a Quantity (to wit, the forty ſecond Power of *a*) which has 42 Aliquot Parts, and 43 Diviſors. The like is to be underſtood of other Quantities, when the multitude of Aliquot Parts deſired being increaſed with Unity makes a prime Number.

For further Illuſtration of the Premiſes, the Learner may view the following Table, which ſhews all the various Quantities expreſs'd by Letters, that have a given multitude of Aliquot Parts not exceeding 50; and upon the grounds before explained the Table may be continued as far as you pleaſe.

Quantities.		Aliquot Parts.
a	has	1
aa	has	2
ab, a^3	have each	3
a^4	*&c.*	4
aab, a^5		5
a^6		6
a^3b, abc, a^7		7
$aabb, a^8$		8
a^4b, a^9		9
a^{10}		10
$a^2bc, a^3b^2, a^5b, a^{11}$		11
a^{12}		12
a^6b, a^{13}		13
a^4bb, a^{14}		14
$a^3bc, abcd, a^3b^3, a^7b, a^{15}$		15
a^{16}		16
$a^2b^2c, a^5b^2, a^8b, a^{17}$		17
a^{18}		18
$a^4bc, a^4b^3.a^9b, a^{19}$		19
a^6b^2, a^{20}		20
$a^{10}b, a^{21}$		21
a^{22}		22
$a^3b^2c, a^2bcd, a^5bc, a^5b^3, a^7b^2, a^{11}b, a^{23}$		23
a^4b^4, a^{24}		24
$a^{12}b, a^{25}$		25
$a^{10}b^2c^2, a^8b^2, a^{26}$		26
$a^6bc, a^6b^3, a^{13}b, a^{27}$		27
a^{28}		28
$a^4b^2c, a^5b^4, a^9b^2, a^{14}b, a^{29}$		29
a^{30}		30
$a^3bcd, a^3b^3c, a^7bc, abcde, a^7b^3, a^{15}b, a^{31}$		31
$a^{10}b^2a^{32}$		32
$a^{16}b, a^{33}$		33
a^5b^4, a^{34}		34
$a^2b^2cd, a^5b^2c, a^3b^2c^2, a^8bc, a^8b^3, a^5b^5, a^{11}b^2, a^{17}b, a^{35}$		35
a^{36}		36
$a^{18}b, a^{37}$		37
$a^{12}b^2, a^{38}$		38
$a^4bcd, a^4b^3c, a^9bc, a^5b^4, a^9b^3, a^{19}b, a^{39}$		39
a^{40}		40
$a^6b^2c, a^6b^5, a^{13}b^2, a^{20}b, b^{41}$		41
a^{42}		42
$a^{10}bc, a^{10}b^3, a^{21}b, a^{43}$		43
$a^4b^2c^2, a^8b^4, a^{14}b^2, a^{44}$		44
$a^{22}b, a^{45}$		45
a^{46}		46
$a^3b^2cd, a^5bcd, a^5b^3c, a^2bcde, a^3b^3c^2, a^7b^2c, a^{11}bc, a^7b^5, a^{11}b^3, a^{15}b^2, a^{23}b, a^{47}$		47
a^6b^6, a^{48}		48
$a^4b^4c, a^9b^4, a^{24}b, a^{49}$		49
$a^{16}b^2, a^{50}$		50

X. *How*

X. *How to find out the ſmalleſt Number that ſhall have a given multitude of Aliquot Parts.*

Firſt, by the foregoing *Sect.* 9. ſearch out all the Quantities expreſſible by Letters, every one of which may have the Number of Aliquot Parts deſired; then to the different Letters by which every one of thoſe Quantities is expreſs'd, aſſign the ſmalleſt prime Numbers, and find out by continual Multiplication the Products of thoſe prime Numbers correſpondent to the ſaid Quantities. Again, let the values of thoſe Letters be expreſs'd by the ſame prime Numbers varied as many ways as is poſſible, and find out their reſpective Products, as before. Laſtly, all thoſe Products being compared to one another, the leaſt of them ſhall be the ſmalleſt Number that has the preſcribed multitude of Aliquot Parts.

Example 1.

Let it be required to find the ſmalleſt Number that has 15 Aliquot Parts.

Firſt, all the different Quantities that can be found to have ſeverally 15 Aliquot Parts (as appears by the precedent *Sect.* 9) are theſe, to wit, $abcd$, a^3bc, a^3b^3, a^7b, a^{15}; then by aſſigning to a,b,c,d the ſmalleſt prime Numbers 2,3,5,7, for $abcd$ there will be found 210, (by multiplying 2,3,5,7 one into the other continually;) for a^3bc 120, for a^3b^3 216, for a^7b 384, and for a^{15} 32768; the leaſt of which Products is 120. But before we can determine whether 120 be the leaſt Number or not that has 15 Aliquot Parts, enquiry muſt be made by exchanging the values of thoſe Letters with the ſaid prime Numbers all manner of ways, *viz.* we may ſuppoſe $a=3$, $b=2$, $c=5$, and $d=7$; or $a=5$, $b=2$, $c=3$, and $d=7$: or again, $a=7$, $b=2$, $c=3$, $d=5$. And many otherways the values of a,b,c,d may be expreſs'd by the ſaid prime Numbers 2,3,5,7; and conſequently from thoſe Variations the Quantities $abcd$, a^3bc, b^3b^3, a^7b, a^{15} will be expounded by various Numbers, which muſt be compared together, and then the leaſt among them all is the Number ſought. So after all Variations are made, it will appear that a^3bc is that Quantity by which 120, the ſmalleſt Number having 15 Aliquot Parts and 16 Diviſors, will be found out.

Example 2.

Again, if the leaſt Number that has 23 Aliquot Parts, or 24 Diviſors, be deſired.

Firſt, by *Sect.* 9. all the Quantities which have ſeverally 23 Parts will be found theſe, to wit, a^2bcd, a^3bbc, a^5bc, a^5b^3, a^7b^2, $a^{11}b$, and a^{23}. Then by aſſuming for the values of a,b,c,d the leaſt prime Numbers 2,3,5,7: for a^2bcd there will be found 420, for a^3b^2c 360, for a^5bc 480, for a^5b^3.864, for a^7b^2 1152, for $a^{11}b$ 6144, and for a^{23} 8388608. And after all other poſſible Variations made with the ſaid Letters and prime Numbers, by taking ſometimes one, ſometimes another of the ſaid Numbers for the value of a, b, &c. it will at length appear that a^3b^2c finds out 360, the leaſt Number that has the deſired multitude of 23 Aliquot Parts and 24 Diviſors.

If there be not occaſion to find the leaſt, but any Number that has a given multitude of Aliquot Parts, ſuppoſe 15, then you may indifferently uſe any one of theſe five Quantities $abcd$, a^3bc, a^3b^3, a^7b, a^{15}, by aſſigning to a,b,c,d prime Numbers at pleaſure, and taking ſometimes one, ſometimes another of thoſe Numbers, or always new prime Numbers for the values of a,b,c,d; whence innumerable Numbers may be found out, every one of which ſhall have Aliquot Parts. As if we ſuppoſe $a=2$, $b=3$, and $c=5$, there will be found for a^3bc 120; but by putting $a=3$, $b=2$, and $c=5$, there will be found for a^3bc 270. Or alſo by aſſuming $a=7$, $b=11$, and $c=13$, there will be produced for a^3bc 49049. Or if we put $a=17$, $b=19$, and $c=23$, then $a^3bc=2146981$. And in like manner you may uſe every one of the other four Quantities $abcd$, a^3b^3, a^7b, and a^{15}. The like alſo is to be underſtood of every one of theſe a^2bcd, a^3b^2c, a^5bc, a^5b^3, a^7b^2, $a^{11}b$, and a^{23}, for the finding out innumerable Numbers, which have ſeverally 23 Aliquot Parts and 24 Diviſors.

Laſtly, to find the leaſt Number that has 42 Parts and 43 Diviſors; foraſmuch as a Quantity having this multitude of Parts and Diviſors can be deſigned only in one manner, *viz* by writing a^{42}; let the leaſt prime Number 2 be taken for the value of a, and then ſeek the forty ſecond Power of the Root 2, by writing down 2 forty two times ſeparately, and multiplying thoſe Numbers one into another, according to the Rule of continual Multiplication, ſo the laſt Product will be 4398046511104, which is the leaſt Number that has the deſired multitude of 42 Aliquot Parts. And ſo of others.

For further illuſtration the Learner may view the following Table, which ſhews the leaſt Number that has any given multitude of Aliquot Parts under 51. *Note*, That the number of Diviſors to any number is always more by one than its number of Aliquot Parts; for albeit a number cannot properly be called a Part of itſelf, yet 'tis contained in it ſelf once, and therefore may be ſaid to be a Diviſor to itſelf.

Each number in the firſt of theſe Columels is the ſmalleſt that can be found to have ſuch a multitude of Aliquot Parts as is expreſs'd in the latter Columel.

2	has	1	Aliquot Part.
4	has	2	Aliquot Parts.
6		3	
16		4	
12		5	
64		6	
24		7	
36		8	
48		9	
1024		10	
60		11	
4096		12	
192		13	
144		14	
120		15	
65536		16	
180		17	
262144		18	
240		19	
576		20	
3072		21	
4194304		22	
360		23	
1296		24	
12288		25	
900		26	
960		27	
268435456		28	
720		29	
1073741824		30	
840		31	
9216		32	
196608		33	
5184		34	
1260		35	
68719476736		36	
786432		37	
36864		38	
1680		39	
1099511627776		40	
2880		41	
4398046511104		42	
15360		43	
3600		44	
12582912		45	
70368744177664		46	
2520		47	
46656		48	
6480		49	
589824		50	

CHAP. IX.

The Arithmetic both of Surd Numbers and Surd Quantities expreſs'd by Letters. The Conſtitution and Invention of ſix Binomials in numbers, agreeable to thoſe expounded in Prop. 49,50,51,52,53.54. Elem. 10. Euclid. *with Rules to extract the Square Root out of every one of them; as alſo what Root you pleaſe out of any Binomial in Numbers, having ſuch a Binomial Root as is deſired.*

Sect. I. *Definitions concerning Surd Roots, and their Fundamental Operations.*

EVery Abſolute (or Ordinary) Number, whether it be a whole Number or a Fraction, or a whole Number with a Fraction annex'd to it, is called *Rational*: As 1, 2,3,4, *&c.* alſo $\frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{15}{21}$, *&c.* and $2\frac{1}{2}$ (or $\frac{5}{2}$,) $5\frac{1}{3}$ (or $\frac{16}{3}$) $20\frac{11}{12}$, *&c.* are called Rational Numbers; ſo alſo $a, ab, \frac{bc}{a}, a+\frac{bc}{a}$, *&c.* repreſent Rational Quantities.

But when the Square Root, Cubic Root, or any other Root, cannot be perfectly extracted out of a Rational Number, that Root is called *Irrational* or *Surd*; and becauſe it cannot be exactly expreſs'd by any Rational Number, it is uſual to ſet ſome Character (which is called the Radical Sign) before the Rational Number out of which the Root ought to be extracted, to deſign or ſignifie the ſame Root: As $\sqrt{}$ or $\sqrt{(2)}$ prefix'd before any Rational Number, ſignifies the Square Root of that Number; $\sqrt{(3)}$ the Cubic Root, $\sqrt{(4)}$ the Biquadratic Root, $\sqrt{(5)}$ the Root of the fifth Power, *&c.*

Hence $\sqrt{(12)}$ or $\sqrt{(2)}12$ denotes or repreſents the Square Root of 12, which Root is called Irrational or Surd, becauſe it cannot be perfectly expreſs'd by any Rational Number; for 3 multiplied by itſelf produces 9, which is leſs than 12; and 4 multiplied by itſelf produces 16, which is greater than 12: and altho there be innumerable mixt Numbers conſiſting of 3, and ſome Fractions which fall between 3 and 4, yet none of them multiplied into itſelf quadraticaly can produce the whole Number 12.

In like manner $\sqrt{(3)}5$, which repreſents the Cubic Root of 5, is called an Irrational or Surd Number, becauſe no Number can be found, which being multiplied into itſelf cubically will produce 5 exactly: ſo alſo $\sqrt{a}, \sqrt{bc}, \sqrt{(3)}bb$, *&c.* repreſent Surd Quantities.

There are two ſorts of Irrational or Surd Numbers, Simple and compound: a Simple Surd Number is expreſs'd by one ſingle Term; ſuch are $\sqrt{5}, \sqrt{10}, \sqrt{(3)}16, \sqrt{(4)}8$, *&c.* but a Compound Surd Number conſiſts of many ſimple or ſingle Terms, and is formed by the Addition or Subtraction of Simple Terms, ſuch are $\sqrt{5}+\sqrt{2}, \sqrt{5}-\sqrt{2}, \sqrt{8}+\sqrt{6}-\sqrt{2}, \sqrt{(3)}:\overline{7+\sqrt{2}}$: which laſt is called an Univerſal Root, and ſignifies the Cubic Root of the Sum of 7, and the ſquare Root of 2. (See *Sect.* 28. *Chap.* 1. *Book* 1. concerning the deſigning of Surd Numbers.

The Arithmetic of Surd Numbers, and Surd Quantities deſign'd by Letters, depends chiefly upon theſe ſix primary or fundamental Operations in Simple Surds, *viz.*

1. The Reduction of Rational Numbers and Rational Quantities expreſs'd by Letters, to the form of Surd Roots, which ſhall have a given Radical Sign.
2. The Reduction of Simple Surd Roots having different Radical Signs, to other Surds which ſhall have one common Radical Sign, and be equal in value to the given Surds.
3. Multiplication in Simple Surds.
4. Diviſion in Simple Surds.
5. The Reduction of a given Surd Number or Quantity to another more ſimple, when it may be done.
6. How to diſcover whether two Simple Surd Numbers or Quantities be *Commenſurable* or not, *viz.* whether their *Reaſon* or *Proportion* can be expreſs'd by Rational Numbers or Quantities, or not. Theſe ſix Operations I ſhall handle in order.

Sect.

Sect. II. *How to reduce Rational Numbers and Quantities deſigned by Letters to the form of Surd Roots, which ſhall have the ſame Radical Sign with any Surd Root preſcribed.*

Multiply the given Rational Number or Quantity into itſelf, ſo often as is requiſite to produce a Power of the ſame degree with that Power which is denoted by the Radical Sign of the preſcribed Surd, and then ſet the ſaid Radical Sign before the Power produced by the ſaid Multiplication.

As to reduce 6 to the form of a Surd Root which ſhall have the ſame Radical Sign with $\sqrt{12}$ (or $\sqrt{(2)}12$,) I multiply 6 into it ſelf quadraticaly, and it makes 36; then $\sqrt{36}$ *(*that is,*6)* and $\sqrt{12}$ have the ſame Radical Sign, to wit, $\sqrt{}$ or $\sqrt{(2)}$.

Again, to reduce 5 to the ſame Radical Sign with $\sqrt{(3)}12$, I multiply 5 into itſelf cubically, *(viz.* 5 into 5, and the Product into 5*)* and it produces 125; then $\sqrt{(3)}125$ (that is, 5) and $\sqrt{(3)}12$ have the ſame Radical Sign, to wit $\sqrt{(3)}$.

Likewiſe to reduce 3 to the ſame Radical Sign with $\sqrt{(4)}12$, I ſeek the fourth Power of 3, which (by multiplying the Square of 3 into itſelf) will be found 81; then $\sqrt{(4)}81$ and $\sqrt{(4)}12$ are of the ſame kind. And ſo of others.

By the help of this Rule, when the Radical Sign of a Simple Surd Fraction has reference only to one of its Terms, we may reduce the Fraction to another, whoſe Radical Sign ſhall refer both to the Numerator and Denominator: As if $\frac{\sqrt{2}}{5}$ be propoſed, which ſignifies that $\sqrt{2}$ is divided or to be divided by 5, we may take $\sqrt{25}$ inſtead of 5, and then that Fraction will be reduced to this $\sqrt{\frac{2}{25}}$, whoſe Radical Sign refers as well to the Denominator as the Numerator, *viz.* $\sqrt{\frac{2}{25}}$ ſignifies that $\sqrt{2}$ is divided by $\sqrt{25}$.

Likewiſe $\frac{5}{\sqrt{(3)}4}$ may be reduced to $\sqrt{(3)}\frac{125}{4}$, by ſetting 125 the Cube of 5 for a Numerator inſtead of 5, and the Radical Sign $\sqrt{(3)}$ againſt the middle of the Fraction; ſo that $\sqrt{(3)}\frac{125}{4}$ (which ſignifies that $\sqrt{(3)}125$ is divided by $\sqrt{(3)}4$) imports as much as $\frac{5}{\sqrt{(3)}4}$ that is, 5 divided by $\sqrt{(3)}4$.

Nor will the Operation be otherwiſe in reducing Rational Quantities deſigned by Letters to the form of Surd Quantities; (reſpect being had to the Rules of Algebraical Multiplication before delivered.) As to reduce the Quantity a, ſo as it may have the ſame Radical Sign with $\sqrt{b}$, I multiply a into itſelf quadraticaly, and it makes aa; then $\sqrt{aa}$ (that is, a) and $\sqrt{b}$ have the ſame Radical Sign.

Again, to reduce $a+b$ to the ſame Radical Sign with $\sqrt{bc}$, I ſquare $a+b$, and it makes $aa+2ab+bb$; then $\sqrt{:\overline{aa+2ab+bb}:}$ (that is, $a+b$) and $\sqrt{bc}$ have the ſame Radical Sign.

Likewiſe to reduce b to the ſame Radical Sign with $\sqrt{(3)}ab$, I multiply b into itſelf cubically, and it makes bbb; then $\sqrt{(3)}bbb$ (that is, b) and $\sqrt{(3)}ab$ have the ſame Radical Sign, to wit, $\sqrt{(3)}$.

Hence alſo $\frac{a}{\sqrt{b}}$ may be reduced to $\sqrt{\frac{aa}{b}}$, and $\frac{\sqrt{(3)}ab}{3c}$ to $\sqrt{(3)}\frac{ab}{27ccc}$.

Sect. III. *How to reduce two ſimple Surd Numbers or Quantities having different Radical Signs, to two others that may have a common Radical Sign.*

This Reduction is like that of reducing Vulgar Fractions to a common Denominator; but how 'tis wrought I ſhall ſhew by Examples, firſt in Surd Numbers, and then in Surd Quantities expreſs'd by Letters.

Example 1.

Let it be required to reduce $\sqrt{(4)}10$ and $\sqrt{(6)}7$ into two other Roots that may have a common Radical Sign, and be equal in value to thoſe given.

Firſt, divide the given Indices *(4)* and *(6)* by their greateſt common Diviſor *(2)*, and ſet the Quotients (2) and (3) under their reſpective Dividends as here you ſee; then multiply croſs-wiſe *viz.* the firſt Dividend or Index *(4)* by the ſecond Quotient *(3)*, (or the ſecond Dividend *(6)* by the

$$\begin{array}{llcl} (2)\ ) & \sqrt{(4)}10 & \times & \sqrt{(6)}7 \\ & \quad(2) & & \quad(3) \\ & \sqrt{(12)}1000 & & \sqrt{(12)}49 \end{array}$$

the firſt Quotient (2), and the Product is (12), before which ſetting $\surd$ it gives $\surd(12)$, which is to be reſerved for the common radical Sign ſought. Then multiply the Powers of the given Roots according to the altern Quotients, *viz.* multiply the firſt Power 10 cubically, becauſe the ſecond Quotient is (3); and the latter Power 7 quadraticaly, becauſe the firſt Quotient is (2): ſo the Products will be 1000 and 49, before each of which prefixing $\surd(12)$ the common Radical Sign before found, there ariſe $\surd(12)1000$ and $\surd(12)49$, the two Surd Roots ſought, which are equal in value to the given Surds reſpectively, *viz.* $\surd(12)1000$ is equal to $\surd(4)10$, and $\surd(12)49$ is equal to $\surd(6)7$; and the Surds found out have a common Radical Sign, as was required.

Example 2.

In like manner $\surd(2)5$ and $\surd(3)6$ will be reduced to $\surd(6)125$ and $\surd(6)36$; and the Work will ſtand as here you ſee underneath.

$$\begin{array}{cccc} (1)\,) & \surd(2)5 & \times & \surd(3)6 \\ & (2) & & (3) \\ & \surd(6)125 & & \surd(6)36 \end{array}$$

Example 3.

Again, if $\frac{\surd 7}{3}$ and $\frac{5}{\surd(3)4}$ be propoſed to be reduced to a common Radical Sign, firſt by the Rule in the preceding *Sect.* 2. I reduce them to $\surd\frac{7}{9}$ (or $\surd(2)\frac{7}{9}$) and $\surd(3)\frac{125}{4}$, which according to the Rule in the firſt Example of this Section will be reduced to theſe, to wit, $\surd(6)\frac{343}{729}$ and $\surd(6)\frac{15625}{16}$, and the Work will ſtand as here you ſee.

$$\begin{array}{cccc} (1)\,) & \surd(2)\frac{7}{9} & \times & \surd(3)\frac{125}{4} \\ & (2) & & (3) \\ & \surd(6)\frac{343}{729} & & \surd(6)\frac{15625}{16} \end{array}$$

The like Work is to be done in reducing two Surd Quantities expreſs'd by Letters, which have different Radical Signs, to two others which ſhall have a common Radical Sign, as will appear in the following Examples.

Example 4.

Suppoſe it be deſired to reduce $\surd(2)a$ and $\surd(6)aa$ to a common Radical Sign.

Firſt, I divide the given Indices (2) and (6) ſeverally by their greateſt common Diviſor (2) and ſet the Quotient (1) and (3) under their reſpective Dividends, as here you ſee; then I multiply croſs-wiſe, *viz.* the firſt Dividend (2) by the ſecond Quotient (3), or the latter Dividend (6) by the firſt Quotient (1), and the Product is (6); before which ſetting $\surd$ it gives $\surd(6)$ for the common Radical Sign ſought. Then I multiply the Powers of the given Roots according to the alternate Quotients, *viz.* the firſt Power a cubically, becauſe the latter Quotient is (3), but the ſecond Power aa, becauſe the firſt Quotient (1) is a lateral Index, is not to be multiplied into itſelf at all. So the Products are aaa and aa, before each of which prefixing $\surd(6)$, (the common Radical Sign before found) there ariſe $\surd(6)aaa$ and $\surd(6)aa$ the two Surd Roots ſought; which are equal in value to the given Surds reſpectively, *viz.* $\surd(6)aaa$ is equal to $\surd(2)a$, and $\surd(6)aa$ is equal to $\surd(6)aa$; and the Surd Roots found out have a common Radical Sign, to wit, $\surd(6)$. Therefore that is done which was required.

$$\begin{array}{cccc} (2)\,) & \surd(2)a & \times & \surd(6)aa \\ & (1) & & (3) \\ & \surd(6)aaa & & \surd(6)aa \end{array}$$

Example 5.

After the ſame manner $\surd(4)3b$ and $\surd(10)5ac$ will be reduced to $\surd(20)243bbbbb$ and $\surd(20)25aacc$, and the Work will ſtand as here you ſee.

$$\begin{array}{cccc} (2)\,) & \surd(4)3b & \times & \surd(10)5ac \\ & (2) & & (5) \\ & \surd(20)243bbbbb & & \surd(20)25aacc \end{array}$$

Sect. IV. *Multiplication in ſimple Surd Quantities.*

Before Addition and Subtraction can be performed in Surd Quantities, the manner of their Multiplication and Diviſion muſt firſt be learnt; I ſhall therefore begin with Multiplication, which requires that the Surd Roots propoſed to be multiplied be of the ſame kind; and therefore if they be of different kinds, they muſt firſt of all be reduced to the ſame radical Sign, (by the Rule in the foregoing *Sect.* 3.) Then,

1. Multiply the Numbers or Quantities ſtanding next after their common radical Sign one into another, without any regard had to the ſaid Sign; and to the Product of that Multiplication prefix the common radical Sign: ſo this new Root ſhall be the Product ſought.

As for Example, to multiply $\surd 5$ by $\surd 3$, I multiply 5 by 3 and it makes 15; to which I prefix $\surd$, (the radical Sign of each of the Surds given to be multiplied) and then ariſes $\surd 15$ for the Product ſought.

Likewiſe if $\surd 6$ be multiplied by $\surd 5$ it produces $\surd 30$.

Alſo $\surd\frac{3}{4}$ multiplied by $\surd\frac{1}{2}$ makes $\surd\frac{3}{8}$.

And $\surd 2\frac{1}{2}$ (or $\surd\frac{5}{2}$) into $\surd 2\frac{1}{3}$ (or $\surd\frac{7}{3}$) gives $\surd\frac{35}{6}$.

Again, $\surd(3)4$ multiplied by $\surd(3)5$ produces $\surd(3)20$.

Likewiſe $\surd(4)\frac{5}{2}$ into $\surd(4)2$ produces $\surd(4)5$.

And if $\surd(2)5$ be to be multiplied into $\surd(3)6$, the Product will be $\surd(6)4500$; for, firſt, the given Roots being of different kinds are reduced to theſe, to wit, $\surd(6)125$ and $\surd(6)36$, which multiplied one into another make $\surd(6)4500$.

After the ſame manner Multiplication in ſimple Surd Quantities expreſs'd by Letters is performed: as if $\surd a$ be to be multiplied by $\surd b$, the Product will $\surd ab$. For (according to the Rule of Algebraical Multiplication) the quantity a multiplied by the quantity b produces ab, to which I prefix the given radical Sign $\surd$, and it gives $\surd ab$ the Product ſought.

Likewiſe $\surd ab$ into $\surd cd$ produces $\surd abcd$.

And $\surd\frac{2ab}{3c}$ multiplied by $\surd\frac{9ad}{2b}$ makes $\surd\frac{3aad}{c}$.

Again, to multiply $\surd(2)d$ by $\surd(3)ab$, firſt, (by the Rule in the foregoing *Sect.* 3.) I reduce them to $\surd(6)ddd$ and $\surd(6)aabb$, which multiplied one into another give $\surd(6)dddaabb$ for the Product required.

2. When any Surd Root is to be multiplied into it ſelf according to the Index of its own Power, *viz.* if a Surd ſquare Root be to be ſquared, or a Surd cubic Root to be cubed, caſt away the radical Sign, and take the number or quantity remaining for the Product ſought, which in this caſe is always rational: as to multiply $\surd 5$ into it ſelf I caſt away the radical Sign $\surd$, and take 5 for the Product or Square of $\surd 5$, (or $\surd 5$ into $\surd 5$ makes $\surd 25$, that is, 5.) Likewiſe the Square of $\surd 8$ is 8, and the Square of $\surd 4$ is 4.

In like manner to multiply $\surd(3)5$ into it ſelf cubically, I take 5 for the Product, to wit, the Cube of $\surd(3)5$: for $\surd(3)5$ into $\surd(3)5$ makes $\surd(3)25$, and this again into $\surd(3)5$ produces $\surd(3)125$, that is, 5.

Again, $\surd(4)12$ multiplied into itſelf biquadraticaly produces 12; for $\surd(4)12$ into $\surd(4)12$ makes $\surd(4)144$, (which is the Square of $\surd(4)12$;) then $\surd(4)144$ again into $\surd(4)12$ makes $\surd(4)1728$, (which is the Cube of $\surd(4)12$.) Laſtly, $\surd(4)1728$ again into $\surd(4)12$ produces $\surd(4)20736$, that is 12, which is the fourth Power of $\surd(4)12$ the Root propoſed.

The like is to be done in Surd Quantities expreſs'd by Letters; as if $\surd ab$ be to be multiplied into itſelf, or ſquared, I caſt away the radical Sign, and write ab for the Product or Square of $\surd ab$. Likewiſe if $\surd(3)bcd$ be to be multiplied into itſelf cubically, the Product or Cube thereof will be bcd.

3. When a Surd Quantity is given to be multiplied by a Rational Quantity, reduce the Rational into the form of a Surd of the ſame kind with the given Surd, (by the foregoing Rule in *Sect.* 2.) and then multiply according to the firſt Rule of this fourth Section; as to multiply $\surd 8$ by 2, I firſt reduce 2 to $\surd 4$, then $\surd 8$ into $\surd 4$ gives $\surd 32$, the Product deſired. Likewiſe $\surd 7$ multiplied by 5, that is, by $\surd 25$, gives the Product $\surd 175$.

Again, if $\surd(3)6$ be to be multiplied by 2, I reduce 2 to $\surd(3)8$, (by multiplying 2 into it ſelf cubically;) then $\surd(3)6$ multiplied by $\surd(3)8$, gives $\surd(3)48$ for the Product deſired.

Likewiſe $\surd(4)8$ multiplied by 5, that is, by $\surd(4)625$, gives $\surd(4)5000$ for the Product ſought.

After the ſame manner to multiply the Surd quantity $\surd a$ by the Rational quantity b, I firſt reduce b to $\surd bb$, then $\surd a$ into $\surd bb$, makes $\surd abb$ the Product ſought. Likewiſe $\surd(3)a$ into b makes $\surd(3)abbb$, (b being firſt reduced to $\surd(3)bbb$.)

Again $\surd 3$ into $4a$ gives the Product $\surd 48aa$.

4. But when a Surd quantity is given to be multiplied by a Rational quantity, it will oftentimes be very convenient to omit their Multiplication, and only to connect them ſo as that the Rational quantity may ſtand on the left hand of the given Surd, to ſignifie the Product of their Multiplication; as to Multiply $\surd 8$ by 2, I write $2\surd 8$ for the Product, which ſignifies twice the ſquare Root of 8. Likewiſe $20\surd 3$ repreſents the Product of the Multiplication of $\surd 3$ by 20, *viz.* it imports $\surd 3$ to be taken 20 times, which amounts to as much as $\surd 1200$, found out by the preceding third Rule of this Section.

Again, $\frac{8}{3}\surd 7$ ſignifies the Product of $\surd 7$ multiplied by $\frac{8}{3}$, (or $\frac{8}{3}$ by $\surd 7$;) and $\frac{2}{3}\surd\frac{7}{5}$ denotes the Product of $\frac{2}{3}$ multiplied into $\surd\frac{7}{5}$, (or $\surd\frac{7}{5}$ into $\frac{2}{3}$;) alſo 4 into $20\surd 3$ makes $80\surd 3$, that is, $20\surd 3$ taken four times. Likewiſe $2\surd(3)6$, ſignifies twice the Cubic Root of 6, and is of equal value with $\surd(3)48$. Likewiſe $\frac{5}{3}\surd(3)80$ denotes the Product of the Cubic Root of 80 multiplied by $\frac{5}{3}$, or $\frac{5}{3}$ of $\surd(3)80$, which is equivalent to $\surd(3)\frac{10000}{27}$; and $3\surd(3)5$ multiplied by 6 makes $18\surd(3)5$, that is, $\surd(3)29160$.

The like may be done in Surd quantities expreſt by Letters; as if $\surd a$ be to be multiplied by b, I write $b\surd a$ to ſignifie the Product; alſo 5 into $b\surd a$ makes $5b\surd a$; and c into $b\surd a$ gives the Product $cb\surd a$; likewiſe $4a$ into $\surd 3$ makes $4a\surd 3$.

Again, if $\surd ab$ be to be multiplied by $b-d$, the Product may be expreſt thus, $\overline{b-d}\times\surd ab$, or thus, $\overline{b-d}\surd ab$.

Alſo if $\surd(3)\frac{2ab}{c}$ be to be multiplied by d, the Product may be expreſt thus, $d\surd(3)\frac{2ab}{c}$ and $\surd(3)a$ into b, makes $b\surd(3)a$, which is equivalent to $\surd(3)abbb$.

5. When two Rational quantities, whether they be equal or unequal, are multiplied ſeverally into one common Surd Square Root, according to the method in the preceding fourth Rule, and it is deſired to multiply thoſe Products one into the other, (which Products are called Commenſurable Quantities, for the reaſon hereafter given in *Sect.* 7.) multiply the Rational by the Rational, and that which is produced multiply by the ſaid common Surd, omitting its Radical Sign; ſo the laſt Product is that which is ſought, and will be intirely Rational.

As for example, to multiply $3\surd 5$ by $2\surd 5$ I multiply 3 by 2, and the Product 6 by 5, ſo it makes 30, which is the Product of $3\surd 5$ multiplied by $2\surd 5$, (or of $\surd 45$ into $\surd 20$.) Likewiſe $2\surd 3$ multiplied by $2\surd 3$, (*viz.* the ſquare of $2\surd 3$) makes 12; and $20\surd 3$ into $8\surd 3$ makes 480, (by multiplying 20, 8, and 3, one into another continually;) again, $\frac{8}{3}\surd 12$ into $5\surd 12$ produces 160.

After the ſame manner to multiply $a\surd c$ by $b\surd c$, I multiply a by b, and the Product ab by c; ſo there ariſes abc for the Product ſought. The Reaſon of this Rule is evident, for $\surd aac$, (that is, $a\surd c$) multiplied into $\surd bbc$, (that is, $b\surd c$) makes $\surd aabbcc$, (that is, abc,) as before.

In like manner $5\surd b$ into $5\surd b$ produces $25b$, to wit, the Square of $5\surd b$; and $2a\surd b$ into $5a\surd b$ gives the Product $10aab$. Alſo $5a\surd 12d$ multiplied by $\frac{8}{3}a\surd 12d$ produces $160aad$.

But here is to be noted, that this fifth Rule of Multiplication takes place only when the common Surd Root into which Rational Numbers are multiplied is a Surd ſquare Root; ſo that if $4\surd(3)5$ be to be multiplied by $2\surd(3)5$, the ſaid fifth Rule will be ineffective, and the Product is to be found out by the following ſixth Rule.

6. When two Rational Quantities, whether they be equal or unequal, are multiplied into two unequal Surd Roots of the ſame kind, or into one common Surd above the quadratic kind, according to the Method in the foregoing fourth Rule of this *Sect.* and it is deſir'd to multiply thoſe Products one into another, multiply the Rational by the Rational and the Surd by the Surd, and joyn theſe Products together, ſo as the Rational Product may ſtand on the left hand; then thoſe 2 Products ſo connected ſhall be the Product ſought.

As for Example, to multiply $5\surd 8$ by $2\surd 3$ I multiply 5 by 2, and the Product is 10; alſo $\surd 8$ into $\surd 3$ makes $\surd 24$; then thoſe 2 Products connected make $10\surd 24$, (that is, $\surd 2400$)

$\surd 2400$) the Product ſought. In like manner $2\surd 8$ into $2\surd 3$ makes $4\surd 24$, that is, $\surd 384$.

Again, $20\surd 5$ multiplied by $18\surd 3$ produces $360\surd 15$; and $8\surd 27$ into $2\surd 3$ makes $16\surd 81$, that is, 144; alſo $5\surd(3)4$ into $3\surd(3)5$ produces $15\surd(3)20$, that is, $\surd(3)3375$; likewiſe $4\surd(3)5$ into $2\surd(3)5$ makes $8\surd(3)25$; and $3\surd(4)5$ into $2\surd(4)6$ makes $6\surd(4)30$.

After the ſame manner to multiply $a\surd bc$ into $g\surd ad$, firſt I multiply a by g, and it makes ag; then $\surd bc$ into $\surd ad$ produces $\surd bcad$. Laſtly, ag into $\surd bcad$ gives $ag\surd bcad$, the Product ſought.

Likewiſe $2\surd ab$ multiplied by $3c\surd bc$ produces $6c\surd abbc$; and $2\surd a$ into $2\surd b$ makes $4\surd ab$.

Alſo $\frac{2bc}{a}\surd ddd$ multiplied by $\frac{aa}{2c}\surd ac$, gives the Product $ab\surd acddd$; and $b\surd(3)dd$ into $c\surd(3)f$ makes $bc\surd(3)ddf$; again, $a\surd(3)c$ into $b\surd(3)c$ makes $ab\surd(3)cc$.

7. When a ſimple Surd Quantity whoſe Radical Sign has for its Index ſome even Number greater than 2 is to be ſquared, prefix a Radical Sign whoſe Index is half the given Index, before the Power of the given Surd; ſo ſhall this new Surd be the ſquare of that given. As if $\surd(4)5$ be to be ſquared or multiplied into it ſelf, take $\surd(2)5$ or $\surd 5$, for the Square or Product ſought. Likewiſe the Square of $\surd(6)10$ is $\surd(3)10$, and $\surd(8)10$ into $\surd(8)10$ makes $\surd(4)10$.

After the ſame manner to multiply $\surd(4)bc$ into itſelf quadraticaly, I write $\surd(2)bc$ or $\surd bc$ for the Product or Square of $\surd(4)bc$. Likewiſe the Square of $\surd(8)10bc$ is $\surd(4)10bc$, and $\surd(10)a$ into $\surd(10)a$ makes $\surd(5)a$. Moreover, $2ab\surd(4)d$ into $3\surd(4)d$ makes $6ab\surd d$; for $2ab$ into 3 makes $6ab$ and $\surd(4)d$ being ſquared makes $\surd(2)$ or $\surd d$.

But when a ſimple Surd Quantity, whoſe Radical ſign has for its Index ſome Ternary Number greater than 3, as 6, 9, &c. is to be multiplied into itſelf cubically, prefix a Radical Sign with an Index that may be a third part of the given Index before the Power of the given Surd Root, ſo ſhall this new Surd be the Cube of that given; as if $\surd(6)64$ be to be multiplied into itſelf cubically, then $\surd(2)64$ or $\surd 64$ ſhall be the Cube ſought. Likewiſe the Cube of $\surd(9)512$ is $\surd(3)512$.

More Examples to excercife the precedent Rules of Multiplication in Simple Surd Numbers.

Multiply	$\surd 5$	$\surd(3)4$	$\surd(4)8$
by	$\surd 8$	$\surd(3)7$	$\surd(4)2$
Product	$\surd 40$	$\surd(3)28$	$\surd(4)16$ that is, 2.

Multiply	$\surd 32$	Multiply theſe three continually,	$\surd(3)50$
by	$\surd 32$		$\surd(3)50$
			$\surd(3)50$
Product	32		50

Multiply	$\surd 27$	12
by	6	$\surd(3)5$
Product	$6\surd 27$ or $\surd 972$	$12\surd(3)5$ or $\surd(3)8640$

Multiply	$18\surd 5$	$24\surd 6\frac{3}{8}$	$6\surd 7$
by	$4\surd 5$	$5\surd 6\frac{3}{8}$	$5\surd 3$
Product	360	765	$30\surd 21$

Multiply	$\surd 8$	that is,	$\surd(6)512$	$4\surd 5$
by	$\surd(3)4$		$\surd(6)16$	$4\surd 5$
Product	. . . .		$\surd(6)8192$	80

Multiply	$5\surd 8$	$12\surd(3)4$	$\surd(4)12$
by	4	$2\frac{1}{2}$	$\surd(4)12$
Product	$20\surd 8$	$30\surd(3)4$	$\surd 12$

More

More Examples to Exerciſe the precedent Rules of Multiplication in Simple Surd Quantities expreſt by Letters.

Multiply	$\sqrt{12a}$	$\sqrt{\frac{8}{3}ab}$
by	$\sqrt{3a}$	$\sqrt{\frac{3}{2}ac}$
Product	$\sqrt{36aa}$ or $6a$	$\sqrt{4aabc}$ or $2a\sqrt{bc}$

Multiply	$\sqrt{a}$	that is,	$\sqrt{(6)aaa}$
by	$\sqrt{(3)aa}$		$\sqrt{(6)aaaa}$
Product			$\sqrt{(6)a7}$

Multiply	$\sqrt{27aa}$	Multiply theſe three continually,	$\sqrt{(3)aa}$
by	$\sqrt{27aa}$		$\sqrt{(3)aa}$
			$\sqrt{(3)aa}$
Product	$27aa$		aa.

Multiply	$\sqrt{3bc}$	$5b$
by	2	$\sqrt{(2)2a}$
Product	$2\sqrt{3bc}$ or $\sqrt{12bc}$	$5b\sqrt{(3)2a}$ or $\sqrt{(3)250abbb}$

Multiply	$3a\sqrt{5}$	$7\sqrt{bc}$	$\frac{8}{3}a\sqrt{bc}$
by	$2b\sqrt{5}$	$4\sqrt{bc}$	$\frac{3}{4}b\sqrt{bc}$
Product	$30ab$	$28bc$	$2abbc$

Multiply	$5\sqrt{ab}$	$3a\sqrt{5}$	$\frac{2bc}{a}\sqrt{d}$.
by	$3\sqrt{ac}$	$2b\sqrt{6}$	$\frac{aa}{2c}\sqrt{d}$
Product	$15\sqrt{aabc}$	$6ab\sqrt{30}$	abd

The certainty of the firſt Rule of this fourth Section, (upon which all the reſt depend) for the Multiplication of two ſimple Surd Numbers of the ſame kind, may be demonſtrated in manner following: Firſt, let there be two ſquare Roots given to be multiplied, ſuppoſe $\sqrt{5}$ and $\sqrt{3}$, then (by the ſaid Rule) the Product of their Multiplication is $\sqrt{15}$; now we muſt prove that $\sqrt{15}$ is the true Product of $\sqrt{5}$ multiplied by $\sqrt{3}$.

Demonſtration.

By the Definition of Multiplication theſe are Proportionals, *viz.* . .	$1 . \sqrt{5} :: \sqrt{3} .$	Product.
Therefore their Squares ſhall be alſo Proportionals, (*per* 22 *Prop.* 6 *Elem Euclid.*) *viz.*	$1 . 5 :: 3 .$	Square of the Product.
But theſe are Proportionals, (*per* 19 *Prop.* 7 *Elem. Euclid*)	$1 . 5 :: 3 .$	15

Therefore from the two laſt Analogies 15 is equal to the Square of the Product, and conſequently $\sqrt{15}$ is the Product of $\sqrt{5}$ into $\sqrt{3}$; which was to be proved.

Likewiſe in Cubic Roots, if $\sqrt{(3)5}$ be to be multiplied by $\sqrt{(3)4}$, the Product (by the ſame Rule) is $\sqrt{(3)20}$. For,

By the Definition of Multiplication theſe are Proportionals, *viz.* . .	$1 . \sqrt{(3)5} :: \sqrt{(3)4} .$	Product.
Therefore their Cubes are alſo Proportionals, (*per Prop.* 37. *Elem.* 11. *Euclid.*) *viz.*	$1 . 5 :: 4 .$	Cube of the Product.
But as	$1 . 5 :: 4 .$	20

Therefore 20 is equal to the Cube of the Product, and conſequently the Cubic Root of 20, to wit, $\surd(3)20$, is the Product of $\surd(3)5$ multiplied by $\surd(3)4$; which was to be proved.

Moreover, (becauſe (by *Sect.* 11. *Chap.* 5.) if four Numbers be Proportionals, their fourth Powers, fifth Powers, *&c.* are alſo Proportionals, this Demonſtration may be extended to prove the certainty of the ſaid Rule for multiplying any two ſimple Surd Numbers of the ſame kind.

Sect. V. *Diviſion in ſimple Surd Quantities.*

As before in Multiplication, ſo here in Diviſion, if the given Surd Roots, to wit, the Dividend and Diviſor be not of the ſame kind, they muſt be reduced to a common Radical Sign by the preceding *Sect.* 3. Then,

1. Divide the Number or Quantity following the Radical Sign of the Dividend, by the Number or Quantity following the ſame Radical Sign of the Diviſor, without any regard to the Sign, and to the Quotient prefix the ſaid common Radical Sign; ſo this new Root ſhall be the Quotient ſought.

As for Example, to divide $\surd 15$ by $\surd 3$, I divide 15 by 3, and there ariſes 5, before which I prefix $\surd$, (the Radical Sign common to the given Surds) ſo $\surd 5$ is the Quotient ſought.

Likewiſe if $\surd 30$ by divided by $\surd 5$, the Quotient $\surd 6$.

Alſo $\surd\frac{3}{8}$ divided by $\surd\frac{3}{4}$ gives the Quotient $\surd\frac{1}{2}$.

And $\surd 5\frac{5}{6}$, or $\surd\frac{35}{6}$, divided by $2\frac{1}{3}$, or $\frac{7}{3}$, gives the Quotient $2\frac{1}{2}$.

Again, $\surd(3)20$ divided by $\surd(3)5$, gives the Quotient $\surd(3)4$; for 20 divided by 5 gives 4, before which ſetting $\surd(3)$ the Radical Sign belonging to each of the given Surds, there ariſes $\surd(3)4$ for the Quotient ſought.

Likewiſe $\surd(4)5$ divided by $\surd(4)\frac{5}{2}$, gives the Quotient $\surd(4)2$.

Moreover, if $\surd(6)4500$ be given to be divided by $\surd(2)5$, the Quotient will be $\surd(3)6$; for firſt, the given Roots being of different kinds are reduced to theſe, to wit, $\surd(6)4500$ and $\surd(6)125$; then by dividing $\surd(6)4500$ by $\surd(6)125$ there ariſes $\surd(6)36$, whoſe ſquare Root being extracted, (becauſe 36 is a ſquare Number, and the Index (6) an even Number) it gives $\surd(3)6$ for the Quotient ſought.

After the ſame manner Diviſion is performed in ſimple Surd Quantities expreſt by Letters. As to divide $\surd ab$ by $\surd a$, I divide ab by a and there ariſes b, then ſetting $\surd$ before b it gives $\surd b$ for the Quotient ſought, to wit, the Quotient that ariſes by dividing $\surd ab$ $\surd a$.

Alſo $\surd b$ divided by $\surd a$ gives the Quotient $\surd\frac{b}{a}$

Likewiſe $\surd abcd$ divided by $\surd ab$ gives the Quotient $\surd cd$.

Alſo $\surd\frac{3aad}{c}$ divided by $\surd\frac{2ab}{3c}$ gives the Quotient $\surd\frac{9ad}{2b}$

Again, to divide $\surd(6)dddaabb$ by $\surd(3)ab$, I firſt reduce them to $\surd(6)dddaabb$, and $\surd(6)aabb$, then I divide $\surd(6)dddaabb$ by $\surd(6)aabb$, and there ariſes $\surd(6)ddd$, that is, $\surd(2)d$ for the Quotient ſought.

2. When a Rational Number or Quantity is to be divided by its ſquare Root, that Root is the Quotient; as if 5 be divided by its ſquare Root, to wit, by $\surd 5$, the Quotient will be $\surd 5$. Alſo 8 divided by $\surd 8$ gives $\surd 8$ for the Quotient.

In like manner if the Quantity bc be divided by its ſquare Root, to wit, by $\surd bc$, the Quotient will be $\surd bc$. And $5a$ divided by $\surd 5a$ gives the Quotient $\surd 5a$.

3. When a Surd number or quantity is to be divided by a Rational number or quantity, or a rational number or quantity by a Surd, reduce the rational into the form of a Surd, (by *Sect.* 2. of this *Chap.*) and then divide according to the firſt rule of this *Sect.* 5.

As to divide $\surd 32$ by 2, I firſt reduce 2 to $\surd 4$; then by dividing $\surd 32$ by $\surd 4$ there ariſes $\surd 8$ for the Quotient.

Likewiſe $\surd 175$ divided by 5, that is $\surd 25$, gives rhe Quotient $\surd 7$.

Alſo 12, that is $\surd 144$, divided by $\surd 3$, gives the Quotient $\surd 48$.

Again, if $\surd(3)48$ be to be divided by 2, I firſt reduce 2 to $\surd(3)8$, then by dividing $\surd(3)48$ by $\surd(3)8$ there ariſes $\surd(3)6$ for the Quotient ſought. Alſo $\surd(4)5000$ divided by 5, (that is, by $\surd(4)625$) gives the Quotient $\surd(4)8$.

After the ſame manner to divide the quantity $\surd abb$ by b, I firſt reduce b to $\surd bb$; and then by dividing $\surd abb$ by $\surd bb$, there ariſes $\surd a$ the Quotient ſought. Again, $\surd 48aa$ divided by $4a$, that is by $\surd 16aa$, gives the Quotient $\surd 3$. Alſo $\surd(3)abbb$ divided by b, that is by $\surd(3)bbb$, gives the Quotient $\surd(3)a$.

Likewiſe to divide the Rational Quantity $\frac{bc}{a}$ by $\surd(3)bbcc$, I firſt reduce $\frac{bc}{a}$ to $\surd(3)\frac{bbbccc}{aaa}$, then I divide $\surd(3)\frac{bbbccc}{aaa}$ by $\surd(3)bbcc$, and there ariſes $\surd(3)\,\frac{bc}{aaa}$ or $\frac{\surd(3)bc}{a}$ the Quotient ſought.

4. When the Product of a Rational Number or Quantity multiplied into a Surd Number or Quantity is to be divided by the ſame Surd, the Quotient will be the ſaid multiplying Rational Number or Quautity. As $5\surd 3$ divided by $\surd 3$ gives the Quotient 5; alſo 20 $\surd(3)4$ gives the Quotient 20.

In like manner $5a\surd b$ divided by $\surd b$ gives the Quotient $5a$; and $4b\surd(3)12$ divided by $\surd(3)12$ gives the Quotient $4b$.

5. When the Dividend and Diviſor are the Products of two Rational Numbers or Quantities multiplied ſeverally into one common Surd, according to the fourth Rule of Multiplication in *Sect.* 4. (which Products are called Commenſurable Surd Roots, as hereafter will appear in *Sect.* 7. of this *Chap.*) divide the Rational part of the Dividend by the Rational part of the Diviſor, and that which ariſes ſhall be the Quotient ſought. As for Example, to divide $6\surd 3$ by $2\surd 3$, I divide 6 by 2, and there ariſes 3 the Quotient ſought; (for $2\surd 3$ multiplied by 3 produces $6\surd 3$.)

Again, $5\surd 6$ divided by $2\surd 6$ gives the Quotient $\frac{5}{2}$ or $2\frac{1}{2}$.

Alſo $2\surd 6$ divided by $5\surd 6$ gives the Quotient $\frac{2}{5}$, and $2\surd 5$ divided by $2\surd 5$ gives the Quotient 1.

So alſo $8\surd(3)7$ divided by $4\surd(3)7$ gives the Quotient 2; and $3\surd(4)5$ divided by $4\surd(4)5$ gives $\frac{3}{4}$ for the Quotient.

In like manner to divide $4a\surd 7$ by $2a\surd 7$, I divide $4a$ by $2a$, and there ariſes 2 the Quotient ſought; (for $2a\surd 7$ into 2 produces $4a\surd 7$: alſo $3\surd b$ divided by $5\surd 6$ gives the Quotient $\frac{3}{5}$, and $2\surd b$ divided by $2\surd 5$ gives the Quotient 1.

Again, $5a\surd 3b$ divided by $3a\surd 3b$ gives the Quotient $\frac{5}{3}$.

And $7ab\surd(3)dd$ divided by $3b\surd(3)dd$ gives the Quotient $\frac{7}{3}a$.

6. When the Dividend and Diviſor are the Products of two Rational Numbers or Quantities multiplied into two unequal Surd Numbers or Quantities, according to the fourth Rule of Multiplication in the preceding *Sect.* 4. (which Products are called Incommenſurable Surd Roots, as hereafter will appear;) divide the Rational part of the Dividend by the Rational part of the Diviſor, and the Surd part by the Surd part, then connect the Quotients ſo as the Rational Quotient may ſtand on the left hand, and this new Quantity ſhall be the Quotient ſought.

As for Example, if $4\surd 15$ be to be divided by $2\surd 5$, firſt I divide 4 by 2, and there ariſes 2; alſo I divide $\surd 15$ by $\surd 5$, and there ariſes $\surd 3$: then thoſe two Quotients joyned together make $2\surd 3$ (or $\surd 12$) the Quotient ſought.

In like manner $4\surd 12$ divided by $3\surd 2$ gives the Quotient $\frac{4}{3}\surd 6$; for 4 divided by 3 (to wit, the Rational by the Rational) gives $\frac{4}{3}$; and $\surd 12$ divided by $\surd 2$, (to wit, the Surd by the Surd) gives $\surd 6$: then by joyning together thoſe two Quotients there ariſes $\frac{4}{3}\surd 6$, or $1\frac{1}{3}\surd 6$, (or $\surd\frac{32}{3}$) for the Quotient ſought.

Again, $2\surd 7$ divided by $3\surd 5$ gives the Quotient $\frac{2}{3}\surd\frac{7}{5}$; and $2\surd 3$ divided by $2\surd 5$ gives the Quotient $1\surd\frac{3}{5}$ or $\surd\frac{3}{5}$.

Likewiſe to divide $4\surd(3)64$ by $2\surd(3)8$, I divide 4 by 2, and it gives 2: alſo $\surd(3)64$ divided by $\surd(3)8$ gives $\surd(3)8$; then thoſe two Quotients joyned together make $2\surd(3)8$, that is 4, the Quotient ſought. Moreover, $5\surd(3)20$ divided by $3\surd(3)4$ gives the Quotient $\frac{5}{3}\surd 35$.

After the ſame manner $4a\surd fb$ divided by $2a\surd f$ gives the Quotient $2\surd b$; for $4a$ divided by $2a$ gives 2, and $\surd fb$ divided by $\surd f$ gives $\surd b$; then connecting thoſe two Quotients there ariſes $2\surd b$ for the Quotient ſought.

So alſo $6ab\surd cd$ divided by $6a\surd df$ gives the Quotient $b\surd\frac{c}{f}$.

And $a\surd(3)cc$ divided by $b\surd(3)dd$, gives the Quotient $\frac{a}{b}\surd(3)\frac{cc}{dd}$

The Demonſtration of the aforeſaid firſt Rule of Diviſion (which is the riſe of all the reſt) may be formed like that of Multiplication in the preceding *Sect.* 4 if there be laid as a ground-work this Analogy, *viz.* As the Diviſor is to 1 (or Unity) ſo is the Dividend to the Quotient. But waving the Demonſtration, I ſhall give more Examples of Diviſion in ſimple Surds, both in Numbers and Quantities expreſt by Letters.

More Examples to exerciſe Diviſion in ſimple Surd Numbers.

Dividend	$\surd 117$	$\surd(3)16\frac{1}{3}$ or $\surd(3)\frac{49}{3}$	$\surd(4)256$
Diviſor	$\surd 6\frac{1}{2}$	$\surd(3)3\frac{1}{2}$ or $\surd(3)\frac{7}{2}$	$\surd(4)16$
Quotient	$\surd 18$	$\surd(3)4\frac{2}{3}$ or $\surd(3)\frac{14}{3}$	2

Dividend	$\surd(12)6125$	that is,	$\surd(12)6125$	$\surd(6)8192$
Diviſor	$\surd(4)5$		$\surd(12)125$	$\surd(2)8$
Quotient	. . . . . .		$\surd(12)49$ or $\surd(6)7$	$\surd(3)4$

Dividend	12	$5\surd 8$	$16\surd(3)25$
Diviſor	$\surd 12$	$\surd 8$	$\surd(3)25$
Quotient	$\surd 12$	5	16

Dividend	$\surd 245$	$\surd(3)686$	$\surd(5)23328$
Diviſor	$3\frac{1}{2}$	$3\frac{1}{2}$	6
Quotient	$\surd 20$	$\surd(3)16$	$\surd(5)3$

Dividend	$20\surd 14$	$\frac{2}{3}\surd 20$	$5\surd(3)3$
Diviſor	$2\surd 14$	$\frac{2}{15}\surd 20$	$2\surd(3)2$
Quotient	10	5	$\frac{5}{2}$ or $2\frac{1}{2}$

Dividend	$15\surd 18$	$3\surd 8$	$6\surd(3)24$
Diviſor	$3\surd 6$	$3\surd 3$	$9\surd(3)4$
Quotient	$5\surd 3$	$\surd\frac{8}{3}$	$\frac{2}{3}\surd(3)6$

More Examples to Exerciſe Diviſion in ſimple Surd Quantities expreſt by Letters.

Dividend	$\surd 15bc$	$\surd(3)4bbddd$	$\surd(4)32aa$
Diviſor	$\surd 3a$	$\surd(3)4bb$	$\surd(4)2aa$
Quotient	$5\frac{bc}{a}$	$\surd(3)ddd$ or d	$\surd(4)16$ or 2

Dividend	$\surd(6)675aaaaabbbbb$	that is,	$\surd(6)675a^5b^5$
Diviſor	$\surd(2)3ab$		$\surd(6)27a^3b^3$
Quotient	. . . .		$\surd(6)25aabb$ or $\surd(3)5ab$

Dividend	$\surd 80aaabbb$	$9bcdd$ (or $\surd 81bbccd^4$)
Diviſor	$4ab$ (or $\surd 16aabb$)	$\surd 27bcd$
Quotient	$\surd 5ab$	$\surd 3bcdd$

Dividend	bc	$b\surd df$	$2d\surd(3)bb$
Diviſor	$\surd bc$	$\surd df$	$\surd(3)bb$
Quotient	$\surd bc$	b	$2d$

Divi-

Dividend	$12\surd dc$	$\frac{2bc}{a}\surd d$	$ab\surd(3)f$
Diviſor	$3\surd dc$	$\frac{2c}{b}\surd d$	$b\surd(3)f$
Quotient	4	$\frac{bb}{a}$	a

Dividend	$2bc\surd d$	$b\surd af$	$6aa\surd(3)bbbd$
Diviſor	$c\surd a$	$c\surd f$	$2a\surd(3)d$
Quotient	$db\surd\frac{d}{a}$	$\frac{b}{c}\surd a$	$3ab$

Note, By the help of Diviſors Surd Quantities may oftentimes he reduced into others more ſimple, which being a very uſeful Work I ſhall explain it in the next Section.

Sect. VI. *How to reduce a Surd Quantity to another more ſimple, when it may be done.*

When the Power of a Surd Quantity, the Radical Sign being omitted, can be divided juſt without any Remainder, by a Power which has a Rational Root of the ſame kind with that which is denoted by the ſaid Radical Sign; then divide the Surd Quantity propoſed by that Rational Root, and prefix this Root before the Quotient; ſo you have a new Surd Quantity equal to that propoſed, and in more ſimple Terms.

As if $\surd 63$ be propoſed, becauſe 63 may be divided by the ſquare Number 9 without any Remainder, I divide $\surd 63$ by $\surd 9$, (that is, by 3) and it gives the Quotient $\surd 7$, before which I ſet the Rational Diviſor 3, and it makes $3\surd 7$, (that is, 3 into the ſquare Root of 7, or thrice the ſquare Root of 7) which is equal to $\surd 63$ firſt propoſed; (for the Quotient $\surd 7$ multiplied by the Diviſor 3 makes the Dividend $\surd 63$; ſo that inſtead of $\surd 63$ I write $3\surd 7$.

Likewiſe inſtead of $\surd 50$ we may write $5\surd 2$, (which ſignifies five times the ſquare Root of 2;) for in regard 50 divided by the Square 25 gives 2, I divide $\surd 50$ by $\surd 25$, that is, by 5, and the Quotient is $\surd 2$: and becauſe every Quotient multiplied by the Diviſor, produces the Dividend, therefore $5\surd 2$ ſhall be equal to the Dividend $\surd 50$.

After the ſame manner inſtead of $\frac{\surd 75}{2}$, or $\frac{75}{4}$, we may write $\frac{5}{2}\surd 3$; for $\frac{75}{4}$ divided by the ſquare Number $\frac{25}{4}$ gives the Quotient 3; and conſequently $\surd\frac{75}{4}$ divided by $\surd\frac{25}{4}$, that is, by $\frac{5}{2}$, gives the Quotient $\surd 3$: Therefore $\frac{5}{2}\surd 3$ ſhall be equal to $\frac{\surd 75}{2}$ or $\surd\frac{75}{4}$.

Again, inſtead of $\surd(3)40$ we may write $2\surd(3)5$, (which ſignifies twice the Cubic Root of 5;) for 40 divided by the Cube 8 gives the Quotient 5; and conſequently $\surd(3)40$ divided by $\surd(3)8$, that is, by 2, gives $\surd(3)5$; Therefore $2\surd(3)5$ ſhall be equal to $\surd(3)40$.

Likewiſe for $\surd(3)\frac{54}{8}$, (or $\frac{\surd(3)54)}{2}$ we may write $\frac{3}{2}\surd(3)2$; for $\frac{54}{8}$ divided by the Cube $\frac{27}{8}$ gives 2; and conſequently $\surd(3)\frac{54}{8}$ divided by $\surd(3)\frac{27}{8}$, that is, by $\frac{2}{3}$, will give $\surd(3)2$: Wherefore $\frac{3}{2}\surd(3)2$ ſhall be equal to $\surd(3)\frac{54}{8}$.

The like Operation is to be done in reducing Surd Quantities expreſt by Letters to others more ſimple: as if $\surd 75aa$ be propoſed, foraſmuch as $75aa$ divided by the Square $25aa$ gives the Quotient 3, and conſequently $\surd 75aa$ divided by $\surd 25aa$, that is, by $5a$, will give $\surd 3$; therefore the Diviſor $5a$ multiplied into the Quotient $\surd 3$, produces $5a\surd 3$, equal to the Dividend $\surd 75aa$, and therefore inſtead of $\surd 75aa$, we may write $5a\surd 3$.

After the ſame manner $\surd 10aabb$ may be reduced to $ab\surd 10$, alſo $\surd 5aa$ to $a\surd 5$, and $\surd(3)4ddd$ to $d\surd(3)4$.

Again, foraſmuch as $aaab+aabb$ may be divided by the Square aa, and there ariſes $ab+bb$, and conſequently $\surd:\overline{aaab+aabb}:$ divided by $\surd aa$, that is, by a, gives the Quotient $\surd:\overline{ab+bb}:$ therefore a into $\surd:\overline{ab+bb}:$ ſhall be equal to $\surd:\overline{aaab+aabb}:$ So that inſtead of $\surd:\overline{aaab+aabb}:$ we may write a into $\surd:\overline{ab+bb}:$ or $a\surd:\overline{ab+bb}:$

Likewiſe for $\surd:\overline{aabbc+2afbbc+ffbbc}:$ we may write $\overline{a+f}$ into $\surd bbc$, or $\overline{a+f}\surd bbc$; for $aabbc+2afbbc+ffbbc$ divided by the Square $aa+2af+ff$ gives bbc, and conſequently $\surd:\overline{aabbc+2afbbc+ffbbc}:$ divided by $\surd:\overline{aa+2af+ff}:$ that is, by $a+f$, gives the Quotient $\surd bbc$. Therefore $\overline{a+f}\surd bbc$ imports as much as $\surd:\overline{aabbc+2afbbc+gfbbc}:$

After the ſame manner inſtead of $\surd(3)\frac{27aaaabbb}{8b-8a}$ we may write $\frac{3ab}{2}$ into $\surd(3)\frac{a}{b-a}$ or $\frac{3ab}{2}\surd(3)\,\frac{a}{b-a}$; for ſince the Power of the Surd propoſed is produced by the Multiplication of $\frac{a}{b-a}$ into the Cube $\frac{27aaabbb}{8}$, whoſe Cubic Root is $\frac{3ab}{2}$, and conſequently $\surd(3)\frac{27aaaabbb}{8b-8a}$ divided by $\surd(3)\,\frac{27aaabbb}{8}$, that is, by $\frac{3ab}{2}$, gives the Quotient $\surd(3)\frac{a}{b-a}$. Therefore $\frac{3ab}{2}\surd(3)\frac{a}{b-a}$ ſhall be equal to $\surd(3)\frac{27aaaabbb}{8b-8a}$.

So alſo for $\surd:\frac{aaoomm+4aammmp}{ppzz}:$ we may write $\frac{am}{pz}\surd:\overline{oo+4mp}:$ for if the Power of the Surd propoſed be divided by the Square $\frac{aamm}{ppzz}$ the Quotient will be $oo+4mp$; and conſequently if the Surd propoſed be divided by $\surd:\frac{aamm}{ppzz}:$ that is by $\frac{am}{pz}$, the Quotient will be $\surd:\overline{oo+4mp}:$ therefore the Diviſor $\frac{am}{pz}$ multiplied into the Quotient $\surd:\overline{oo+4mp}:$ viz. $\frac{am}{pz}\surd:\overline{oo+4mp}:$ denotes as much as $\surd:\frac{aaoomm+4aammmp}{ppzz}$ the Surd propoſed.

Likewiſe for $\surd:\frac{oozz+4mpzz}{aa}:$ we may write $\frac{z}{a}\surd:\overline{oo+4mp}:$

But when a Square or Cube, *&c.* by which the Diviſion neceſſary to ſuch Contraction is to be performed, cannot be readily diſcerned, firſt, (by the Rules of the preceding eighth Chapter) ſearch out all the Diviſors of the Power of the Surd Quantity propoſed, and then ſee whether any of them be a Square or Cube, *&c.* to wit, ſuch a Power as the Radical Sign denotes, which if you find you may uſe in the aforeſaid manner to free the Surd Quantity in part from the Radical Sign.

As if $\surd 288$ be propoſed, becauſe among the Diviſors of 288 there are found the Square Numbers 4, 9, 16, 36, and 144, which dividing 288 will give the Quotients 72, 32, 18, 8 and 2; inſtead of $\surd 288$ we may write $2\surd 72$, or $3\surd 32$, or $4\surd 18$, or $6\surd 8$, or laſtly $12\surd 2$.

In like manner if $\surd:\overline{aaab+aabb}:$ be propoſed, becauſe among the Diviſors of the Quantity $aaab+aabb$, there is found the Square aa, the ſaid $\surd:\overline{aaab+aabb}:$ may be reduced to $a\surd:\overline{aa+bb}:$ as before.

Again, for as much as $a^3b-aabb+2aabc+abcc-ab^3+bbcc-2b^3c+b^4$ is produced by the Multiplication of $ab+bb$ into the Square $aa+2ac+cc-2ab-2bc+bb$, whoſe Root is $a+c-b$; we may inſtead of $\surd:\overline{a^3b-aabb+2aabc+abcc-ab^3+bbcc-2b^3c+b}:$ write $\overline{a+c-b}$ into $\surd:\overline{ab+bb}:$ or $\overline{a+c-b}\surd:\overline{ab+bb}:$

Likewiſe, becauſe among the Diviſors of $1200aabb$ there are found the Squares $4aabb$, $16aabb$, $25aabb$, $100aabb$, and $400aabb$; which dividing the ſaid $1200aabb$ will give the Quotients 300, 75, 48, 12, and 3, we may for $\surd 1200aabb$ write $2ab\surd 300$, or $4ab\surd 75$, or $5ab\surd 48$, or $10ab\surd 12$, or laſtly $20ab\surd 3$.

Sect. VII. *Two Surd Roots being given, to find whether they be Commenſurable or Incommenſurable.*

Commenſurable Surd Roots are ſuch whoſe Reaſon or Proportion to one another may be expreſt by Rational Numbers or Quantities; and thoſe Surd Roots whoſe Proportion cannot be expreſt by Rational Numbers or Quantities, are called Incommenſurable.

The Rule to try whether two Surd Roots of the ſame kind, (that is, ſuch as have a common Radical Sign) be Commenſurable or not, is this that follows, *viz.*

Divide the given Roots ſeverally by their greateſt common Diviſor, then if the Quotients be Rational Numbers or Quantities, the Roots propoſed are Commenſurable; but if the Quotient be Irrational or Surd, the given Roots are Incommenſurable.

As for Example, to try whether $\surd 12$ and $\surd 3$ be Commenſurable or not, I divide them ſeverally by their greateſt common Diviſor $\surd 3$, and find the Quotient $\surd 4$ and $\surd 1$, that is, 2 and 1 to be Rational Numbers; whence I conclude that $\surd 12$, that is $2\surd 3$, has ſuch proportion to $\surd 3$, that is $1\surd 3$, as 2 to 1, *viz.* as a Rational Number to a Rational Number; and conſequently $\surd 12$ and $\surd 3$ (according to the Definition above given) are Commenſurable. But that $\surd 12$ is to $\surd 3$ as 2 to 1, may be demonſtrated thus, *viz.* It is evident (by reaſon of the common Factor $\surd 3$) that $2\surd 3 \,.\, 1\surd 3 :: 2 \,.\, 1$, and (by Diviſion as above) $\surd 12 = 2\surd 3$, and $\surd 3 = 1\surd 3$; therefore $\surd 12 \,.\, \surd 3 :: 2 \,.\, 1$. Otherwiſe thus:

Foraſmuch as 12 and 3 divided ſeverally by their common Diviſor 3 give the Quotients 4 and 1; therefore as . . . $12 \,.\, 3 :: 4 \,.\, 1$

Wherefore the ſquare Roots of thoſe Proportionals ſhall be Proportionals alſo, (*per 22 Prop. 6. Elem. Euclid.*) *viz.* $\surd 12 \,.\, \surd 3 :: 2 \,.\, 1$

Which was to be demonſtrated.

After the ſame manner $\surd 18$ and $\surd 8$ will be found Commenſurable; for the former is to the latter as 3 to 2, to wit, as a Rational Number to a Rational Number: for if $\surd 18$ and $\surd 8$ be ſeverally divided by their greateſt common Diviſor $\surd 2$, the Quotients will be $\surd 9$ and $\surd 4$, that is 3 and 2. Therefore $\surd 18$ is to $\surd 8$ as 3 to 2, and inſtead of $\surd 18$ and $\surd 8$ we may write $3\surd 2$ and $2\surd 2$, to wit, the Products of the Rational Quantities 3 and 2, multiplied into the common Diviſor $\surd 3$.

Again, $\surd 48$ and $\surd 75$ (that is, $4\surd 3$ and $5\surd 3$) are Commenſurable; for the former is to the latter as 4 to 5, to wit, as a Rational Number to a Rational Number: for $\surd 48$ and $\surd 75$ being ſeverally divided by their greateſt common Diviſor $\surd 3$, give the Quotients $\surd 16$ and $\surd 25$, to wit, 4 and 5. Therefore $\surd 48 \,.\, \surd 75 :: 4 \,.\, 5 :: 4\surd 3 \,.\, 5\surd 3$.

Moreover, $\surd(3)320$ and $\surd(3)135$ (that is, $4\surd(3)5$) and $3\surd(3)5$) having ſuch proportion one to the other as 4 to 3 are Commenſurable; for $\surd(3)320$ and $\surd(3)135$ being ſeverally divided by their greateſt common Diviſor $\surd(3)5$, will give the Quotient $\surd(3)64$ and $\surd(3)27$, to wit, 4 and 3. Therefore $\surd(3)320 \,.\, \surd(3)135 :: 4 \,.\, 3 :: 4\surd(3)5 \,.\, 3\surd(3)5$.

So alſo $\surd(4)3888$ and $\surd(4)243$ (that is, $2\surd(4)243$ and $1\surd(4)243$) are Commenſurable, the former having ſuch proportion to the latter as 2 to 1; for if they be ſeverally divided by their greateſt Common Diviſor $\surd(4)243$, the Quotients will be $\surd(4\;16$ and $\surd(4)1$, to wit, 2 and 1. Therefore $\surd(4)3888 \,.\, \surd(4)243 :: 2 \,.\, 1 :: 2\surd(4)243 \,.\, 1\surd(4)243$.

If two Surd Fractions, or mix'd Numbers ſtanding Fraction-wiſe, be propoſed, and have not a common Denominator, reduce them to their ſmalleſt common Denominator, and then try (in like manner as before) whether the new Surd Numerators be Commenſurable or not; for if theſe be Commenſurable, the Surd Fractions firſt propoſed ſhall be alſo Commenſurable. As if $\surd\frac{2}{3}$ and $\surd\frac{24}{25}$ be propoſed, I reduce them to $\surd\frac{50}{75}$ and $\surd\frac{72}{75}$; then I divide the new Numerators only, to wit, $\surd 50$ and $\surd 72$, by their greateſt Common Diviſor $\surd 2$, and the Quotients $\surd 25$ and $\surd 36$, that is, 5 and 6 are Rational Numbers. Therefore $\surd\frac{2}{3}$ and $\surd\frac{24}{25}$ firſt propoſed are Commenſurable, and the former has ſuch proportion to the latter as 5 to 6. For,

As $\frac{50}{75} \,.\, \frac{72}{75} :: 50 \,.\, 72 :: 25 \,.\, 36$
Therefore $\surd\frac{50}{75} \,.\, \surd\frac{72}{75} :: \surd 50 \,.\, \surd 72 :: 5 \,.\, 6$
And becauſe $\surd\frac{2}{3} = \surd\frac{50}{75}$ and $\surd\frac{24}{25} = \surd\frac{72}{75}$
Therefore $\surd\frac{2}{3} \,.\, \surd\frac{24}{25} :: 5 \,.\, 6$

But if either the Numerators or Denominators of two Surd Fractions or mix'd Numbers ſtanding Fraction-wiſe, (the Radical Sign being neglected) be Squares or Cubes, &c, *viz.* Powers of that kind which is denoted by the Radical Sign, then you need not reduce the Surd Fractions to a common Denominator, but try whether their Numerators or Denominators be Commenſurable or not; for if theſe be Commenſurable, the

Surd Fractions propoſed ſhall be alſo Commenſurable. As if $\surd\frac{50}{16}$ and $\surd\frac{72}{25}$ be propoſed, becauſe the Denominators (the Radical Sign being neglected) are Squares, (to wit, Powers of that kind which the Radical Sign denotes) and the Numerators $\surd 50$ and $\surd 72$ are Commenſurable; (for if theſe be divided by their common Diviſor $\surd 2$, the Quotients are rational, to wit 5 and 6.) Therefore the Surd Fractions propoſed are alſo Commenſurable, and have ſuch proportion as $\frac{6}{4}$ to $\frac{4}{5}$, (whoſe Denominators 4 and 5, to wit, $\surd 16$ and $\surd 25$, are the given Denominators) or as 25 to 24; and (according to the preceding *Sect.* 6.) the Surd Fractions propoſed may be expreſs'd thus, $\frac{5}{4}\surd 2$ and $\frac{6}{5}\surd 2$.

When two Surd Roots propoſed be of different kinds, they muſt firſt of all be reduced to a common Radical Sign, (by the preceding *Sect.* 3. of this *Chap.*) before the Rules aforeſaid be uſed, to try whether they be Commenſurable or not. As if $\surd(6)64$ and $\surd(3)27$ be given, they may be reduced to $\surd(6)64$ and $\surd(6)729$, which divided by their greateſt common Diviſor $\surd(6)1$, the Quotient will be the ſame with the Dividends. Now if $\surd(6)64$ and $\surd(6)729$ be Rational, then the Surds firſt given are Commenſurable; but $\surd(6)64$ is 2, and $\surd(6)729$ is 3. Therefore the Surd Roots propoſed are Commenſurable, and have ſuch proportion as 2 to 3.

But if the Quotients ariſing by the Diviſion of two Surd Roots by their greateſt common Diviſor as aforeſaid, happen to be Irrational or Surd, then the Roots propoſed are Incommenſurable; ſuch are $\surd 48$ and $\surd 8$, for if they be divided ſeverally by their greateſt common Diviſor $\surd 8$, the Quotients are $\surd 6$ and 1: but $\surd 6$ is Irrational, therefore the proportion which $\surd 48$ has to $\surd 8$ is not as a Rational Number to a Rational Number, and conſequently $\surd 48$ and $\surd 8$ are Incommenſurable, and ſo are all other Surd Roots whoſe proportion cannot be expreſs'd by Rational Numbers.

I ſhall now ſhew how by the help of the preceding Rules we may diſcover whether two Surd Quantities expreſs'd by Letters be Commenſurable or not. As if $\surd 27aa$ and $\surd 12aa$ be propoſed, they will be found Commenſurable; for if they be ſeverally divided by their greateſt common Diviſor $\surd 3aa$, the Quotients $\surd 9$ and $\surd 4$, that is 3 and 2, are Rational Numbers, and ſhew that $\surd 27aa$ is to $\surd 12aa$ as 3 to 2, to wit, as a Rational Number to a Rational Number; wherefore $\surd 27aa$ and $\surd 12aa$ are Commenſurable, and may be expreſs'd thus, $3\surd 3aa$ and $2\surd 3aa$.

Note, If two Surd Quantities be divided by ſome common Diviſor, though it be not the greateſt, yet if there come forth Rational Quotients, we may thence conclude thoſe Surd Quantities to be Commenſurable, and oftentimes expreſs them various ways. As if $\surd 27aa$ and $\surd 12aa$ be again propoſed, by dividing them ſeverally by their common Diviſor $\surd 3$, there will come forth the Quotients $\surd 9aa$ and $\surd 4aa$, that is, $3a$ and $2a$; whence it is evident, that $\surd 27aa$ is to $\surd 12aa$ as $3a$ to $2a$, to wit, as a Rational Quantity to a Rational Quantity, and conſequently $\surd 27aa$ and $\surd 12aa$ are Commenſurable. Moreover, according to this latter Diviſion we may write $3a\surd 3$ for $\surd 27aa$, and $2a\surd 3$ for $\surd 12aa$.

Again, $\surd\overline{:aaaa+aabb:}$ and $\surd\overline{:aabb+bbbb:}$ are Commenſurable; for each of them being divided by $\surd\overline{:aa+bb:}$ there ariſe $\surd aa$ and $\surd bb$, that is a and b, which are Rational Quantities, each of which being multiplied into the common Diviſor $\surd\overline{:aa+bb:}$ will give, inſtead of the Surds propoſed, $a\surd\overline{aa+bb}$ and $b\surd\overline{aa+bb}$, which have the ſame proportion to one another as there is between a and b.

Likewiſe $\surd\frac{oozz+4mpzz}{aa}$ and $\surd\frac{aaoomm+4aammmp}{ppzz}$ are Commenſurable, for each of them being divided by their common Diviſor $\surd\overline{:oo+4mp:}$ there will ariſe $\surd\frac{zz}{aa}$ and $\surd\frac{aamm}{ppzz}$ that is, $\frac{z}{a}$ and $\frac{am}{pz}$, (to wit, Rational Quantities) each of which multiplied into the common Diviſor $\surd\overline{:oo+4mp:}$ will produce $\frac{z}{a}\surd\overline{:oo+4mp:}$ and $\frac{am}{pz}\surd\overline{:oo+4mp:}$ which are equal to, but more ſimply expreſs'd than the Surd Quantites propoſed, and have that proportion one to another as is between $\frac{z}{a}$ and $\frac{am}{pz}$.

So alſo $\surd\overline{:aaaa+6aaa+21aa+72a+108:}$ and $\surd\overline{:aaaa-10aaa+37aa-120a+300:}$ are Commenſurable, for if they be ſeverally divided by their common Diviſor $\surd\overline{:aa+12:}$ there will ariſe $\surd\overline{:aa+6a+9:}$ and $\surd\overline{:aa+10a+25:}$ that is, $a+3$ and $a \backsim 5$, each of which multiplied into the common Diviſor $\sqrt{aa+12}$ will produce $\overline{a+3\surd:aa+12}$: and $a \backsim 5\surd:aa+12$: which have the ſame proportion between themſelves, as that of $a+3$ to $a \backsim 5$, and are of the ſame value with the Surd Quantities firſt propoſed.

Again, $\surd(3)81abbb$ and $\surd(3)24abbb$ are Commenſurable, for if each of them be divided by their common Diviſor $\surd(3)3a$, there will ariſe $\surd(3)27bbb$ and $\surd(3)8bbb$, that is, $3b$ and $2b$; therefore the Surds propoſed may be reduced to $3b\surd(3)3a$ and $2b\surd(3)3a$, the former of which is to be the latter as $3b$ to $2b$: and ſo of others.

Sect. VIII. *Addition and Subtraction in ſimple Surd Quantities.*

When two or more equal Surd Roots are to be added together, multiply one of them by the Number which expreſſes the Multitude of the Roots propoſed, and the Product ſhall be their ſum: as the ſum of $\surd 6$ and $\surd 6$ is $\surd 24$; for $\surd 6$ multiplied by 2, that is by $\surd 4$, produces $\surd 24$. Alſo $\surd(3)6$, $\surd(3)6$, and $\surd(3)6$, added into one make $\surd(3)162$; for $\surd(3)6$ multiplied by 3, that is, by $\surd(3)27$, makes $\surd(3)162$.

But when two unequal Surd Roots of the ſame kind, that is, ſuch as have the ſame Radical Sign prefix'd before each of them, be to be added together; alſo when the leſſer is to be ſubtracted from the greater, obſerve this Rule, *viz.* Firſt, (by the preceding *Sect.* 7. of this *Chap.*) you muſt try whether they be Commenſurable or not; then if they be Commenſurable, that is, if after they have been ſeverally divided by their greateſt common Diviſor, the Quotients be Rational Quantities, multiply the ſum of thoſe Rational Quantities by the ſaid common Diviſor, and the Product ſhall be the ſum of the Surd Roots propoſed; but if the Difference of thoſe Rational Quotients be multiplied by the ſaid common Diviſor, the Product ſhall be the Difference of the Roots propoſed.

As for Example, if the Sum and Difference of $\surd 50$, and $\surd 8$ be deſired, firſt, I divide each of them by their greateſt common Diviſor $\surd 2$, and the Quotients are $\surd 25$ and $\surd 4$, that is 5 and 2, (which are Rational Numbers expreſſing the proportion of the given Roots one to the other;) whoſe ſum 7 multiplied by the common Diviſor $\surd 2$ produces $7\surd 2$, or if you pleaſe $\surd 98$, (for 7, to wit, $\surd 49$ into $\surd 2$, makes $\surd 98$;) which is the deſired ſum of the given Roots $\surd 50$ and $\surd 8$. And if $5-2$, that is 3, (the Difference of the Rational Quotients before found) be multiplied by the ſaid common Diviſor $\surd 2$, the Product will be $3\surd 2$, that is $\surd 18$; which is the deſired Difference of $\surd 50$ and $\surd 8$, the Roots firſt propoſed.

Likewiſe the ſum of $\surd(3)500$ and $\surd(3)108$ will be found $8\surd(3)4$, that is, $\surd(3)2048$; and their Difference $2\surd(3)4$, that is $\surd(3)32$, as will appear by the following Work, *viz* firſt, I divide each of the given Roots $\surd(3)500$ and $\surd(3)108$ by their greateſt common Diviſor $\surd(3)4$, and the Quotients are $\surd(3)125$ and $\surd(3)27$, that is 5 and 3; then by multiplying 8 (to wit $5+3$, the ſum of the Rational Quotients) by the common Diviſor $\surd(3)4$, the Product $8\surd(3)4$, that is, $\surd(3)2048$; (for 8, to wit, $\surd(3)512$ into $\surd(3)4$ makes $\surd(3)2048$) which is the ſum of $\surd(3)500$ and $\surd(3)108$, the Roots propoſed.

And by multiplying 2, (that is, $5-3$ the Difference of the Rational Quotients) by the ſaid common Diviſor $\surd(3)4$, the Product is $2\surd(3)4$, that is, $\surd(3)32$; (for 2, to wit, $\surd(3)8$ into $\surd(3)4$ makes $\surd(3)32$) which is the Difference of $\surd(3)500$ and $\surd(3)108$, the Roots propoſed.

Here follow Contractions of the Work in the two laſt preceding Examples, with others of like nature, to illuſtrate the Rule before given for the Addition and Subtraction of ſuch ſimple Surd Roots as are Commenſurable.

Example 1.

What is the Sum and Difference of . . . . . . $\surd 50$ and $\surd 8$?

The Operation.

$\surd 2)\ \surd 50\ (\surd 25$, that is, 5. Therefore $5\surd 2=\surd 50$.
$\surd 2)\ \surd 8\ (\surd 4$, that is, 2. Therefore $2\surd 2=\surd 8$.

The Sum, $7\surd 2=\surd 50+\surd 8$.
Or, $\surd 98=\surd 50+\surd 8$.

The Difference, $3\surd 2=\surd 50-\surd 8$.
Or, $\surd 18=\surd 50-\surd 8$.

Example 2.

What is the Sum and Difference of . . . . . $\surd(3)500$ and $\surd(3)108$?

The Operation.

I. $\surd(3)4)\ \surd(3)500\ (\surd(3)125$, that is, 5.
II. $\surd(3)4)\ \surd(3)108\ (\surd(3)\ 27$, that is, 3.
From Diviſion I. $5\surd(3)4=\surd(3)500$.
From Diviſion II. $3\surd(3)4=\surd(3)108$.
The Sum, $8\surd(3)4=\surd(3)500+\surd(3)108$.
Or, $\surd(3)2048=\surd(3)500+\surd(3)108$.
The Difference, $2\surd(3)4=\surd(3)500-\surd(3)108$.
Or, $\surd(3)32=\surd(3)500-\surd(3)108$.

Example 3.

What is the Sum and Difference of . . . . . . $\surd 147$ and $\surd 12$?

The Operation.

$\surd 3)\ \surd 147\ (\surd 49$, that is, 7. Therefore $7\surd 3=\surd 147$.
$\surd 3)\ \surd\ 12\ (\surd 4$, that is, 2. Therefore $2\surd 3=\surd\ 12$.
The Sum, $9\surd 3=\surd 147+\surd 12$.
Or, $\surd 243=\surd 147+\surd 12$.
The Difference, $5\surd 3=\surd 147-\surd 12$.
Or, $\surd 75=\surd 147-\surd 12$.

Example 4.

What is the Sum and Difference of . . . . $\surd(3)1715$ and $\surd(3)40$?

The Operation.

I. $\surd(3)5)\ \surd(3)1715\ (\surd(3)343$, that is, 7.
II. $\surd(3)5)\ \surd(3)\ 40\ (\surd(3)\ 8$, that is, 2.
From Diviſion I. $7\surd(3)5=\surd(3)1715$.
From Diviſion II. $2\surd(3)5=\surd(3)\ 40$.
The Sum, $9\surd(3)5=\surd(3)1715+\surd(3)40$.
Or, $\surd(3)3645=\surd(3)1715+\surd(3)40$.
The Difference, $5\surd(3)5=\surd(3)1715-\surd(3)40$.
Or, $\surd(3)625=\surd(3)1715-\surd(3)40$.

Note, When two Commenſurable Surd Roots propoſed to be added or ſubtracted are Fractions, or mix'd Numbers reduced into the form of Fractions, if they have not a common Denominator, reduce them into others which may have a common Denominator in the leaſt Terms, then to find out the Rational Quotients divide only the two new Numerators ſeverally by their greateſt Common Diviſor, and continue the Proceſs as before. The Practice of this Note will be evident in the two following Examples.

Example 5.

What is the Sum and Difference of . . . . $\left\{\begin{array}{l}\surd\frac{24}{25} \text{ and } \surd\frac{2}{3}. \\ \text{Or, } \surd\frac{72}{75} \text{ and } \surd\frac{50}{75}.\end{array}\right.$

The Operation.

$\surd\frac{2}{75})\ \surd\frac{72}{75}\ (\surd 36$, that is, 6. Therefore $6\surd\frac{2}{75}=\surd\frac{72}{75}$.
$\surd\frac{2}{75})\ \surd\frac{50}{75}\ (\surd 25$, that is, 5. Therefore $5\surd\frac{2}{75}=\surd\frac{50}{75}$.
The Sum, $11\surd\frac{2}{75}=\surd\frac{72}{75}+\surd\frac{50}{75}$.
Or, $\surd\frac{242}{75}=\surd\frac{72}{75}+\surd\frac{50}{75}$.
The Difference $\surd\frac{2}{75}=\surd\frac{72}{75}-\surd\frac{50}{75}$.

Example 6.

What is the Sum and Difference of . . . . $\left\{\begin{array}{l}\sqrt{12} \text{ and } \sqrt{\frac{27}{4}}\,? \\ \text{Or, } \sqrt{\frac{48}{4}} \text{ and } \sqrt{\frac{27}{4}}\,?\end{array}\right.$

The Operation.

$\sqrt{\frac{3}{4}})\ \sqrt{\frac{48}{4}}\ (\sqrt{16}$, that is, 4. Therefore $4\sqrt{\frac{3}{4}}=\sqrt{\frac{48}{4}}$.

$\sqrt{\frac{3}{4}})\ \sqrt{\frac{27}{4}}\ (\sqrt{9}$, that is, 3. Therefore $3\sqrt{\frac{3}{4}}=\sqrt{\frac{27}{4}}$.

The Sum, $7\sqrt{\frac{3}{4}}=\sqrt{\frac{48}{4}}+\sqrt{\frac{27}{4}}$.

Or, $\sqrt{\frac{147}{4}}=\sqrt{\frac{48}{4}}+\sqrt{\frac{27}{4}}$.

The Difference, $\sqrt{\frac{3}{4}}=\sqrt{\frac{48}{4}}-\sqrt{\frac{27}{4}}$.

When two ſimple Surd Roots given to be added or ſubtracted be Incommenſurable, neither their Sum nor their Difference can be expreſs'd by any ſimple Root, but they are to be added by +, and to be ſubtracted by —. As to add $\sqrt{5}$ and $\sqrt{3}$, I write $\sqrt{5}+\sqrt{3}$ for the Sum; but to ſubtract $\sqrt{3}$ from $\sqrt{5}$, I write $\sqrt{5}-\sqrt{3}$ for the Remainder. So alſo the Sum of $\surd(3)40$ and $\surd(3)12$ is $\surd(3)40+\surd(3)12$, and their Difference is $\surd(3)40-\surd(3)12$.

But Incommenſurable ſquare Roots may be added or ſubtracted by this following Rule, (which is deduced from *Prop.* 4. *&* 7. *lib.* 2. *Euclid.*)

To the Sum of the Squares of the given Surd ſquare Roots, add the double Product of the Multiplication of thoſe Roots one into another; ſo ſhall the ſquare Root of the Sum be the Sum of the Roots propoſed to be added. But if the ſaid double Product be ſubtracted from the ſaid ſum of the Squares, the ſquare Root of the Remainder ſhall be the Difference of the given Surd ſquare Roots. As if the Sum and Difference of $\sqrt{6}$ and $\sqrt{3}$ be deſired, their Sums ſhall be $\sqrt{\overline{:9+\sqrt{72}:}}$ and their Difference $\sqrt{\overline{:9-\sqrt{72}:}}$ for the Sum of the Squares of the given ſquare Roots $\sqrt{6}$ and $\sqrt{3}$ is 9, and the double Product of their Multiplication is $\sqrt{72}$, which I add to and ſubtract from 9; ſo the ſquare Root of the ſum, to wit, $\sqrt{\overline{:9+\sqrt{72}:}}$ is the Sum deſired; and the ſquare Root of the Remainder, to wit, $\sqrt{\overline{:9-\sqrt{72}:}}$ is the Difference.

After the ſame manner the Addition and Subtraction of ſimple Surd Quantities expreſs'd by Letters may be performed; as to add $\sqrt{75aa}$ and $\sqrt{27aa}$, firſt, (by the preceding *Sect.* 7.) I find them to be Commenſurable; for if $\sqrt{75aa}$ and $\sqrt{27aa}$ be ſeverally divided by their greateſt common Diviſor $\sqrt{3aa}$, the Quotients are $\sqrt{25}$ and $\sqrt{9}$, that is, 5 and 3, whoſe ſum 8 multiplied into the common Diviſor $\sqrt{3aa}$ makes $8\sqrt{3aa}$, (that is, $\sqrt{192aa}$) for the ſum of $\sqrt{75aa}$ and $\sqrt{27aa}$. But if the Difference of the ſame Rational Quotients 5 and 3, to wit 2, be multiplied into the ſaid common Diviſor $\sqrt{3aa}$, it makes $2\sqrt{3aa}$, (that is, $\sqrt{12aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$, the Roots firſt propoſed.

Or we may write $8a\sqrt{3}$ (inſtead of $8\sqrt{3aa}$) for the Sum, and $2a\sqrt{3}$ inſtead of $2\sqrt{3aa}$) for the Difference of $\sqrt{75aa}$ and $\sqrt{27aa}$ before propoſed; for theſe divided ſeverally by their common Diviſor $\sqrt{3}$, give Rational Quotients, to wit $\sqrt{25aa}$ and $\sqrt{9aa}$, that is, $5a$ and $3a$; whoſe Sum $8a$ multiplied into the common Diviſor $\sqrt{3}$, gives $8a\sqrt{3}$ for the Sum of $\sqrt{75aa}$ and $\sqrt{27aa}$; but if the Difference of the ſaid Rational Quotients $5a$ and $3a$, to wit, $2a$, be multiplied into the ſaid common Diviſor $\sqrt{3}$, the Product $2a\sqrt{3}$ is the Difference of the ſaid $\sqrt{75aa}$ and $\sqrt{27aa}$.

Again, to add $\surd(3)256aaa$ and $\surd(3)32aaa$, firſt, (by *Sect.* 7.) I find them to be Commenſurable, for if each of them be divided by their common Diviſor $\surd(3)4$, the Quotients are Rational, to wit, $\surd(3)64aaa$ and $\surd(3)8aaa$, that is, $4a$ and $2a$; theſe added together make $6a$, which multiplied into the common Diviſor $\surd(3)4$, makes $6a\surd(3)4$ (that is, $\surd(3)864aaa$) for the deſired Sum of $\surd(3)256aaa$ and $\surd(3)32aaa$; but if $2a$, the Difference of the ſame Rational Quotients $4a$ and $2a$, be multiplied into the ſaid common Diviſor $\surd(3)4$, the Product $2a\surd(3)4$, (that is, $\surd(3)32aaa$) ſhall be the Difference of $\surd(3)256aaa$ and $\surd(3)32aaa$ firſt propoſed.

More Examples of the Addition and Subtraction of Commenſurable ſimple Surd Quantities expreſs'd by Letters.

Example 1.

What is the Sum and Difference of . . . . . . $\surd 28aa$ and $\surd 7aa$?

The Operation.

I. $\surd 7)\ \surd 28aa\ (\surd 4aa$, that is, $2a$.

II. $\surd 7)\ \surd 7aa\ (\surd aa$, that is, a.

From Diviſion I. $2a\surd 7 = \surd 28aa$.

From Diviſion II. $a\surd 7 = \surd 7aa$.

The Sum, $3a\surd 7 = \surd 28aa + \surd 7aa$.

The Difference, $a\surd 7 = \surd 28aa - \surd 7aa$.

Example 2.

What is the Sum and Difference of . . . . . . $\surd 45aabc$ and $\surd 20aabc$?

The Operation.

I. $\surd 5bc)\ \surd 45aabc\ (\surd 9aa$, that is, $3a$.

II. $\surd 5bc)\ \surd 20aabc\ (\surd 4aa$, that is, $2a$.

From Diviſion I. $3a\surd 5bc = \surd 45aabc$.

From Diviſion II. $2a\surd 5bc = \surd 20aabc$.

The Sum, $5a\surd 5bc = \surd 45aabc + \surd 20aabc$.

The Difference, $a\surd 5bc = \surd 45aabc - \surd 20aabc$.

Example 3.

What is the Sum and Difference of . . . . $\surd(3)81abbb$ and $\surd(3)24abbb$?

The Operation.

I. $\surd(3)3a)\ \surd(3)81abbb\ (\surd(3)27bbb$, that is, $3b$.

II. $\surd(3)2a)\ \surd(3)24abbb\ (\surd(3)8bbb$, that is, $2b$.

From Diviſion I. $3b\surd(3)3a = \surd(3)81abbb$.

From Diviſion II. $2b\surd(3)3a = \surd(3)24abbb$.

The Sum, $5b\surd(3)3a = \surd(3)81abbb + \surd(3)24abbb$.

The Difference, $b\surd(3)3a = \surd(3)81abbb - \surd(3)24abbb$.

Example 4.

What is the Sum and Difference of . . . . $\left\{\begin{array}{l} \surd\frac{15}{4}aad \text{ and } \surd\frac{125}{48}aad, \\ \text{Or, } \surd\frac{180}{48}aad \text{ and } \surd\frac{125}{48}aad? \end{array}\right.$

The Operation.

I. $\surd\frac{5}{48}d)\ \surd\frac{180}{48}aad\ (\surd 36aa$, that is, $6a$.

II. $\surd\frac{5}{48}d)\ \surd\frac{125}{48}aad\ (\surd 25aa$, that is, $5a$.

From Diviſion I. $6a\surd\frac{5}{48}d = \surd\frac{180}{48}aad$.

From Diviſion II. $5a\surd\frac{5}{48}d = \surd\frac{125}{48}aad$.

The Sum, $11a\surd\frac{5}{48}d = \surd\frac{180}{48}aad + \surd\frac{125}{48}aad$.

The Difference, $a\surd\frac{15}{48}d = \surd\frac{180}{48}aad - \surd\frac{125}{48}aad$.

If two Surd Quantities expreſs'd by Letters be Incommenſurable, their ſum is given by +, and their Difference by —; as to add $\surd 5a$ and $\surd 3a$, I write $\surd 5a + \surd 3a$ for the Sum; and to ſubtract $\surd 3a$ from $\surd 5a$, I write $\surd 5a - \surd 3a$ for the Remainder or Difference.

Sect. IX. *Addition and Subtraction in Compound Surd Roots.*

The Arithmetic of Compound Surds depends upon the Rules of the Simple, and the Rules of + and — in Algebraical Addition, Subtraction, Multiplication, and Diviſion; but how thoſe Rules are applied to the Arithmetic of Compound Surds, I ſhall ſhew in this and the following tenth and eleventh Sections, by Examples both in Surd Numbers and Surd Quantities expreſs'd by Letters.

Examples of Addition and Subtraction in Commenſurable ſimple Surd Numbers connected to Rational Numbers by + or —, as alſo in Compound Surd Numbers compoſed of Commenſurable ſimple Surds.

To and from	$6+\sqrt{18}(3\sqrt{2})$	$\sqrt{192}(8\sqrt{3})+3$
Add and Subtr.	$4+\sqrt{8}(2\sqrt{2})$	$\sqrt{75}(5\sqrt{3})-3$
The Sum	$10+\sqrt{50}(5\sqrt{2})$	$\sqrt{507}(13\sqrt{3})+0$
Difference	$2-\sqrt{2}$	$\sqrt{27}(3\sqrt{3})+6$

To and from	$+\sqrt{242}(11\sqrt{2})-12$	$15-2\sqrt{2}(\sqrt{8})$
Add and Subtr.	$-\sqrt{50}(-5\sqrt{2})+8$	$7+\sqrt{2}$
Sum	$+\sqrt{72}(6\sqrt{2})-4$	$22-\sqrt{2}$
Difference	$+\sqrt{512}(16\sqrt{2})-20$	$8-3\sqrt{2}(\sqrt{18})$

To and from	$\sqrt{242}+\sqrt{192}$	that is,	$11\sqrt{2}+8\sqrt{3}$
Add and Subtr.	$\sqrt{50}+\sqrt{75}$		$5\sqrt{2}+5\sqrt{3}$
Sum	$\sqrt{512}+\sqrt{507}$	that is,	$16\sqrt{2}+13\sqrt{3}$
Difference	$\sqrt{72}+\sqrt{27}$		$6\sqrt{2}+3\sqrt{3}$

To and from	$\sqrt{320}-\sqrt{108}$	that is,	$8\sqrt{5}-6\sqrt{3}$
Add and Subtr.	$\sqrt{80}-\sqrt{27}$		$4\sqrt{5}-3\sqrt{3}$
Sum	$\sqrt{720}-\sqrt{243}$	that is,	$12\sqrt{5}-9\sqrt{3}$
Difference	$\sqrt{80}-\sqrt{27}$		$4\sqrt{5}-3\sqrt{3}$

To and from	$\sqrt{320}+\sqrt{108}$	that is,	$8\sqrt{5}+6\sqrt{3}$
Add and Subtr.	$\sqrt{80}-\sqrt{27}$		$4\sqrt{5}-3\sqrt{3}$
Sum	$\sqrt{720}+\sqrt{27}$	that is,	$12\sqrt{5}+3\sqrt{3}$
Difference	$\sqrt{80}+\sqrt{243}$		$4\sqrt{5}+9\sqrt{3}$

To and from	$\sqrt[3]{2058}+\sqrt[3]{54}$	that is,	$7\sqrt[3]{6}+3\sqrt[3]{2}$
Add and Subtr.	$\sqrt[3]{162}+\sqrt[3]{16}$		$3\sqrt[3]{6}+2\sqrt[3]{2}$
Sum	$\sqrt[3]{6000}+\sqrt[3]{250}$	that is,	$10\sqrt[3]{6}+5\sqrt[3]{2}$
Difference	$\sqrt[3]{384}-\sqrt[3]{2}$		$4\sqrt[3]{6}-\sqrt[3]{2}$

To and from	$\sqrt[4]{1875}+\sqrt[3]{250}$	that is,	$5\sqrt[4]{3}+5\sqrt[3]{2}$
Add and Subtr.	$\sqrt[4]{48}-\sqrt[3]{16}$		$2\sqrt[4]{3}-2\sqrt[3]{2}$
Sum	$\sqrt[4]{7203}+\sqrt[3]{54}$	that is,	$7\sqrt[4]{3}+3\sqrt[3]{2}$
Difference	$\sqrt[4]{243}+\sqrt[3]{686}$		$3\sqrt[4]{3}+7\sqrt[3]{2}$

EXPLICATION.

In the firſt Example the Rational Numbers 6 and 4 added together make 10, and their difference is 2; then foraſmuch as $\sqrt{18}$ and $\sqrt{8}$ (that is, $3\sqrt{2}$ and $2\sqrt{2}$) are Commenſurable, (for the former is to the latter as 3 to 2) their Sum is $\sqrt{50}$ (that is, $5\sqrt{2}$) and their Difference $\sqrt{2}$ (by *Sect.* 8.) Wherefore $10+\sqrt{50}(5\sqrt{2})$ is the Sum, and $2-\sqrt{2}$ the Difference of the two Binomials $6+\sqrt{18}$ and $4+\sqrt{8}$, propoſed in the firſt Example.

Likewiſe in the ſecond Example the two Commenſurable Surd Roots $\sqrt{192}$ and $\sqrt{75}$, (that is, $8\sqrt{3}$ and $5\sqrt{3}$) added into one ſimple Surd make $\sqrt{507}$, (that is, $13\sqrt{3}$) but their Difference is $\sqrt{27}$, (that is, $3\sqrt{3}$;) alſo $+3$ and -3 added together make 0, but -3 ſubtracted from $+3$ makes $+6$. Wherefore $\sqrt{507}$ (that is, $13\sqrt{3}$) is the Sum, and $\sqrt{27}$ (that is, $3\sqrt{3}$) $+6$ is the Difference of the Binomial $\sqrt{192}+3$, and the Reſidual $\sqrt{75}-3$ propoſed in the ſecond Example.

Again, in the third Example, where $-\sqrt{50}+8$ is propoſed to be added to $\sqrt{242}-12$, and alſo to be ſubtracted from the ſame; firſt, $-\sqrt{50}$ added to $+\sqrt{242}$ (that is, $5\sqrt{2}$ to $+11\sqrt{2}$) makes $+\sqrt{72}$ (that is, $6\sqrt{2}$;) but $-\sqrt{50}$ ſubtracted from $+\sqrt{242}$ (that is, $-5\sqrt{2}$ from $+11\sqrt{2}$) leaves the Remainder or Difference $+\sqrt{512}$, (that is, $16\sqrt{2}$; alſo $+8$ added to -12 makes -4, but $+8$ ſubtracted from -12 leaves the Remainder or Difference -20. Wherefore $\sqrt{72}$ (that is, $6\sqrt{2}$)-4 is the Sum, and $\sqrt{512}$ (that is, $16\sqrt{2}$)-20 is the Difference of the two Reſiduals propoſed in the third Example. The Operation in the reſt of the preceding Examples is after the ſame manner.

Examples of Addition and Subtraction in Compound and Surd Numbers, partly Commenſurable and partly Incommenſurable.

To and from	$\sqrt{27}(3\sqrt{3})+\sqrt{8}$	$\sqrt{10}+\sqrt{8}(2\sqrt{2})$
Add and Subtr.	$\sqrt{12}(2\sqrt{3})+\sqrt{5}$	$\sqrt{3}-\sqrt{2}$
The Sum,	$\sqrt{75}(5\sqrt{3})+\sqrt{8}+\sqrt{5}$	$\sqrt{10}+\sqrt{3}+\sqrt{2}$
Or,	$\sqrt{75}(5\sqrt{3})+\sqrt{:13+\sqrt{160}:}$	$\sqrt{:13+\sqrt{120}:}+\sqrt{2}$
The Difference,	$\sqrt{3}+\sqrt{8}-\sqrt{5}$	$\sqrt{10}-\sqrt{3}+\sqrt{18}(3\sqrt{2})$
Or,	$\sqrt{3}+\sqrt{:13-\sqrt{160}:}$	$\sqrt{:13-\sqrt{120}:}+\sqrt{18}(3\sqrt{2})$

To and from	$\sqrt{(3)}56+\sqrt{(3)}16$	$\sqrt{(4)}405-\sqrt{(3)}2$
Add and Subtr.	$\sqrt{(3)}\,7-\sqrt{(3)}12$	$\sqrt{(4)}\,80+\sqrt{(3)}5$
Sum	$3\sqrt{(3)}\,7+\sqrt{(3)}16-\sqrt{(3)}12$	$5\sqrt{(4)}\,5+\sqrt{(3)}5-\sqrt{(3)}2$
Difference	$\sqrt{(3)}\,7+\sqrt{(3)}16+\sqrt{(3)}12$	$\sqrt{(4)}\,5-\sqrt{(3)}5-\sqrt{(3)}2$

EXPLICATION.

In the firſt of the four laſt preceding Examples the Sum of the two Commenſurable Surd Roots $\sqrt{27}$ and $\sqrt{12}$ (that is, $3\sqrt{3}$ and $2\sqrt{3}$) is $\sqrt{75}$, (that is, $5\sqrt{3}$;) but their Difference is $\sqrt{3}$: and the Sum of the two Incommenſurable Roots $\sqrt{8}$ and $\sqrt{5}$ is $\sqrt{8}+\sqrt{5}$, or $\sqrt{:13+\sqrt{160}:}$ but their Difference is $\sqrt{8}-\sqrt{5}$, or $\sqrt{:13-\sqrt{160}:}$ (according to the Rule before given in *Sect.* 8. for adding and ſubtracting two Incommenſurable ſquare Roots. Therefore $5\sqrt{3}+\sqrt{8}+\sqrt{5}$, or $5\sqrt{3}+\sqrt{:13-\sqrt{160}:}$ is the Sum; and $\sqrt{3}+\sqrt{8}-\sqrt{5}$, or $\sqrt{3}+\sqrt{:13-\sqrt{160}:}$ is the Difference of the two Binomials $\sqrt{27}+\sqrt{8}$ and $\sqrt{12}+\sqrt{5}$, propoſed in the ſaid firſt Example.

Again, in the third of the ſaid four Examples, where $\sqrt{(3)}56+\sqrt{(3)}16$ and $\sqrt{(3)}7-\sqrt{(3)}12$ are propoſed to be added and ſubtracted; the Sum of the two Commenſurable Surd Cubic Roots $\sqrt{(3)}56$ and $\sqrt{(3)}7$ is $3\sqrt{(3)}7$, and their Difference is $\sqrt{(3)}7$; alſo the Sum of the two Incommenſurable Cubic Roots $\sqrt{(3)}16$ and $-\sqrt{(3)}12$ is $\sqrt{(3)}16-\sqrt{(3)}12$; but $-\sqrt{(3)}12$ ſubtracted from $\sqrt{(3)}16$ leaves $\sqrt{(3)}15+\sqrt{(3)}12$. Wherefore $3\sqrt{(3)}7+\sqrt{(3)}16-\sqrt{(3)}12$ is the Sum, and $\sqrt{(3)}7+\sqrt{(3)}16+\sqrt{(3)}12$ is the Difference of the ſaid Binomial and Reſidual propoſed in the third Example.

Examples of Addition and Subtraction in Compound Surd Quantities expreſs'd by Letters.

Example 1.

To and From	$\sqrt{75aa}+\sqrt{8bb}$	viz.	$5a\sqrt{3}+2b\sqrt{2}$
Add and Subtr.	$\sqrt{12aa}+\sqrt{2bb}$		$2a\sqrt{3}+b\sqrt{2}$
The Sum is			$7a\sqrt{3}+3b\sqrt{2}$
The Difference is			$3a\sqrt{3}+b\sqrt{2}$

EXPLICATION.

Firſt, (by *Sect.* 7.) I find that $\sqrt{75aa}$ and $\sqrt{12aa}$ are Commenſurable, and may be reduced to $5a\sqrt{3}$ and $2a\sqrt{3}$; likewiſe $\sqrt{8bb}$ and $\sqrt{2bb}$ are Commenſurable, and may be reduced to $2b\sqrt{2}$ and $b\sqrt{2}$: then the ſum of $5a\sqrt{3}$ and $2a\sqrt{3}$ is $7a\sqrt{3}$; alſo the Sum of $2b\sqrt{2}$ and $b\sqrt{2}$ is $3b\sqrt{2}$: therefore the Sum of the two Binomials propoſed in the Example is $7a\sqrt{3}+3b\sqrt{2}$. But by ſubtracting $2a\sqrt{3}$ from $5a\sqrt{3}$, the Remainder is $3a\sqrt{3}$; and by ſubtracting $b\sqrt{2}$ from $2b\sqrt{2}$ the Remainder is $b\sqrt{2}$. Therefore the Difference of the two Binomials propoſed is $3a\sqrt{3}+b\sqrt{2}$.

Example 2.

$$\begin{array}{ll}
\text{What is the Sum and Difference of this Binomial} & \left\{\begin{array}{l}\surd(3)1715a^3b^3+\surd(3)bcd,\\ \surd(3)\ 40a^3b^3-\surd(3)bcd\,?\end{array}\right.\\
\text{and Reſidual, } \ldots\ldots & \\
\text{Thoſe reduced give theſe, to wit, } \ldots\ldots & \left\{\begin{array}{r}7ab\surd(3)5+\surd(3)bcd,\\ 2ab\surd(3)5-\surd(3)bcd.\end{array}\right.\\
\quad\text{The Sum, } \ldots\ldots & 9ab\surd(3)5\\
\quad\text{The Difference, } \ldots\ldots & 5ab\surd(3)5+2\surd(3)bcd.
\end{array}$$

Examples of Addition and Subtraction in Compound Surd Numbers altogether Incommenſurable.

$$\begin{array}{rl}
\text{To and from} & \surd 10+\surd 7\\
\text{Add and Subtr.} & \surd\ 3+\surd 2\\
\hline
\text{Sum,} & \surd 10+\surd 7+\surd 3+\surd 2\\
\text{Or,} & \surd{:}17+\surd 280{:}+\surd{:}5+\surd 24{:}\\
\hline
\text{Difference,} & \surd 10+\surd 7-\surd 3-\surd 2\\
\text{Or,} & \surd{:}17+\surd 280{:}-\surd{:}5+\surd 24{:}
\end{array}$$

$$\begin{array}{rl}
\text{To and from} & \surd(3)10+\surd(3)7\\
\text{Add and Subtr.} & \surd(3)\ 3-\surd(3)2\\
\hline
\text{Sum,} & \surd(3)10+\surd(3)7+\surd(3)3-\surd(3)2\\
\text{Difference,} & \surd(3)10+\surd(3)7-\surd(3)3+\surd(3)2.
\end{array}$$

Sect. X. *Of Multiplication in Compound Surds.*

Example 1.

$$\begin{array}{ll}
\begin{array}{l}\text{Multiplicand,}\\ \text{Multiplicator,}\end{array} \quad \left.\begin{array}{l}\surd 180+\surd 48\\ \surd 125+\surd 12\end{array}\right\}\ \text{that is,} & \left\{\begin{array}{l}6\surd 5+4\surd 3\\ 5\surd 5+2\surd 3\end{array}\right.\\
\hline
 & 150+20\surd 15\\
 & \qquad\quad +12\surd 15+24\\
\hline
\text{Product,} & 150+32\surd 15+24\\
\text{That is,} & 174+32\surd 15.
\end{array}$$

Example 2.

$$\begin{array}{ll}
\begin{array}{l}\text{Multiplicand,}\\ \text{Multiplicator,}\end{array} \quad \left.\begin{array}{l}6-\surd 20\\ 8-\surd 45\end{array}\right\}\ \text{that is,} & \left\{\begin{array}{l}6-2\surd 5\\ 8-3\surd 5\end{array}\right.\\
\hline
 & 48-16\surd 5\\
 & \qquad -18\surd 5+30\\
\hline
\text{Product,} & 78-34\surd 5
\end{array}$$

Example 3.

$$\begin{array}{ll}
\begin{array}{l}\text{Multiplicand}\\ \text{Multiplicator,}\end{array} \quad \left.\begin{array}{l}\surd 18-\ 3\\ \surd\ 8+\ 2\end{array}\right\}\ \text{that is,} & \left\{\begin{array}{l}3\surd 2-3\\ 2\surd 2+2\end{array}\right.\\
\hline
 & 12-6\surd 2\\
 & \qquad +6\surd 2-6\\
\hline
\text{Product,} & 12 \qquad\qquad -6\\
\text{That is,} & 6.
\end{array}$$

Example 4.

$$\begin{array}{ll}
\begin{array}{l}\text{Multiplicand,}\\ \text{Multiplicator,}\end{array} \quad \left.\begin{array}{l}4\surd 5+3\surd 5\\ 4\surd 5+3\surd 5\end{array}\right\}\ \text{that is,} & \left\{\begin{array}{l}7\surd 5\\ 7\surd 5\end{array}\right.\\
\hline
\text{Product,} & 245
\end{array}$$

EXPLICATION.

In the firſt Example, the two Compound Surd Numbers propos'd to be multiplied are $\surd 180+\surd 48$ and $\surd 125+\surd 12$, which are reduced to $6\surd 5+4\surd 3$ and $5\surd 5+2\surd 3$; (by *Sect.* 6. of this *Chap.*) then $6\surd 5$ multiplied by $5\surd 5$, (according to Rule 5. in *Sect.* 4. of this *Chap.*) produces 150; alſo $4\surd 3$ multiplied by $5\surd 5$ (according to Rule 6. in *Sect.* 4.) produces $20\surd 15$; again, $6\surd 5$ into $2\surd 3$ makes $12\surd 15$, and $4\surd 3$ into $2\surd 3$ produces 24; laſtly, thoſe Products added together make $174+32\surd 15$, the Product ſought. The reſt of the Examples are wrought in like manner.

When the Multiplicand has not the ſame Radical Sign with the Multiplier, they muſt firſt be reduced to the ſame Radical Sign, (by *Sect.* 3. of this *Chap*) and then the Multiplication is to be made by ſome of the Rules in *Sect.* 4 as will be manifeſt in the following Example.

Multiplicand,	$\surd(5)6+\surd(3)7+5$
Multiplicator,	$\surd 3$
Product,	$\surd(10)8748+\surd(6)1323+5\surd 3$.

EXPLICATION.

1. $\surd(5)6$ and $\surd 3$ are reduced to theſe having a common Radical Sign, to wit, $\surd(10)36$ and $\surd(10)243$, which multiplied one into the other produce $\surd(10)8748$.

2. $\surd(3)7$ and $\surd 3$ are reduced to $\surd(6)49$ and $\surd(6)27$, which multiplied one by the other produce $\surd(6)1323$.

3. The Rational Number 5 multiplied into $\surd 3$ makes $5\surd 3$ or $\surd 75$.

Laſtly, thoſe three ſimple Products added together give the Product ſought, to wit, $\surd(10)8748+\surd(6)1323+5\surd 3(\surd 75.)$

Three Compendious Rules, very uſeful in the Multiplication of Binomials and Reſiduals.

1. Becauſe $a+e$ multiplied by $a+e$ produces $aa+2ae+ee$, it is evident that the ſum of the Squares of the Parts (or Names) or any Binomial, together with twice the Product of the Parts multiplied one into the other is equal to the Square of the Sum of the Parts. Therefore to multiply any Binomial by itſelf (or to ſquare it) take the Squares of the Parts, and twice the Product of the Parts for the Square ſought.

2. Becauſe $a-e$ multiplied by $a-e$ produces $aa-2ae+ee$, it is manifeſt that the ſum of the ſquares of the Parts of any Reſidual, leſs by the double Product of the Parts, is equal to the ſquare of the difference of the Parts. Therefore to ſquare any Reſidual from the Sum of the Squares of the Parts ſubtract twice the Product of the Parts, and take the remainder for the Square ſought.

3. Becauſe $a+e$ multiplied by $a-e$ produces $aa-ee$, it is evident that the difference of the Square of the Parts of any Binomial, is equal to the Product made by the Multiplication of the Sum of the Parts into their difference. Therefore if a Binomial be to be multiplied by its correſpondent Reſidual, that is, by the difference of the Parts of the Binomial, take the difference of the Squares of the Parts for the Product ſought. Theſe three Rules will be exerciſed by the ſix Examples next following, and by divers other Examples in this and the following Sections of this Chapter.

Multiplicand,	$3+\surd 5$	$3-\surd 5$
Multiplicator,	$3+\surd 5$	$3-\surd 5$
Product	$9+6\surd 5+5$	$9-6\surd 5+5$
That is,	$14+6\surd 5$	$14-6\surd 5$

Multiplicand,	$3+\surd 5$	$\surd(3)\,27+\surd(3)\,8$
Multiplicator,	$3+\surd 5$	$\surd(3)\,27-\surd(3)\,8$
Product,	$9-5$	$\surd(3)729-\surd(3)64$
That is,	4	5

Multiplicand,	$\surd(6)7-\surd(6)5$	$\surd(10)7+\surd(10)3$
Multiplicator,	$\surd(6)7+\surd(6)5$	$\surd(10)7-\surd(10)3$
Product,	$\surd(3)7-\surd(3)5$	$\surd(5)7-\surd(5)3$

EX-

EXPLICATION.

In the firſt of the ſix laſt Examples the Binomial $3+\surd 5$ multiplied into it ſelf, or ſquared, produces $14+6\surd 5$; for the Squares of the Parts 3 and $\surd 5$ are 9 and 5, and twice the Product of 3 into $\surd 5$ makes $6\surd 5$, to wit $\surd 180$; therefore (by the firſt of the three preceding Rules) $9+5+6\surd 5$, that is, $14+6\surd 5$ is the Square of the given Binomial $3+\surd 5$.

In the ſecond Example the Reſidual $3-\surd 5$ ſquared or multiplied by itſelf produces $14-6\surd 5$, (by the ſecond of the ſaid three Rules.)

In the third Example the Binomial $3+\surd 5$ multiplied by its correſpondent Reſidual $3-\surd 5$ produces 4, which (by the laſt of the ſaid three Rules) is equal to the difference of the Squares of the Parts 3 and $\surd 5$.

Likewiſe in the fourth Example the Binomial $\surd(3)27+\surd(3)8$ multiplied by its correſpondent Reſidual $\surd(3)27-\surd(3)8$ produces $\surd(3)729-\surd(3)64$, to wit, the difference of the Squares of the Parts of the given Binomial or Reſidual.

And in the fifth Example the Reſidual $\surd(6)7-\surd(6)5$, multiplied by its correſpondent Binomial $\surd(6)7+\surd(6)5$, produces $\surd(3)7-\surd(3)5$; which is equal to the difference of the Squares of the parts of the given Reſidual or Binomial. For (by the ſeventh Rule in *Sect.* 4. of this *Chap.*) the Square of $\surd(6)7$ is $\surd(3)7$, and the Square of $\surd(6)5$ is $\surd(3)5$.

Examples of Multiplication in Compound Surd Quantities expreſt by Letters.

$$\left.\begin{array}{ll} \text{Multiplicand} & \surd abb+\surd cff \\ \text{Multiplicator,} & \surd add+\surd caa \end{array}\right\} \text{ that is, } \left\{\begin{array}{l} b\surd a+f\surd c \\ d\surd a+a\surd c \end{array}\right.$$

$$\begin{array}{rl} & bda+fd\surd ca \\ & \quad +ba\surd ca+fac \\ \hline \text{Product,} & bda+\overline{fd+ba}\times\surd ca+fac. \end{array}$$

$$\begin{array}{ll|l} \text{Multiplicand,} & 2a+3a\surd d & \surd bc+a \\ \text{Multiplicator,} & 3c-2c\surd d & \surd bc-a \\ \hline & 6ac+9ac\surd d & bc+a\surd bc \\ & \quad -4ac\surd d-6acd & \quad -a\surd bc-aa \\ \hline \text{Product,} & 6ac+5ac\surd d-6acd & bc \qquad -aa \end{array}$$

$$\begin{array}{ll|l} \text{Multiplicand,} & a+\surd b & \surd ab+\surd c \\ \text{Multiplicator,} & a+\surd b & \surd ac+\surd d \\ \hline \text{Product,} & aa+2a\surd b+b & a\surd bc+c\surd a+\surd abd+\surd cd \end{array}$$

$$\left.\begin{array}{ll} \text{Multiplicand,} & 3bb\surd d+d\surd d \\ \text{Multiplicator,} & 3bb\surd d+d\surd d \end{array}\right\} \text{ that is, } \left\{\begin{array}{l} \overline{3bb+d}\times\surd d \\ \overline{3bb+d}\times\surd d \end{array}\right.$$

$$\text{Product,} \quad 9bbbbd+6bbdd+ddd \quad \text{or} \quad \overline{9bbbb+6bbd+dd}\times d.$$

The Operation in theſe ſix laſt Examples will be familiar to him that underſtands the Rules and Examples before delivered concerning the Multiplication of Surd Numbers and Quantities expreſt by Letters.

Sect. XI. *Diviſion in Compound Surds.*

Examples of Diviſion where the Dividend is a Compound Quantity, and the Diviſor a Simple Quantity.

$$\begin{array}{ll|l} \text{Dividend,} & \surd 21+\surd 15 & \surd(3)14-\surd(3)28 \\ \text{Diviſor,} & \surd\ 3 & \surd(3)\ 7 \\ \hline \text{Quotient,} & \surd\ 7+\surd\ 5 & \surd(3)\ 2-\surd(3)\ 4 \end{array}$$

Dividend,	$12\surd 6+6\surd 18-2\surd 12$	$\surd 20-\surd(3)10$
Diviſor	$3\surd 6$	3
Quotient,	$4+2\surd 3-\frac{2}{3}\surd 2$	$\surd\frac{20}{9}-\surd(3)\frac{10}{27}$
Dividend,	$\surd(4)8+\surd(5)3$	$\surd(4)23328-\surd(4)10368$
Diviſor,	$\surd 2$	6
Quotient,	$\surd(4)2+\surd(10)\frac{9}{32}$	$\surd(4)18-\surd(4)8$

EXPLICATION.

The firſt Example is wrought according to Rule 1. in *Sect.* 5. of this *Chap.* For firſt, $\surd 21$ divided by $\surd 3$ gives the Quotient $\surd 7$, then $\surd 15$ divided by $\surd 3$ gives the Quotient $\surd 5$. Therefore $\surd 21+\surd 15$ divided by $\surd 3$ gives $\surd 7+\surd 5$, the Quotient ſought in the firſt Example.

The ſecond Example is wrought like the firſt; for $\surd(3)14$ divided by $\surd(3)7$ gives $\surd(3)2$, and $-\surd(3)28$ divided by $\surd(3)7$ gives $-\surd(3)4$. Therefore $\surd(3)14-\surd(3)28$ divided by $\surd(3)7$, gives $\surd(3)2-\surd(3)4$, the Quotient ſought in the ſecond Example.

The third Example is wrought according to the fifth and ſixth Rules of *Sect.* 5. of this *Chap.* For firſt, $12\surd 6$ divided by $3\surd 6$ give the Quotient 4, (by the ſaid fifth Rule;) then $6\surd 18$ divided by $3\surd 6$ gives $2\surd 3$, (by the ſaid ſixth Rule;) likewiſe $-2\surd 12$ divided by $3\surd 6$ gives $-\frac{2}{3}\surd 2$; (for 2 divided by 3 gives $\frac{2}{3}$, and $\surd 12$ divided by $\surd 6$ gives $\surd 2$.) Therefore $12\surd 6+6\surd 18-2\surd 12$ divided by $3\surd 6$ gives $4+2\surd 3-\frac{2}{3}\surd 2$, the Quotient ſought in the third Example.

In the fourth Example $\surd 20$ divided by $\surd 3$, (that is, by $\surd 9$) gives $\surd\frac{20}{9}$, or $\surd 2\frac{2}{9}$; and $-\surd(3)10$ divided by 3, (that is, by $\surd(3)27$) gives $-\surd(3)\frac{10}{27}$.

In the fifth Example $\surd(4)8$ and $\surd 2$ are firſt reduced to $\surd(4)8$ and $\surd(4)4$; then $\surd(4)8$ divided by $\surd(4)4$ gives $\surd(4)2$; likewiſe $\surd(5)3$ and $\surd 2$ are reduced to $\surd(10)9$ and $\surd(10)32$; then $\surd(10)9$ divided by $\surd(10)32$ gives the Quotient $\surd(10)\frac{9}{32}$. Therefore $\surd(4)8+\surd(5)3$ divided by $\surd 2$, gives $\surd(4)2+\surd(10)\frac{9}{32}$, the Quotient ſought in the fifth Example. The ſixth Example is wrought in like manner, and the Proof in theſe or the like Examples of Diviſion may be made by Multiplication.

Propoſitions concerning Diviſion in Surd Quantities, when the Diviſor is a Binomial or Trinomial, &c.

When the Diviſor is a Binomial or Reſidual conſiſting of two Square Roots or Biquadratic Roots, or of one Square Root or Biquadratic Root, and of a Rational Number; as alſo when the Diviſor is a Trinomial or Quadrinomial, and none of its Radical Signs exceeds that of the Square Root, the work of Diviſion in thoſe caſes is grounded upon theſe five following Propoſitions, *viz.*

1. If a Binomial conſiſting of two ſimple ſquare Roots connected by +, be multiplied by its correſpondent Reſidual, that is, by the difference of thoſe Roots; or if a Reſidual conſiſting of two ſimple ſquare Roots connected by —, be multiplied by its correſpondent Binomial, that is, by the Sum of the ſame Roots, the Product will be entirely Rational. So the Binomial $\surd 5+\surd 3$ multiplied by $\surd 5-\surd 3$, (or the Reſidual $\surd 5-\surd 3$ by $\surd 5+\surd 3$) gives the Rational Product 2, (by the laſt of the three Rules before delivered in *Sect.* 10. of this *Chap.*)

Likewiſe $\surd a+\surd b$ multiplied by $\surd a-\surd b$ gives the Rational Product $a-b$.

2. If a Binomial conſiſting of two Biquadratic ſimple Roots connected by +, be multiplied by its correſpondent Reſidual, to wit, by the difference of thoſe Roots the Product will be alſo a Reſidual conſiſting of two ſquare Roots connected by —, and if this Reſidual be multiplied by the ſum of its Names (or Parts,) it will give a Product entirely Rational.

As for Example, the Binomial $\surd(4)5+\surd(4)3$ multiplied by $\surd(4)5-\surd(4)3$ makes $\surd 5-\surd 3$, which multiplied by $\surd 5+\surd 3$ gives the Rational Product 2.

Likewiſe $\surd(4)81-2$ or $\surd(4)81-\surd(4)16$ multiplied by $\surd(4)81+\surd(4)16$ makes $\surd 81-\surd 16$, which multiplied by $\surd 81+\surd 16$ gives the Rational Product 65.

3. If a Trinomial conſiſting of three ſimple ſquare Roots connected by +, or by + and −, be multiplied by the ſame Trinomial, after any one Sign + is changed into −, or any one Sign − into +, the Product will conſiſt of two Names (or Parts;) and then if this Product be multiplied by its correſpondent Binomial or Reſidual, (according to the preceding *Prop.* 1.) the laſt Product will be entirely Rational.

As for Example, the Trinomial $\sqrt{5}+\sqrt{3}+\sqrt{2}$ multiplied by $\sqrt{5}+\sqrt{3}-\sqrt{2}$ gives $2\sqrt{15}+6$, and this multiplied by $2\sqrt{15}-6$ gives the Rational Product 24.

Likewiſe $\sqrt{30}-\sqrt{5}-\sqrt{3}$ multiplied by $\sqrt{30}+\sqrt{5}-\sqrt{3}$ produces $28-2\sqrt{90}$, and this multiplied by $28+2\sqrt{90}$ gives the Rational Product 424.

After the ſame manner $\sqrt{a}+\sqrt{b}-\sqrt{c}$ multiplied by $\sqrt{a}+\sqrt{b}+\sqrt{c}$ gives the Product $2\sqrt{ab}+a+b-c$, whoſe Rational Part $a+b-c$ we may ſuppoſe to be equal to ſome ſingle Quantity d, and then the ſaid Product will be a Binomial $2\sqrt{ab}+d$, which multiplied by its correſpondent Reſidual $2\sqrt{ab} \backsim d$ gives a Product entirely Rational, to wit, $4ab \backsim dd$. And ſo of other Trinomials that are qualified as before is ſuppoſed.

4. If a Quadrinomial conſiſting of four ſimple ſquare Roots connected by +, or by + and −, be multiplied by the ſame Quadrinomial after two Signs + are changed into −, or two Signs − into +, the Product will conſiſt of three Names (or Parts; (then if this Product be multiplied by its correſpondent Trinomial (according to *Prop.* 3.) there will come forth a Binomial or Reſidual. And laſtly, this Binomial or Reſidual multiplied by its correſpondent Reſidual or Binomial will give a Rational Product.

As for Example, the Quadrinomial $\sqrt{6}+\sqrt{5}+\sqrt{3}+\sqrt{2}$ multiplied by $\sqrt{6}+\sqrt{5}-\sqrt{3}-\sqrt{2}$, produces the Trinomial $6+2\sqrt{30}-2\sqrt{6}$; which multiplied by its correſpondent Trinomial $6+2\sqrt{30}+2\sqrt{6}$, (according to the precedent *Prop.* 3.) gives the Binomial $132+24\sqrt{30}$; and this multiplied by its correſpondent Reſidual $132-24\sqrt{30}$, gives the Rational Product 144.

After the ſame manner the Quadrinomial $\sqrt{a}+\sqrt{b}+\sqrt{c}-\sqrt{d}$ multiplied by $\sqrt{a}-\sqrt{b}-\sqrt{c}-\sqrt{d}$ gives the Product $a+d-b-c-2\sqrt{ad}-2\sqrt{bc}$, whoſe Rational Part $a+d-b-c$ we may ſuppoſe to be equal to ſome ſingle Quantity f, and then the ſaid Product will be a Trimonial, to wit, $f-2\sqrt{ad}-2\sqrt{bc}$; this multiplied by it ſelf after one of its Signs − is changed into + (according to *Prop.* 3.) will produce a Reſidual of two Names (or Parts,) and this Reſidual multiplied by its correſpondent Binomial will give a Rational Product.

5 If two Numbers be given for a Dividend and Diviſor, and each be multiplied by ſome Number, the firſt Product divided by the later will give the ſame Quotient that ariſes by dividing the given Dividend by the given Diviſor As if 6 be to be divided by 2, if you multiply each by 4, and divide the firſt Product 24 by the later 8, the Quotient 3 is the ſame that ariſes by dividing 6 by 2. For (by 17 *Prop* 7. *Elem. Euclid*) if a Number a multiplying two numbers b, c, produce two other Numbers ab and ac, the Numbers produced ſhall be in the ſame proportion that the numbers multiplied are, *viz.* as $b \,.\, c :: ab \,.\, ac$, and therefore $\frac{ab}{ac}=\frac{b}{c}$; alſo $\frac{ac}{ab}=\frac{c}{b}$. From the foregoing five Propoſitions the following Rule is deduced, *viz.*

6. *A Rule for Diviſion in Surd Quantities when the Diviſor is a Binomial, Trinomial or Quadrinomial of ſuch kind as before is declared.*

Reduce the given Diviſor to a new Diviſor that may be a ſimple Rational Quantity; reduce alſo the given Dividend to a new Dividend, by multiplying the former by the ſame Quantity or Quantities that were Multiplicators in reducing the given Diviſor to a Rational Quantity; then divide the new Dividend by the new Diviſor, (according to the Method in the Examples at the beginning of this *Sect.* 11.) ſo the Quotient ſhall be the ſame with that which would ariſe by dividing the given Dividend by the given Diviſor.

As for Example, to divide $\sqrt{8}+\sqrt{6}$ by $\sqrt{4}+\sqrt{2}$, I firſt multiply the Diviſor $\sqrt{4}+\sqrt{2}$ by its correſpondent Reſidual $\sqrt{4}-\sqrt{2}$, and it produces 2 fot a new Diviſor; alſo I multiply the Dividend $\sqrt{8}+\sqrt{6}$ by the ſaid $\sqrt{4}-\sqrt{2}$, and it gives the Product $\sqrt{32}+\sqrt{24}-\sqrt{16}-\sqrt{13}$ for a new Dividend, this divided by 2 (the Diviſor before found) gives $\sqrt{8}+\sqrt{6}-2+\sqrt{3}$ the Quotient ſought, being equal to that which would ariſe by dividing $\sqrt{8}+\sqrt{6}$ by $\sqrt{4}+\sqrt{2}$, as will be evident by the Proof;

for if the ſaid Quotient $\sqrt{8}+\sqrt{6}-2-\sqrt{3}$ be multiplied by the given Diviſor $\sqrt{4}+\sqrt{2}$, it produce the given Dividend $\sqrt{8}+\sqrt{6}$.

Likewiſe to divide $ab+b\sqrt{bc}$, by $a+\sqrt{bc}$, I multiply each by $a-\sqrt{bc}$ (the Reſidual Correſpondent to the Diviſor) and it produces $aa-bc$ for a new Diviſor, and $aab-bbc$ for a new Dividend, this divided by that gives b for the Quotient ſought; for b multiplied into the given Diviſor $a+\sqrt{bc}$ makes the given Dividend $ab+b\sqrt{bc}$. Another way of finding out the Quotient in this laſt Example, is ſhewn in the firſt of the ſix Examples at the latter end of this *Sect.* 11.

Again, to divide 10 by $\sqrt{(4)}5+\sqrt{(4)}3$, I multiply each by $\sqrt{(4)}5-\sqrt{(4)}3$, and there comes forth a new Dividend $\sqrt{(4)}50000-\sqrt{(4)}30000$, and a new Diviſor $\sqrt{5}-\sqrt{3}$; but this Diviſor not being a Rational Number, I multiply again both the ſaid new Dividend and Diviſor by $\sqrt{5}+\sqrt{3}$, and it produces another new Dividend $\sqrt{(4)}1250000-\sqrt{(4)}750000+\sqrt{(4)}450000-\sqrt{(4)}270000$, and another new Diviſor **2**; by this I divide the laſt Dividend, and there ariſes $\sqrt{(4)}78125-\sqrt{(4)}46875+\sqrt{(4)}28125-\sqrt{(4)}16875$ the Quotient ſought; for if it be multiplied by the propoſed Diviſor $\sqrt{(4)}5+\sqrt{(4)}3$ it will produce the given Dividend 10.

Again, to divide $\sqrt{8}$ by $\sqrt{3}+\sqrt{2}+1$, I firſt multiply the Diviſor by $\sqrt{3}+\sqrt{2}-1$, and it makes $\sqrt{24}+4$, this multiplied by its correſpondent Reſidual $\sqrt{24}-4$ gives the Product 8 for a new Diviſor. Now becauſe the given Diviſor was firſt multiplied by $\sqrt{3}+\sqrt{2}-1$, and the Product by $\sqrt{24}-4$, the given Dividend muſt likewiſe be multiplied firſt by $\sqrt{3}+\sqrt{2}-1$, and the Product $\sqrt{24}+4-\sqrt{8}$ by $\sqrt{24}-4$, and there will be produced $8+\sqrt{128}-\sqrt{192}$ for a new Dividend; ſo inſtead of the given Dividend and Diviſor we have other Numbers in the ſame proportion, *viz.* $8+\sqrt{128}-\sqrt{192}$ and 8. Therefore (by *Prop.* 5.) the former divided by the latter will give the Quotient ſought, to wit, $1+\sqrt{2}-\sqrt{3}$; but that this is the true Quotient will appear by Multiplication, for if $1+\sqrt{2}-\sqrt{3}$ be multiplied by the propoſed Diviſor $\sqrt{3}+\sqrt{2}+1$, it will produce the given Dividend $\sqrt{8}$.

Note, Although the new Diviſor and Dividend found out as aforeſaid, may ſometimes happen to be Negative Quantities, (that is, ſuch whoſe values are leſs than nothing) yet Diviſion being made by them with reſpect to the Rules of + and —, they will give the true Quotient ſought. As for Example, ſuppoſe 30 be to be divided by $2+\sqrt{9}$, (that is 30 by 5;) firſt the Diviſor $2+\sqrt{9}$ being multiplied by $2-\sqrt{9}$ gives $4-9$, that is, -5 for a new Diviſor, and the Dividend 30 multiplied by the ſaid $2-\sqrt{9}$ gives $60-\sqrt{8100}$ for a new Dividend, which divided by -5 gives $+6$, which is the ſame with the Quotient that ariſes by dividing 30 by $2+\sqrt{9}$, that is, by 5.

Again, let $4+\sqrt{25}$ be divided by $1+\sqrt{9}$, (that is, 9 by 4, where the Quotient is manifeſtly $2\frac{1}{4}$;) firſt, the Diviſor $1+\sqrt{9}$ multiplied by $1-\sqrt{9}$ produces $1-9$, that is, -8 for a new Diviſor; and the Dividend $4+2\sqrt{5}$ multiplied by the ſaid $1-\sqrt{9}$ makes $4+\sqrt{25}-4\sqrt{9}-\sqrt{225}$ for a new Dividend, which divided by -8, (according to the Examples at the beginning of this *Sect.* 11.) gives $-\frac{1}{2}-\sqrt{\frac{25}{64}}+\frac{1}{2}\sqrt{9}+\sqrt{\frac{225}{64}}$ the Quotient ſought, which after due contraction makes $2\frac{1}{4}$. For $\frac{1}{2}\sqrt{9}$, that is, $\sqrt{\frac{144}{64}}$ is equal to $\frac{12}{8}$, and $\sqrt{\frac{225}{64}}$ is $\frac{15}{8}$, which added to the ſaid $\frac{12}{8}$ makes $\frac{27}{8}$; alſo $-\sqrt{\frac{25}{64}}$ is $-\frac{5}{8}$, which added to $-\frac{1}{2}$, (or $-\frac{4}{8}$) makes $-\frac{9}{8}$, this added to $\frac{27}{8}$ gives $\frac{18}{8}$ (or $2\frac{1}{4}$) the Quotient before found.

7. When the Diviſor is a Binomial or a Reſidual, conſiſting of two ſimple Cubic or Biquadratic, &c. Roots, it may be reduced to a Rational Diviſor by this following Propoſition, *viz.*

If in the Proportion of the Names (or Parts) of a Binomial or Reſidual, there be found ſo many continual Proportionals in multitude as there be Units in the Index of the Radical Sign, and that the Radical Signs of the Parts of the Binomial or Reſidual, and alſo of the Proportionals be the ſame, but connected in the Binomial by +, and in the Proportionals by + and — alternately; or contrarily, in the Proportionals by +, and in the Reſidual by + and —; the Product made by the Multiplication of the Proportionals by the Binomial or Reſidual ſhall be Rational.

As for example, if there be propoſed the Binomial $\sqrt{(3)}7+\sqrt{(3)}5$; find three continual Proportionals, that the firſt may be to the ſecond, and the ſecond to the third, as $\sqrt{(3)}7$ to $\sqrt{(3)}5$, which may be done by the help of *Sect.* 8. *Chap.* 5. of this Book; where it has been ſhewn, that aa, ae, and ee, are continual Proportionals in the Reaſon of a to e. Therefore if we ſuppoſe $\sqrt{(3)}7$ to be a, and $\sqrt{(3)}5$ to be e, then the Square of $\surd(3)7$, to wit, $\surd(3)49$, ſhall be the firſt Proportional (aa); the Product of $\surd(3)7$ into $\surd(3)5$, to wit, $\surd(3)35$, ſhall be the ſecond Proportional (ae); and the Square of $\surd(3)5$, to wit, $\surd(3)25$, ſhall be the third Proportional (ee): ſo that theſe three Cubic Roots, to wit, $\surd(3)49$, $\surd(3)35$, and $\surd(3)25$, are continual Proportionals in the Reaſon of $\surd(3)7$ and $\surd(3)5$. Now I ſay, (according to the Propoſition) if $\surd(3)49 - \surd(3)35 + \surd(3)25$ be multiplied by $\surd(3)7 + \surd(3)5$, the Product ſhall be Rational; alſo if $\surd(3)49 + \surd(3)35 + \surd(3)25$ be multiplied by $\surd(3)7 - \surd(3)5$, the Product ſhall be Rational, as will appear by the following Operation.

Multiplicand, $\surd(3)49 - \surd(3)35 + \surd(3)25$
Multiplicator, $\surd(3)7 + \surd(3)5$
$7 - \surd(3)245 + \surd(3)175$
$+ \surd(3)245 - \surd(3)175 + 5$
The Product, 12 is Rational.

Multiplicand, $\surd(3)49 + \surd(3)35 + \surd(3)25$
Multiplicator, $\surd(3)7 - \surd(3)5$
$7 + \surd(3)245 + \surd(3)175$
$- \surd(3)245 - \surd(3)175 - 5$
The Product, 2 is Rational.

But for the greater Evidence of the certainty of this Propoſition in a Binomial and Reſidual conſiſting of any two ſimple Cubic Roots whatever, let there be propoſed this Binomial $\surd(3)b + \surd(3)d$, and ſuppoſe b greater than d; then three continual Proportionals in the Proportion of $\surd(3)b$ to $\surd(3)d$ will be found $\surd(3)bb$, $\surd(3)bd$, and $\surd(3)dd$; then multiply as before, *viz.*

Multiplicand, $\surd(3)bb - \surd(3)bd + \surd(3)dd$
Multiplicator, $\surd(3)b + \surd(3)d$
$b - \surd(3)bbd + \surd(3)bdd$
$+ \surd(3)bbd - \surd(3)bdd + d$
The Product, $b + d$ is Rational.

Again,
Multiplicand, $\surd(3)bb + \surd(3)bd + \surd(3)dd$
Multiplicator, $\surd(3)b - \surd(3)d$
$b + \surd(3)bbd + \surd(3)bdd$
$- \surd(3)bbd - \surd(3)bdd - d$
Product, $b - d$ is Rational.

Whence you may obſerve, that the firſt Rational Product is the ſum of the Names (or Parts,) omitting the Radical Signs, of the Cubic Binomial propoſed; and the latter Rational Product is the difference of the Parts, omitting the Radical Signs, of the Cubic Reſidual propoſed: ſo that the Rational Product made by the Multiplication of the ſaid Proportionals and Binomial or Reſidual may be diſcovered without any Multiplication.

8. Now that the uſe of the laſt preceding Propoſition may appear, let it be required to divide 10 by $\surd(3)7 - \surd(3)5$; firſt, becauſe the Index of the Radical Sign is 3, I ſeek three continual Proportionals in the Proportion of $\surd(3)7$ to $\surd(3)5$; which Proportionals as before has been ſhewn are $\surd(3)49$, $\surd(3)35$, and $\surd(3)25$; theſe I connect by +, becauſe the Parts of the given Diviſor are connected by —, and there ariſes $\surd(3)49 + \surd(3(35 + \surd(3)25$: then by this common Multiplicator I multiply as well the Dividend 10, as the Diviſor $\surd(3)7 - \surd(3)5$, and it produces $\surd(3)49000 + \surd(3)35000 + \surd(3)25000$ for a new Dividend, and 2 for a new Diviſor. Laſtly, by dividing the ſaid new Dividend by the new Diviſor, there ariſes $\surd(3)6125 + \surd(3)4375 - \surd(3)3125$ the Quotient ſought: for if it be multiplied by the great Diviſor $\surd(3)7 - \surd(3)5$, it will produce the given Dividend 10.

In like manner, to divide 10 by this Binomial $\surd(3)5+\surd(3)3$, firſt, I ſeek three continual Proportionals in the Reaſon of $\surd(3)5$ to $\surd(3)3$, which Proportionals will be found $\surd(3)25$, $\surd(3)15$, and $\surd(3)9$; theſe I connect by + and — alternately, becauſe the Parts of the given Diviſor are connected by +, *viz.* to the firſt Proportional I prefix +, to the ſecond —, and to the third +; ſo they make $\surd(3)25-\surd(3)15+\surd(3)9$. By this as a common Multiplicator I multiply as well the Dividend 10 as the Diviſor $\surd(3)5+\surd(3)3$, and there ariſes a new Dividend $\surd(3)25000-\surd(3)15000+\surd(3)9000$, and a new Diviſor 8, by which I divide the ſaid new Dividend, and there comes forth $\surd(3)\frac{3125}{64}-\surd(3)\frac{1875}{64}+\surd(3)\frac{1125}{64}$, the Quotient ſought.

The ſame Method is to be obſerved when the Diviſor is a Binomial or a Reſidual conſiſting of two ſimple Biquadratic Roots.

As for Example, to divide 10 by $\surd(4)5+\surd(4)3$, (which has already been done after another manner in the third Example of the Rule in the ſixth ſtep of this Section;) firſt, becauſe the Index of the Radical Sign is 4, I ſearch out four continual Proportionals in the Reaſon of $\surd(4)5$ to $\surd(4)3$ in this manner, *viz.* Foraſmuch as (by *Sect.* 8. *Chap.* 5. of this Book) theſe are continual Proportionals, to wit, aaa, aae, aee, and eee; I ſuppoſe $\surd(4)5$ to be a, and $\surd(4)3$ to be e, then I multiply $\surd(4)5$ into it ſelf cubically, and it gives the firſt Proportional $\surd(4)125$, to wit, aaa;) alſo I multiply the Square of $\surd(4)5$ into $\surd(4)3$, and it gives the ſecond Proportional $\surd(4)75$, (to wit, aae;) again, I multiply $\surd(4)5$ into the Square of $\surd(4)3$, and it gives the third Proportional $\surd(4)45$, (to wit, aee;) laſtly, I multiply $\surd(4)3$ into itſelf cubically, and it gives the fourth Proportional $\surd(4)27$, (to wit, eee:) Then becauſe the two Parts of the given Diviſor are connected by +, I connect thoſe four Proportionals by + and — alternately; ſo there ariſes this Compound Number $\surd(4)125-\surd(4)75+\surd(4)45-\surd(4)27$, by which as a common Multiplicator I multiply as well the given Dividend 10, as the given Diviſor $\surd(4)5+\surd(4)3$, and there ariſes a new Dividend $\surd(4)1250000-\surd(4)750000+\surd(4)450000-\surd(4)270000$, and a New Diviſor 2; which are the ſame in every reſpect with thoſe found in the place before cited.

After the ſame manner, when the Diviſor is a Binomial or a Reſidual having 5 or 6, &c. for the Index of the common Radical Sign of the Roots, it may be reduced to a new Diviſor that ſhall be Rational. But it muſt be remembred, that when the Roots are of different kinds they muſt firſt be reduced to a common Radical Sign.

But when the Diviſor cannot be reduced to a ſimple Rational Number by any of the foregoing Rules, (which are all that I have met with in Algebraical Authors) the Dividend may be ſet as a Numerator over the Diviſor as a Denominator, and the Fraction ſo conſtituted ſhall be equal to the Quotient. As for Example, if $\surd48+\surd(3)3$ be to be divided by $\surd15+\surd(3)6-\surd3$, the Quotient may be repreſented by this Fraction, to wit,

$$\frac{\surd48+\surd(3)3}{\surd15+\surd(3)6-\surd3.}$$

Examples of Diviſion in Compound Surd Quantities expreſt by Letters.

Diviſion in Compound Surd Quantities expreſt by Letters depends upon the Rules of ſimple Surds before delivered; as alſo upon the general Method of Diviſion in *Sect.* 9. *Chap.* 5. *Book* 1. as will appear by the following Examples, ſome of which I ſhall afterwards explain.

$$\begin{array}{lll}
\text{Diviſor.} & \text{Dividend.} & \\
a+\surd bc) & ab+b\surd bc\ (b & \text{Quotient.} \\
 & ab+b\surd bc & \\
\hline
 & 0 \quad\quad 0 &
\end{array}$$

$$\begin{array}{lll}
a+\surd bc) & aa-bc & (a-\surd bc. \\
 & aa+a\surd bc & \\
\hline
 & -bc-a\surd bc & \\
 & -bc-a\surd bc & \\
\hline
 & 0 \quad\quad 0 &
\end{array}$$

$\surd ab-$

$$\begin{array}{rl}
\sqrt{ab}-\sqrt{cd}\;) & ab - cd \quad (\;\sqrt{ab}+\sqrt{cd} \\
 & ab - \sqrt{abcd} \\ \cline{2-2}
 & \quad - cd + \sqrt{abcd} \\
 & \quad - cd + \sqrt{abcd} \\ \cline{2-2}
 & \quad\;\; 0 \qquad 0
\end{array}$$

$$\begin{array}{rl}
a+\sqrt{bc}\;) & aaa + bc\sqrt{bc} \quad (\;aa + bc - a\sqrt{bc} \\
 & aaa + aa\sqrt{bc} \\ \cline{2-2}
 & \quad + bc\sqrt{bc} - aa\sqrt{bc} \\
 & \quad + bc\sqrt{bc} + abc \\ \cline{2-2}
 & \qquad\qquad - aa\sqrt{bc} - abc \\
 & \qquad\qquad - aa\sqrt{bc} - abc \\ \cline{2-2}
 & \qquad\qquad\quad 0 \qquad 0
\end{array}$$

$$\begin{array}{rl}
aa+a\sqrt{bc}\;) & aaab - abbc \quad (\;ab - b\sqrt{bc} \\
 & aaab + aab\sqrt{bc} \\ \cline{2-2}
 & \quad - abbc - aab\sqrt{bc} \\
 & \quad - abbc - aab\sqrt{bc} \\ \cline{2-2}
 & \quad\;\; 0 \qquad 0
\end{array}$$

$$\begin{array}{rl}
a-\sqrt{bc}\;) & aab - bbc - ab\sqrt{bc} + \frac{bbc}{a}\sqrt{bc} \quad (\;ab - \frac{bbc}{a} \\
 & aab \qquad\quad - ab\sqrt{bc} \\ \cline{2-2}
 & \quad - bbc \qquad + \frac{bbc}{a}\sqrt{bc} \\
 & \quad - bbc \qquad + \frac{bbc}{a}\sqrt{bc} \\ \cline{2-2}
 & \quad\;\; 0 \qquad\qquad 0
\end{array}$$

EXPLICATION.

In the firſt Example, firſt, ab divided by a gives the Quotient b, by which I multiply the whole Diviſor $a+\sqrt{bc}$, and it makes $ab+b\sqrt{bc}$, this ſubtracted from the given Dividend $ab+b\sqrt{bc}$, there remains 0; ſo the Quotient ſought is b.

In the third Example, firſt, ab divided by $\sqrt{ab}$ gives the Quotient $\sqrt{ab}$, by which I multiply the whole Diviſor $\sqrt{ab}-\sqrt{cd}$, and the Product is $ab-\sqrt{abcd}$, this ſubtracted from the given Dividend $ab-cd$, there remains to be yet divided $-cd+\sqrt{abcd}$; then I divide $-cd$ by $-\sqrt{cd}$, and it gives the Quotient $+\sqrt{cd}$, by which I multiply the whole Diviſor $\sqrt{ab}-\sqrt{cd}$, and it makes $-cd+\sqrt{abcd}$, this ſubtracted from the remaining Dividend $-cd+\sqrt{abcd}$ leaves 0; ſo the Diviſion is finiſh'd, and the Quotient ſought is $\sqrt{ab}+\sqrt{cd}$.

In the ſixth and laſt Example, firſt, aab divided by a gives the Quotient ab, this multiplying the whole Diviſor $a-\sqrt{bc}$ produces $aab-ab\sqrt{bc}$, which ſubtracted from the given Dividend leaves to be yet divided $-bbc+\frac{bbc}{a}\sqrt{bc}$; then I divide $+\frac{bbc}{a}\sqrt{bc}$ by $-\sqrt{bc}$, and it gives the Quotient $-\frac{bbc}{a}$, by which I multiply the whole Diviſor $a-\sqrt{bc}$, and it produces $-bbc+\frac{bbc}{a}\sqrt{bc}$, which ſubtracted from the remaining Dividend $-bbc+\frac{bbc}{a}\sqrt{bc}$ leaves nothing; ſo the Quotient ſought is $ab-\frac{bbc}{a}$.

The Arithmetic of Univerſal Surd Roots, both in Numbers and Quantities expreſs'd by Letters.

Sect. XII. *Multiplication in Univerſal Surds.*

Univerſal Roots are the Roots of Compound Numbers or Quantities. How to expreſs Univerſal Roots, and to find out their values, has already been ſhewn in *Sect.* 28. *Chap.* 1. *Book* 1. I ſhall therefore proceed to their Multiplication.

1. If the ſquare Root of any compound Number to be ſquared, or multiplied into itſelf, caſt away the univerſal Radical Sign $\surd$ or $\surd(2)$, as alſo the Line that is drawn over the compound Number, and the compound Number itſelf ſhall be the Square of the univerſal Root propoſed. Alſo the Cube of the Cubic Root of any compound Number is the compound Number itſelf, the Line drawn over it, and the univerſal Radical Sign $\surd(3)$ being caſt away; and ſo of others.

As for Example, the ſquare of this univerſal ſquare Root $\surd:\overline{12+\surd 3}:$ is $12+\surd 3$; likewiſe the ſquare of $\surd:\overline{12-\surd 3}:$ is $12-\surd 3$; alſo the ſquare of $\surd:\overline{15+\surd 3+\surd 2}:$ is $15+\surd 3+\surd 2$; and the ſquare of $\surd:\overline{15-\surd 3-\surd 2}:$ is $15-\surd 3-\surd 2$.

After the ſame manner the Cube of this univerſal Cubic Root $\surd(3):\overline{\surd 25+\surd 9}:$ is $\surd 25+\surd 9$, that is 8.

Likewiſe the Square of $\surd:\overline{aa+bb}:$ is $aa+bb$, and the Cube of $\surd(3):\overline{bbb+ccc}:$ is $bbb+ccc$; alſo the ſquare of $\surd:\overline{\frac{1}{2}c+\frac{1}{4}cc-n}:$ is $\frac{1}{2}c+\surd\overline{\frac{1}{4}cc-n}:$ and ſo of others.

2. When an univerſal Root is to be multiplied by a Rational Quantity, or by a ſimple or compound Surd, or by any univerſal Root; multiply the ſquare of the Multiplicand by the ſquare of the Multiplier, when the univerſal Radical Sign is Quadratic; or the Cube of the one by the Cube of the other, when the univerſal Radical Sign is Cubic, *&c.* then before that Cubic prefix the given univerſal Radical Sign; ſo ſhall this new univerſal Root be the Product ſought,

As for Example, if it be deſired to double or multiply by 2 this univerſal ſquare Root $\surd:\overline{10+\surd 40}:$ I take the ſquare of 2 which is 4, and the ſquare of $\surd:\overline{10+\surd 40}:$ which (by the foregoing firſt Rule of this *Sect.*) is $10+\surd 40$; then I multiply $10+\surd 40$ by 4, and it makes $40+4\surd 40$, or $40+\surd 640$, whoſe univerſal ſquare Root, to wit, $\surd:\overline{40+4\surd 40}:$ or $\surd:\overline{40+\surd 640}:$ is the Product of $\surd:\overline{10+\surd 40}:$ multiplied by 2, or the ſaid Product may be expreſs'd thus $2\surd:\overline{10+\surd 40}:$

Likewiſe if $\surd(3):\overline{\surd(3)64+\surd(3)27}:$ be to be doubled or multiplied by 2, I firſt multiply each of thoſe Numbers cubically, becauſe the Radical Sign of the given univerſal Root is $\surd(3)$, and their Cubes will be $\surd(3)64+\surd(3)27$ and 8; which multiplied one into the other make $8\surd(3)64+8\surd(3)27$, to which Product I prefix the univerſal Radical Sign $\surd(3)$ and it gives $\surd(3):\overline{8\surd(3)64+8\surd(3)27}:$ that is, $\surd(3):\overline{32+24}:$ or $\surd(3)56$, which is the Product ſought, to wit, the double of $\surd(3):\overline{\surd(3)64+\surd(3)27}:$

After the ſame manner if $\surd(3):\overline{\surd(3)64+\surd(2)36+3}:$ be to be multiplied by 5, or $\surd(3)125$, the Product will be $\surd(3):\overline{125\surd(3)64+125\surd(2)36+375}:$ that is, $\surd(3)1625$.

Again, to multiply $\surd:\overline{\surd 10+\surd 3}:$ by $\surd 5$, their Squares are $\surd 10+\surd 3$ and 5, which multiplied one into another make $5\surd 10+5\surd 3$, (that is, $\surd 250+\surd 75$) whoſe univerſal ſquare Root, to wit, $\surd:\overline{5\surd 10+5\surd 3}:$ (or $\surd:\overline{\surd 250+\surd 75}:$) is the Product of $\surd:\overline{\surd 10+\surd 3}:$ multiplied by $\surd 5$.

Likewiſe, to multiply $\surd:\overline{13+\surd 9}:$ by $\surd:\overline{5+\surd 16}:$ (that is, 4 by 3, where the Product is manifeſtly 12;) the Squares of the univerſal Roots propoſed are $13+\surd 9$ and $5+\surd 16$, which multiplied one into another make $65+5\surd 9+13\surd 16+\surd 144$; whoſe univerſal ſquare Root, to wit, $\surd:\overline{65+5\surd 9+13\surd 16+\surd 144}:$ that is, $\surd 144$, or 12, is the Product ſought.

Again, to multiply $\surd:\overline{\frac{7}{2}+\surd\frac{29}{4}}:$ into $\surd:\overline{\frac{7}{2}-\surd\frac{29}{4}}:$ I multiply their Squares $\frac{7}{2}+\surd\frac{29}{4}$ and $\frac{7}{2}-\surd\frac{29}{4}$ one into another, according to the laſt of the three compendious Rules in *Sect.* 10. of this *Chap.* and there comes forth $\frac{49}{4}-\frac{29}{4}$, that is 5, (to wit, the difference between the Squares of $\frac{7}{2}$ and $\sqrt{\frac{29}{4}}$.) Laſtly, the Square Root of the ſaid 5 is $\sqrt{5}$ for the Product ſought.

So alſo to multiply $\sqrt{:5+\sqrt{2}:}$ by $\sqrt{5}+\sqrt{2}$, their Squares $5+\sqrt{2}$ and $7+2\sqrt{10}$ multiplied one into another give $35+10\sqrt{10}+7\sqrt{2}+2\sqrt{20}$, whoſe univerſal ſquare Root to wit, $\sqrt{:35+10\sqrt{10}+7\sqrt{2}+2\sqrt{20}:}$ is the Product ſought.

Moreover, to multiply $\sqrt{:\sqrt{144}+4:}-\sqrt{:\sqrt{4}+2:}$ by $\sqrt{:\sqrt{100}-1:}$ (that is, 2 by 3, which will produce 6) I firſt multiply the Square of $\sqrt{:\sqrt{144}+4:}$ by the Square of $\sqrt{:\sqrt{100}-1:}$ *viz.* $\sqrt{144}+4$ by $\sqrt{100}-1$, and it makes $\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4$, before which I prefix the univerſal Radical Sign $\sqrt{}$, and it gives $\sqrt{:\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4:}$ which is one of the Members of the Product ſought; then I multiply in like manner $-\sqrt{:\sqrt{4}+2:}$ by $\sqrt{:\sqrt{100}-1:}$ and it makes $-\sqrt{:\sqrt{400}+2\sqrt{100}-\sqrt{4}-2:}$ for the latter Member of the Product ſought. Laſtly, both thoſe Members being joined together give $\sqrt{:\sqrt{14400}+4\sqrt{100}-\sqrt{144}-4:}$ $-\sqrt{:\sqrt{400}+2\sqrt{160}-\sqrt{4}-2:}$ that is, $\sqrt{144}-\sqrt{36}$, that is, $12-6$ or 6, for the Product required.

3. Sometimes the fourth, fifth, and ſixth Rules in *Sect.* 4. of this *Chap.* will be uſeful in the Multiplication of univerſal Surds. As if it be deſired to multiply $3\sqrt{:2+\sqrt{5}:}$ by $4\sqrt{:2+\sqrt{5}:}$ (which are commenſurable Roots, for they are in proportion one to the other as 3 to 4) I multiply 3 by 4, and the Product 12 into $2+\sqrt{5}$; ſo there is produced $24+12\sqrt{5}$ (that is, $24+\sqrt{720}$) for the Product ſought.

Likewiſe; $5\sqrt{:6+\sqrt{9}:}$ multiplied by $2\sqrt{:6+\sqrt{9}:}$ (that is, 15 by 6) produces $60+10\sqrt{9}$, (that is, 90.)

Moreover, if $5\sqrt{:6+\sqrt{9}:}$ be to be multiplied by $3\sqrt{:19-\sqrt{9}:}$ (that is, 15 by 12) I firſt multiply 5 by 3 and it makes 15, then I multiply $\sqrt{:6+\sqrt{9}:}$ by $\sqrt{:19-\sqrt{9}:}$ and it produces $\sqrt{:105+13\sqrt{9}:}$ which latter Product multiplied into the former Product 15 makes $15\sqrt{:105+13\sqrt{9}:}$ (that is, 180) the Product ſought.

4. Sometimes alſo the three Rules before delivered in *Sect.* 10. of this *Chap.* concerning the multiplying of Binomials and Reſiduals will be uſeful in the Multiplication of univerſal Surd Roots. As if this Binomial Root $\sqrt{:12+\sqrt{6}:}+\sqrt{:12-\sqrt{6}:}$ be to be ſquared or multiplied into itſelf, the Squares of the Parts are $12+\sqrt{6}$ and $12-\sqrt{6}$, whoſe Sum is 24; then the Product made by the Multiplication of the Parts one into the other, *viz.* $\sqrt{:12+\sqrt{6}:}$ into $\sqrt{:12-\sqrt{6}:}$ is $\sqrt{138}$, (for the difference of the Squares of 12 and $\sqrt{6}$ is 138, whoſe ſquare Root is $\sqrt{138}$;) and the double of the ſaid Product is $2\sqrt{138}$, which added to 24 (the Sum of the Squares of the Parts) makes $24+2\sqrt{138}$, which is the Square of $\sqrt{:12+\sqrt{6}:}+\sqrt{:12-\sqrt{6}:}$ Moreover, the ſquare Root of the ſaid $24+2\sqrt{138}$, to wit, $\sqrt{:24+2\sqrt{138}:}$ is the Sum of the two Parts $\sqrt{:12+\sqrt{6}:}$ and $\sqrt{:12-\sqrt{6}:}$ For when the Sum of two Numbers is multiplied into itſelf, the ſquare Root of the Product is equal to the ſame Sum.

Likewiſe if $\sqrt{:10+\sqrt{36}:}-\sqrt{:10-\sqrt{36}:}$ that is 2, be to be ſquared or multiplied into itſelf, the Product will be found $20-2\sqrt{64}$, that is 4, and the ſquare Root of this 4, to wit 2, is the difference of the two Roots or Parts $\sqrt{:10+\sqrt{36}:}$ and $\sqrt{:10-\sqrt{36}:}$ For when the difference of two Numbers is multiplied into itſelf, the ſquare Root of the Product is equal to the ſaid difference.

Again, if $6+\sqrt{:20-\sqrt{16}:}$ be to be multiplied into $6-\sqrt{:20-\sqrt{16}:}$ the Product will be found 20; for (according to Rule 3. in *Sect.* 10. of this *Chap.*) if $20-\sqrt{16}$, which is the Square of $\sqrt{:20-\sqrt{16}:}$ be ſubtracted from 36 the Square of 6, there will remain $16+\sqrt{16}$, that is, 20 the Product ſought.

Likewiſe if $\sqrt{20}+\sqrt{:20-\sqrt{5}:}$ be to be multiplied into $\sqrt{20}-\sqrt{:20-\sqrt{5}:}$ the Product will be $\sqrt{5}$.

So alſo if $\sqrt{:5+\sqrt{:20-\sqrt{16}:}}$ be to be multiplied by $\sqrt{:5-\sqrt{:20-\sqrt{16}:}}$ (that is 3 by 1;) firſt, the Squares of the univerſal Roots propoſed are $5+\sqrt{:20-\sqrt{16}:}$ and $5-\sqrt{:20-\sqrt{16}:}$ theſe multiplied one by the other, by taking the difference of the Squares of 5 and $\sqrt{:20-\sqrt{16}:}$ give the Product $5+\sqrt{16}$, whoſe univerſal ſquare Root, to wit, $\sqrt{:5+\sqrt{16}:}$ that is 3, is the Product of the two univerſal ſquare Roots propoſed to be multiplied.

5. The four preceding Rules of this Section are alſo to be obſerved in the Multiplication of univerſal Surd Roots expreſs'd by Letters. As if it be deſired to multiply $\sqrt{:aa+bb:}$ by a, I multiply their Squares $aa+bb$ and aa one into the other, and there comes forth $aaaa+aabb$, whoſe univerſal ſquare Root $\sqrt{:aaaa+aabb:}$ is the Product ſought; which may more compendiouſly be expreſs'd thus, $a\sqrt{:aa+bb:}$

Likewiſe to multiply $\sqrt{:oo+4mp:}$ into $\frac{z}{a}$, I write $\sqrt{\frac{oozz+4mpzz}{aa}}$, or $\frac{z}{a}\sqrt{:oo+4mp:}$ for the Product.

Again, if $\sqrt{:aa+12:}$ be to be multiplied by $a+3$, the Product may be ſignified by $\overline{a+3}$ into $\sqrt{:aa+12:}$ Or, after the Squares of the Quantities propoſed are multiplied one into the other, and the univerſal Radical Sign prefix'd, the Product may be expreſs'd thus, $\sqrt{:aaaa+6aaa+21aa+72a+108:}$

So alſo $\sqrt{bc}$ multiplied into $\sqrt{:aa+bb:}$ produces $\sqrt{:aabc+bbbc:}$ and $\sqrt{:\sqrt{bc}+\sqrt{a}:}$ multiplied by $\sqrt{:\sqrt{ba}-\sqrt{bc}:}$ produces $\sqrt{:b\sqrt{ca}+a\sqrt{b}-bc-\sqrt{abc}:}$ that is, $\sqrt{:\sqrt{bbca}+\sqrt{aab}-bc-\sqrt{abc}:}$

Again, after the manner of the preceding third Rule of this Section $a\sqrt{:bb-cc:}$ multiplied by $d\sqrt{:bb-cc:}$ produces $adbb-adcc$.

And $a\sqrt{:b+c:}$ into $d\sqrt{:b-c:}$ produces $ad\sqrt{:bb-cc:}$

Moreover, if this Binomial Root $\sqrt{:\sqrt{a}+\sqrt{bc}:}+\sqrt{:\sqrt{a}-\sqrt{bc}:}$ be to be ſquared or multiplied into itſelf, firſt, the Squares of the Names or Parts of the Binomial are $\sqrt{a}+\sqrt{bc}$ and $\sqrt{a}-\sqrt{bc}$, which added together make $2\sqrt{a}$; then the double Product of the Parts is $2\sqrt{:a+\sqrt{bc}:}$ (for the difference of the Squares of $\sqrt{a}$ and $\sqrt{bc}$ is $a-bc$, whoſe univerſal ſquare Root doubled is $2\sqrt{:a-bc:}$) which double Product added to $2\sqrt{a}$, (to wit, the ſum of the Squares of the Parts firſt found) makes $2\sqrt{a}+2\sqrt{:a-bc:}$ which is the Square or Product deſired; and if the ſquare Root of this Product be extracted, it gives $\sqrt{:2\sqrt{a}+2\sqrt{:a-bc:}}$ which is equal to the Sum of the Parts of the Binomial Roots firſt propoſed to be ſquared.

Sect. XIII. *Diviſion in Univerſal Surds.*

Divide the Square of the Dividend by the Square of the Diviſor, when the univerſal Radical Sign is Quadratic, or the Cube of the one by the Cube of the other, when the univerſal Radical Sign is Cubic, *&c.* then prefix the given univerſal Sign to the Quotient, ſo ſhall this new Root be the Quotient ſought.

As for Example, if it be deſired to divide $\sqrt{:40+4\sqrt{40}:}$ by 2, I divide $40+4\sqrt{40}$, which is the Square of the Dividend, by 4 the Square of the Diviſor, (according to *Sect.* 11. of this *Chap.*) and there ariſes $10+\sqrt{40}$, whoſe ſquare Root univerſal, to wit, $\sqrt{:10+\sqrt{40}:}$ is the Quotient ſought.

Again, if it be deſired to divide $\sqrt{:40+4\sqrt{40}:}$ by $\sqrt{:10+\sqrt{40}:}$ firſt, I take their Squares, to wit, $40+4\sqrt{40}$ and $10+\sqrt{40}$ as a Dividend and Diviſor, then becauſe the Diviſor is a Compound Number, a new Dividend and Diviſor muſt be found out, ſuch that the new Diviſor may be a Rational Number; ſo (according to the Rule in the ſixth branch of *Sect.* 11. of this *Chap.*) there will be produced 240 and 60 for a new Dividend and Diviſor, which give the Quotient 4, whoſe ſquare Root is 2 the Quotient ſought, to wit, the Quotient of $\sqrt{:40+4\sqrt{40}:}$ divided by $\sqrt{:10+\sqrt{40}:}$.

Likewiſe, to divide 20 by $\sqrt{:10-\sqrt{5}:}$ firſt, I reduce their Squares 400 and $10-\sqrt{5}$ to a new Dividend and Diviſor, to wit, $4000+400\sqrt{5}$ and 95; then I divide $4000+400\sqrt{5}$ by 95, and there ariſes $42\frac{2}{19}+\frac{80}{19}\sqrt{5}$, whoſe univerſal ſquare Root, to wit, $\sqrt{:42\frac{2}{19}+\frac{80}{19}\sqrt{5}:}$ the Quotient ſought.

Another Example (in Rational Numbers expreſs'd Surd-wiſe) may be this, *viz.* ſuppoſe it be deſired to divide $\sqrt{:4+\sqrt{25}:}$ by $\sqrt{:1+\sqrt{9}:}$ (that is, by 3 and 2, which gives the Quotient $1\frac{1}{2}$;) firſt, I reduce $4+\sqrt{25}$ and $1+\sqrt{9}$, the Squares of the given Dividend and Diviſor, to a new Dividend and Diviſor, to wit, $4+\surd 25-4\surd 9-\surd 225$ and -8, theſe give the Quotient $\frac{9}{4}$, (as has been proved in the latter Example of the Note in the preceding *Sect.* 11.) the ſquare Root whereof, to wit $\frac{3}{2}$, is the Quotient ſought; for if the given Diviſor $\surd:\overline{1+\surd 9}:$ be multiplied by the Quotient $\frac{3}{2}$, it will produce 3, which is equal to the given Dividend $\surd:\overline{4+\surd 25}:$

Again, to divide $\surd(3):\overline{8\surd(3)64+8\surd(2)27}:$ by 2, I divide the Cube of the one by the Cube of the other, *viz.* $8\surd(3)64+8\surd(2)27$ by 8, and there ariſes $\surd(3)64+\surd(2)27$, whoſe univerſal Cubic Root, to wit, $\surd(3):\overline{\surd 64+\surd(2)27}:$ is the Quotient ſought, to wit, the half of the Dividend propoſed.

2. If the given univerſal Roots, to wit, the Dividend and Diviſor be commenſurable, then (according to the fifth Rule of *Sect.* 5. of this *Chap.*) divide the Rational part of the Dividend by the Rational part of the Diviſor, and what ariſes is the Quotient ſought. As to divide $21\surd:\overline{6+\surd 9}:$ by $3\surd:\overline{6+\surd 9}:$ I divide 21 by 3, and there ariſes 7 for the Quotient ſought.

Likewiſe $183\surd:\overline{\surd 3-\surd 2}:$ divided by $\frac{61}{8}\surd:\overline{\surd 3-2}:$ gives the Quotient 24.

3. Diviſion in univerſal Surds expreſs'd by Letters depends upon the Rules before given: as to divide $\surd:\overline{aaaa+aabb}:$ by a, I divide the Square of the Dividend by the Square of the Diviſor, *viz.* $aaaa+aabb$ by aa, and there ariſes $aa+bb$, whoſe ſquare Root univerſal, to wit, $\surd:\overline{aa+bb}:$ is the Quotient ſought.

Again, if it be deſired to divide $\surd:\overline{\surd bbca+\surd aab-bc-\surd abc}:$ by $\surd:\overline{bc+\surd a}:$ I divide the ſquare of the Dividend by the ſquare of the Diviſor, *viz.* $\surd bbca+\surd aab-bc-\surd abc$ by $\surd bc+\surd a$, (according to the Method in the Examples at the latter end of *Sect.* 11. of this *Chap.*) and there ariſes $\surd ba-\surd bc$, whoſe univerſal ſquare Root, to wit, $\surd:\overline{\surd ba-\surd bc}:$ is the Quotient ſought.

Moreover, to divide $d\surd:\overline{bb+cc}:$ by $3a\surd:\overline{bb+cc}:$ becauſe they are commenſurable, I divide only the Rational part by the Rational, and there ariſes $\frac{d}{3a}$ for the Quotient.

4. Laſtly, when the work of Diviſion in univerſal Surds according to the foregoing Rules and Examples in this Section, happens to be intricate, or will not work off juſt without a Remainder, you may ſet the Power of the Dividend (the univerſal Radical Sign being omitted) as a Numerator, over the Power of the Diviſor as a Denominator, and prefix the univerſal Radical Sign before the Line that ſeparates the Numerator from the Denominator; then ſhall the univerſal Root ſo denoted ſignifie the Quotient ſought.

As if it be deſired to divide $\surd:\overline{\surd 5+\surd 8-3}:$ by $\surd:\overline{\surd 7-\surd 2+1}:$ the Quotient may be repreſented by this Fraction $\surd:\frac{\surd 5+\surd 8-3}{\surd 7-\surd 2+1}:$

Likewiſe if $\surd:\overline{\surd abb+bcd}:$ be to be divided by $\surd:\overline{\surd ac-dd}:$ you may write $\surd:\frac{\surd abb+bcd}{\surd ac-dd}:$ to ſignifie the Quotient.

Sect. XIV. *Addition and Subtraction in Univerſal Surds.*

1. When two univerſal Surds propoſed to be added or ſubtracted are commenſurable, they may be added or ſubtracted like ſimple Surds, (according to the Rule in *Sect.* 8. of this *Chap.*) As for Example, if the Sum and Difference of $\surd:\overline{8+4\surd 3}:$ and $\surd:\overline{2+\surd 3}:$ be deſired; becauſe each of them divided by their common Diviſor $\surd:\overline{2+\surd 3}:$ gives $\surd 4$ and $\surd 1$, that is, 2 and 1, which are Rational Numbers expreſſing the proportion of the Surds propoſed. Therefore the Sum of 2 and 1, to wit, 3 multiplied into the ſaid common Diviſor gives $3\surd:\overline{2+\surd 3}:$ for the Sum required, (which may alſo be expreſs'd thus, $\surd:\overline{18+\surd 243}:$) and the difference of the ſaid 2 and 1, to wit, 1 multiplied into the ſaid common Diviſor $\surd:\overline{2+\surd 3}:$ makes $\surd:\overline{2+\surd 3}:$ for the difference of the two Roots firſt propoſed.

Another Example in Rational Numbers expreſs'd Surd-wiſe, *viz.* let it be required to find out the Sum and Difference of $\surd:\overline{99+9\surd 25}:$ and $\surd:\overline{44+4\surd 25}:$ (that is, 12 and 8; firſt, thoſe univerſal Roots being ſeverally divided by the common Diviſor

$\sqrt{:11+\sqrt{25}:}$ give the Quotients $\sqrt{9}$ and $\sqrt{4}$, to wit 3 and 2, which are Rational Numbers expreſſing the Proportion which the given Roots have one to another. Therefore $\overline{3+2}$, to wit 5, multiplied into the common Diviſor $\sqrt{:21+\sqrt{25}:}$ gives $5\sqrt{:11+\sqrt{25}:}$ that is, $\sqrt{:275+\sqrt{15625}:}$ (to wit 20) which is the Sum of the Roots propoſed; and 3—2, that is 1, multiplied into the ſaid $\sqrt{:11+\sqrt{25}:}$ gives $\sqrt{:11+\sqrt{25}:}$ that is 4, for the Difference of the given Roots.

Here follow Contractions of the work of Addition and Subtraction in the two laſt Examples, with others of like nature in Surd Quantities expreſs'd by Letters.

Example 1.

What is the Sum and Difference of . . . . . $\sqrt{:8+4\sqrt{3}:}$ and $\sqrt{:2+\sqrt{3}:}$

The Operation.

I. $\sqrt{:2+\sqrt{3}:}\ )\ \sqrt{:8+4\sqrt{3}:}\ (\ \sqrt{4}$, that is, 2.
II. $\sqrt{:2+\sqrt{3}:}\ )\ \sqrt{:2+\sqrt{3}:}\ (\ \sqrt{1}$, that is, 1.
Therefore from I. $2\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:}$
And from II. $1\sqrt{:2+\sqrt{3}:} = \sqrt{:2+\sqrt{3}:}$
The Sum, $3\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:} + \sqrt{:2+\sqrt{3}:}$
The Difference $1\sqrt{:2+\sqrt{3}:} = \sqrt{:8+4\sqrt{3}:} - \sqrt{:2+\sqrt{3}:}$

Example 2.

What is the Sum and Difference of $\sqrt{:99+9\sqrt{25}:}$ and $\sqrt{:44+4\sqrt{25}:}$

The Operation.

I. $\sqrt{:11+\sqrt{25}:}\ )\ \sqrt{:99+9\sqrt{25}:}\ (\ \sqrt{9}$, that is, 3.
II. $\sqrt{:11+\sqrt{25}:}\ )\ \sqrt{:44+4\sqrt{25}:}\ (\ \sqrt{4}$, that is, 2.
Therefore from I. $3\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:}$
And from II. $2\sqrt{:11+\sqrt{25}:} = \sqrt{:44+4\sqrt{25}:}$
The Sum, $5\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:} + \sqrt{:44+4\sqrt{25}:}$
The Difference, $1\sqrt{:11+\sqrt{25}:} = \sqrt{:99+9\sqrt{25}:} - \sqrt{:44+4\sqrt{25}:}$

Example 3.

What is the Sum and Difference of $\sqrt{:aaaa+aabb}:$ and $\sqrt{:aabb+bbbb}:$
Thoſe reduced (by *Sect.* 6. of this *Chap*) give $a\sqrt{:aa+bb:}$ and $b\sqrt{:aa+bb:}$
Therefore their Sum is . . . . $\overline{a+b}$ into $\sqrt{:aa+bb:}$
And their Difference is . . . . $\overline{a \backsim b}$ into $\sqrt{:aa+bb:}$

Example 4.

What is the Sum and Difference of $\Big\}\ \sqrt{\dfrac{oozz+4mpzz}{aa}}$ and $\sqrt{\dfrac{aaoomm+4aammmp}{ppzz}}$

By dividing each of them by their common Diviſor $\sqrt{:oo+4mp:}$ there will ariſe Rational Quotients, to wit, ... $\Big\}\ \dfrac{z}{a}$ and $\dfrac{am}{pz}$

Therefore the Surds propoſed are Commenſurable and inſtead of them we may write . . $\Big\}\ \dfrac{z}{a}\sqrt{:oo+4mp:}$ and $\dfrac{am}{pz}\sqrt{:oo+4mp:}$

Therefore their Sum ſhall be . $\Big\}\ \overline{\dfrac{z}{a}+\dfrac{am}{pz}}$ into $\sqrt{:oo+4mp:}$

That is, ... $\Big\}\ \dfrac{pzz+aam}{apz}$ into $\sqrt{:oo+4mp:}$

And the Difference of the given Surds ſhall be $\left\{\frac{pzz \backsim aam.}{apz}\right.$ into $\sqrt{:oo+4mp:}$

Example 5.

What is the Sum and Difference of theſe two univerſal Roots?

$\sqrt{:aaaa + 6aaa + 21aa + 72a + 108:}$ and,

$\sqrt{:aaaa + 10aaa + 37aa - 120a + 300:}$

The Operation.

The given Roots are Commenſurable, (as has been ſhewn in the laſt Example but one in *Sect.* 7. of this *Chap.*) and may be expreſs'd thus;

$a + 3\sqrt{:aa + 12:}$ and $a \backsim 5\sqrt{:aa + 12:}$

Therefore their Sum, ſuppoſing a to be greater than 5, ſhall be

$2a - 2$ into $\sqrt{:aa + 12:}$

And their Difference ſhall be $8\sqrt{:aa + 12:}$

But if we ſuppoſe a to be leſs than 5, then the Sum of the given Surds will be $8\sqrt{:aa+12:}$ and their Difference $2a \backsim 2\sqrt{:aa+12:}$ that is, $2a \backsim 2$ into $\sqrt{:aa+12:}$

2. When the Root of a Reſidual is to be added unto, or ſubtracted from, the Root of its correſpondent Binomial, thoſe Roots may be connected together by + or —; and then the whole being multiplied into itſelf, the univerſal Root of the Product ſhall be the Sum or Difference of the Roots given to be added or ſubtracted, as before has been ſhewn in Rule 4. *Sect.* 12. of this *Chap.*

As if theſe two Roots be propoſed to be added, to wit, $\sqrt{:12+\sqrt{6}:}$ and $\sqrt{:12-\sqrt{6}:}$ we may multiply this compoſed Number $\sqrt{:12+\sqrt{6}:} + \sqrt{:12-\sqrt{6}:}$ into itſelf, and there will be produced $24+2\sqrt{138}$, whoſe univerſal ſquare Root, to wit, $\sqrt{:24+2\sqrt{138}:}$ ſhall be the Sum of the two Roots propoſed to be added.

Likewiſe if $\sqrt{:12+\sqrt{6}:} - \sqrt{:12-\sqrt{6}:}$ be multiplied into itſelf, the Product will be $24-2\sqrt{138}$, whoſe univerſal ſquare Root, to wit, $\sqrt{:24-2\sqrt{138}:}$ is the Difference of the two Roots propoſed.

After the ſame manner the Sum of theſe two Roots, $\sqrt{:10+\sqrt{36}:}$ and $\sqrt{:10-\sqrt{36}:}$ will be found $\sqrt{:20+2\sqrt{64}:}$ (that is, $\sqrt{36}$, to wit 6;) but their Difference $\sqrt{:20-2\sqrt{64}:}$ (that is $\sqrt{4}$, to wit 2.)

Likewiſe the Sum of theſe Binomial Roots $\sqrt{:\sqrt{a}+\sqrt{bc}:}$ and $\sqrt{:\sqrt{a}-\sqrt{bc}:}$ will be found $\sqrt{:2\sqrt{a}+2\sqrt{:a-bc}:}$ and their Difference $\sqrt{:2\sqrt{a}-2\sqrt{:a-bc}:}$

3. But if the univerſal Roots propoſed be not Commenſurable, nor ſuch Binomials and Reſiduals as are mentioned in the laſt preceding Rule, then they are to be added by +, and ſubtracted by —.

As if $\sqrt{:5+\sqrt{2}:}$ and $\sqrt{:5-\sqrt{3}:}$ be to be added, I write $\sqrt{:5+\sqrt{2}:} + \sqrt{:-\sqrt{3}:}$ for the Sum, and to ſubtract $\sqrt{:5-\sqrt{3}:}$ from $\sqrt{:5+\sqrt{2}:}$ I write $\sqrt{:5+\sqrt{2}:} - \sqrt{:5-\sqrt{3}:}$ for the Remainder.

Likewiſe the Sum of $\sqrt{:aa+bb:}$ and $\sqrt{:aa-cc:}$ is $\sqrt{:aa+bb:} + \sqrt{:aa-cc:}$ and their Difference is $\sqrt{:aa+bb:} - \sqrt{:aa-cc:}$

Sect. XV. *Concering the Conſtitution and Invention of ſix Binomials in Numbers, agreeable to thoſe expounded in* Prop. 49, 50, 51, 52, 53, 54. Elem. 10. Eucl.

By way of preparation to the Conſtruction of the ſix Binomials in Numbers I ſhall premiſe this

QUESTION.

To find two ſquare Numbers whoſe Difference may be equal to a given Rational Number?

CANON.

Take any two Numbers, which multiplied one by the other will produce the given

Number; then half the Sum of thoſe two Numbers and half their difference ſhall be the Sides or Roots of the two Squares ſought.

As if 5 be given for the difference of two Squares ſought, I take 5 and 1; for the Product of their Multiplication is 5; then the half of their Sum is 3, and the half of their difference is 2; laſtly, the Squares of the ſaid 3 and 2 are 9 and 4, the Squares ſought; for their difference is 5, as was preſcribed.

Again, the ſame Number 5 being given for the difference of two Squares, take a Number at pleaſure, as 2, by this divide the given Number 5, the Quotient is $\frac{5}{2}$, therefore the Product of the Multiplication of the Diviſor 2 by the Quotient $\frac{5}{2}$ is 5; then according to the Canon, half the ſum and half the difference of the ſaid 2 and $\frac{5}{2}$, to wit, $\frac{9}{2}$ and $\frac{1}{4}$, ſhall be the Sides of the Squares ſought; and conſequently the ſquares themſelves are $\frac{81}{16}$ and $\frac{1}{16}$, whoſe difference is 5, as was deſired.

After the ſame manner innumerable pairs of ſquares may be found out in Rational Numbers, and the difference of each pair ſhall be equal to one and the ſame given Number.

The Reaſon of the Canon may be made manifeſt by this

Theorem.

The Product made by the Multiplication of any two unequal Numbers is equal to the difference of two ſquares, to wit, of the ſquare of half the ſum, and the ſquare of half the difference of the ſame two unequal Numbers.

As if *c* be the greater, and *b* the leſſer of two Numbers, then

The Square of $\frac{1}{2}c+\frac{1}{2}b$ is . . . . . $\frac{1}{4}cc+\frac{1}{2}cb+\frac{1}{4}bb$,
The Square of $\frac{1}{2}c-\frac{1}{2}b$ is . . . . . $\frac{1}{4}cc-\frac{1}{2}cb+\frac{1}{4}bb$,
The difference of thoſe two Squares is . . . cb

Which difference is manifeſtly the Product of the Multiplication of the two propoſed Numbers *c* and *b*; wherefore the Theorem, and conſequently the Canon firſt given, is manifeſt.

The Definition of Binomial I.

When the greater Name (or Part) of a Binomial is a Rational Number, and the leſſer part is a Surd ſquare Root of ſome Rational Number, the ſquare Root of the difference of the Squares of the parts is a Rational Number, the ſum of the two parts is called a firſt Binomial.

Explication.

Let this Binomial be propoſed, ... $3+\sqrt{5}$
The Squares of the Names or Parts are . . . . . $\}\begin{matrix}9\\5\end{matrix}$
The difference of thoſe Squares is 4
The ſquare Root of that difference is 2

Becauſe the greater part 3 is a Rational Number, and the leſſer part $\sqrt{5}$ is a Surd ſquare Root of a Rational Number 5, and the difference of the Squares of the Parts, *viz.* 4, is a Square whoſe Root 2 is a Rational Number; the Binomial propoſed, to wit, $3+\sqrt{5}$, is called a firſt Binomial.

How to find out two ſuch Numbers as may conſtitute a firſt Binomial.

1. By the Canon of the preceding Queſtion at the beginning of this 15 *Sect.* find out two ſquare Numbers, whoſe difference may be ſome Rational Number not a Square, ſuch are theſe Squares, . $\}\begin{matrix}9\\4\end{matrix}$
2. Their difference is 5
3. Take ſome Rational Number at pleaſure for the greater part of the Binomial ſought, as . . . . } 6
4. Then ſay, By the Rule of Three if 9 the greater of the two ſquares found out in the firſt ſtep, give 5 the difference in the ſecond, what ſhall 36 the ſquare of the Number taken in the third give? whence the fourth Proportional will be found 20, the ſquare Root whereof is the leſſer part, to wit, ... } $\sqrt{20}$
5. I ſay, The ſum of the two Numbers found out in the third and fourth ſteps, is a firſt Binomial, to wit . . . . } $6+\sqrt{20}$

The

The Definition of Binomial II.

When the leſſer part of a Binomial is a Rational Number, and the greater part is a Surd ſquare Root of a Rational Number, and the ſquare Root of the Difference of the Squares of the Parts is Commenſurable to the greater Part, the Sum of the two Parts is called a ſecond Binomial.

Explication.

Let this Binomial be propoſed . . . . $\sqrt{18}+4$

The Squares of the Parts are . . . . $\begin{cases}18\\16\end{cases}$

The Difference of thoſe Squares is 2

The ſquare Root of the Difference is . . . . $\sqrt{2}$

Becauſe the leſſer Part 4 is a Rational Number, and the greater Part $\sqrt{18}$ is the Surd ſquare Root of a Rational Number 18, and the ſquare Root of the Difference of the Squares of the Parts, *viz* $\sqrt{2}$, is Commenſurable to the greater Part $\sqrt{18}$; (for according to the Definition in *Sect* 7. of this *Chap.* $\sqrt{2} . \sqrt{18} :: 1 . 3$, that is, as a Rational Number to a Rational Number) the propoſed Number $\sqrt{18}+4$ is a ſecond Binomial.

How to find out two ſuch Numbers as may conſtitute a ſecond Binomial.

1. By the foregoing Canon find out two ſquare Numbers, whoſe Difference may be ſome Rational Number not a Square; ſuch are theſe Squares . . . . $\begin{cases}9\\4\end{cases}$
2. Their Difference is 5
3. Take ſome Rational Number at pleaſure for the leſſer Part of the Binomial ſought, as 10
4. Then ſay, If 5 the Difference in the third ſtep gives 9 the greater of the two Squares in the firſt; what ſhall 100 the Square of the Number taken in the third give? Whence you will find 180, whoſe ſquare Root ſhall be the greater part, *viz* $\sqrt{180}$
5. I ſay, The Sum of the two Numbers found out in the third and fourth ſteps is a ſecond Binomial, *viz.* . . . . $\sqrt{180}+10$

The Definition of Binomial III.

When each of the two Parts of a Binomial is a Surd Square of a Rational Number, and the ſquare Root of the Difference of the Squares of the Parts is Commenſurable to the greater Part, the Sum of the two Parts is called a third Binomial.

Explication.

Let this Binomial be propoſed . . . . $\sqrt{50}+\sqrt{32}$

The Squares of the Parts are . . . . $\begin{cases}50\\32\end{cases}$

The Difference of thoſe Squares is 18

The ſquare Root of that Difference is . . . . $\sqrt{18}$

Becauſe the two Parts $\sqrt{50}$ and $\sqrt{32}$ are Surd ſquare Roots of two Rational Numbers 50 and 32, and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{18}$, is Commenſurable to the greater Part $\sqrt{50}$; (for $\sqrt{18} . \sqrt{50} :: 3 . 5$, that is, as a Rational Number to a Rational Number) the propoſed Number $\sqrt{50}+\sqrt{32}$ is a third Binomial.

How to find out two ſuch Numbers as may conſtitute a third Binomial.

1. Find out two ſquare Numbers whoſe Difference may be ſome Rational Number not a Square; ſuch are theſe Squares . . $\begin{cases}9\\4\end{cases}$
2. Their Difference is 5
3. Take ſome Rational Number not a Square, which may exceed the ſaid Differences 5 by an Unit or two, *viz.* by 1, when the ſaid Difference increaſed with 1 makes not a Square; but by 2, when the Difference increaſed with 1 makes a Square: So in this Example I take 6, becauſe 5+1 makes not a Square . . 6
4. Again, take ſome Rational Number at pleaſure, as 12

5. The ſquare thereof is 144
6. Then ſay, If 6 the Number taken in the third ſtep gives 9 the greater of the two ſquares in the firſt, what ſhall 144 the ſquare Number in the fifth give? whence the fourth Proportional is 216, whoſe ſquare Root, to wit $\sqrt{216}$, ſhall be the greater part. $\sqrt{216}$
7. Say again, If the ſaid Square 9 gives 5 the Difference in the ſecond ſtep, what ſhall 216 the fourth Proportional found out in the ſixth give? Whence you will find 120, whoſe ſquare Root, to wit $\sqrt{120}$, ſhall be the leſſer part . . . . . $\sqrt{120}$
8. I ſay, the ſum of the two Numbers found out in the ſixth. and ſeventh ſteps is a third Binomial, to wit, ... $\sqrt{216}+\sqrt{120}$

The Definition of Binomial IV.

When the greater part of a Binomial is a Rational Number, and the leſſer part is a Surd ſquare Root of a Rational Number, and the ſquare Root of the Difference of the ſquares of the parts is Incommenſurable to the greater part, the Sum of the two parts is called a fourth Binomial.

Explication.

Let this Binomial be propoſed . . . . $5+\sqrt{12}$
The Squares of the Parts are 25, 12
The Difference of thoſe Squares is 13
The ſquare Root of that Difference is . . . . $\sqrt{13}$

Becauſe the greater part 5 is a Rational Number, and the leſſer part $\sqrt{12}$ is a Surd ſquare Root of a Rational Number 12, and the ſquare Root of the Difference of the ſquares of the Parts, *viz.* $\sqrt{13}$, is Incommenſurable to the greater part 5; (for $\sqrt{13}$ has not ſuch proportion to 5 as a Rational Number to a Rational Number) the Number $5+\sqrt{12}$ above propoſed is a fourth Binomial.

How to find out two ſuch Numbers as may conſtitute a fourth Binomial.

1. Take any ſquare Number, as 9
2. Divide that ſquare Number 9 into two Numbers not ſquares, as into . . . . 6 and 3
3. Take a Rational Number at pleaſure for the greater part of the Binomial ſought, as 6
4. Then ſay, If 9 the ſquare Number in the firſt ſtep give 6 the greater of the two Numbers in the ſecond, what ſhall 36 the ſquare of the Number taken in the third give? So the fourth Proportional will be found 24, whoſe ſquare Root, to wit $\sqrt{24}$, ſhall be the leſſer part. . . . . $\sqrt{24}$
5. I ſay, The Sum of the two Numbers found out in the third and fourth ſteps is a fourth Binomial, *viz,* . . . . . $6+\sqrt{24}$

The Definition of Binomial V.

When the leſſer part of a Binomial is a Rational Number, and the greater part is a Surd ſquare Root of ſome Rational Number, and the ſquare Root of the Difference of the ſquares of the Parts is Incommenſurable to the greater part, the Sum of the two Parts is called a fifth Binomial.

Explication.

Let this Binomial be propoſed . . . . $\sqrt{6}+2$
The ſquares of the Parts are 6, 4
The Difference of thoſe ſquares is 2
The ſquare Root of the Difference is . . . . . . $\sqrt{2}$

Becauſe the leſſer part 2 is a Rational Number, and the greater part $\sqrt{6}$ is a Surd ſquare Root of a Rational Number 6, and the ſquare Root of the Difference of the ſquares of the parts, *viz.* $\sqrt{2}$, is Incommenſurable to the greater part $\sqrt{6}$; (for $\sqrt{2} . \sqrt{6} :: 1 . \sqrt{3}$, not as a Rational Number to a Rational Number) the propoſed Number $\sqrt{6}+2$ is a fifth Binomial.

How to find out two ſuch Numbers as may conſtitute a fifth Binomiol.

1. Take any ſquare Number, as	9
2. Divide that ſquare Number 9 into two Numbers not ſquares, as into .	6 and 3
3. Take a Rational Number at pleaſure for the leſſer part of the Binomial ſought, as	2
4. Then ſay, If 6 the greater of the two Numbers in the ſecond ſtep gives 9 the ſquare Number in the firſt; what ſhall 4 the ſquare of the Rational Number taken in the third give? Whence you will find the fourth Proportional 6, whoſe ſquare Root, to wit $\sqrt{6}$, ſhall be the greater part ſought, ...	$\sqrt{6}$
5. I ſay, The ſum of the two Numbers found out in the third and fourth ſteps is a fifth Binomial, *viz.*	$2+\sqrt{6}$

The Definition of Binomial VI.

When each of the two parts of a Binomial is a Surd ſquare Root of ſome Rational Number, and the ſquare Root of the Difference of the Squares of the Parts is Incommenſurable to the greater part, the Sum of the two parts is called a ſixth Binomial.

Explication.

Let this Binomial be propoſed	$\sqrt{5}+\sqrt{3}$
The Squares of the Parts are	5 3
The difference of the Squares of the Parts is . . .	2
The ſquare Root of that difference is	$\sqrt{2}$

Becauſe the two Parts $\sqrt{5}$ and $\sqrt{3}$ are Surd ſquare Roots of two Rational Numbers 5 and 3, and the ſquare Root of the Difference of the Squares of the Parts, *viz.* $\sqrt{2}$, is Incommenſurable to the greater part $\sqrt{5}$; (for $\sqrt{2}$ has not ſuch a proportion to $\sqrt{5}$ as a Rational Number to a Rational Number) the Number $\sqrt{5}+\sqrt{3}$ above propoſed is a ſixth Binomial.

How to find out two ſuch Numbers as may conſtitute a ſixth Binomial.

1. Take two ſuch prime Numbers that their Sum may not be a Square as	7 and 5
2. Their Sum is	12
3. Take alſo any ſquare Number, as	9
4. Take again ſome Rational Number at pleaſure, as	6
5. The ſquare thereof is	36
6. Then ſay, If 9 the ſquare Number taken in the third ſtep, gives 12 the ſum of the two prime Numbers in the firſt, what ſhall 36 the ſquare in the fifth ſtep give? Whence you will find 48, whoſe ſquare Root, to wit, $\sqrt{48}$, ſhall be the greater part, ...	$\sqrt{48}$
7. Say again, If 12 the ſum of the two prime Numbers in the firſt ſtep, gives 7 the greater of thoſe prime Numbers, what ſhall 48 the fourth Proportional found out in the ſixth ſtep give? Whence you will find 28, whoſe ſquare Root, *viz.* $\sqrt{28}$, ſhall be the leſſer part,	$\sqrt{28}$
I ſay, the ſum of the two Numbers found out in the ſixth and ſeventh ſteps is a ſixth Binomial, *viz.*	$\sqrt{48}+\sqrt{28}$

If of every one of thoſe ſix Binomials the leſſer part be ſubtracted from the greater by interpoſing the Sign —, the ſix Remainders anſwer to the ſix Lines which *Euclid* in *Prop.* 86, 87, 88, 89, 90, 91. of his *Elem.* 10. calls *Apotomes* or *Reſidual Lines*; as,

Out of Binomial		By changing + into — is made Reſidual	
I.	$3+\sqrt{5}$	I.	$3-\sqrt{5}$
II.	$\sqrt{18}+4$	II.	$\sqrt{18}-4$
III.	$\sqrt{50}+\sqrt{32}$	III.	$\sqrt{50}-\sqrt{32}$
IV.	$5+\sqrt{12}$	IV.	$5-\sqrt{12}$
V.	$\sqrt{6}+2$	V.	$\sqrt{6}-2$
VI.	$\sqrt{5}+\sqrt{3}$	VI.	$\sqrt{5}-\sqrt{3}$

The precedent Conſtructions of the ſaid ſix Binomials are demonſtrated in *Prop.* 49, 50, 51, 52, 53, 54. of 10 *Elem. Euclid.*

Now if any Binomial or Reſidual be given, we may eaſily find out another of the ſame kind in this manner, *viz.* For the firſt and fourth Binomials, if it be made as the greater Name or Part to the leſſer, ſo any Rational Number aſſumed for the greater Part of a new firſt or fourth Binomial, to a fourth Proportional Number, this Number ſhall be the leſſer Part of the new firſt or fourth Binomial. But for the ſecond and fifth, if it be made as the leſſer part to the greater, ſo any Rational Number taken for the leſſer part of a new ſecond or fifth Binomial to a fourth Proportional, the Number ſo produced ſhall be the greater part of the new ſecond or fifth Binomial. And laſtly, for the third and ſixth Binomials, if it be made as the greater Part to the leſſer, (each of which is a Surd ſquare Root) ſo any Surd ſquare Root aſſumed for the greater Part of a new third or ſixth Binomial, to a fourth Proportional, there will come forth the leſſer part of a new third or ſixth Binomial. (The reaſon of this Operation is manifeſt *per Prop.* 15. *Elem.* 10. *Eucl.*) And after a new Binomial is found out, its correſpondent Reſidual is alſo made by changing the Sign + into —, as before has been ſaid.

As for Example, if a firſt Binomial $3+\sqrt{5}$ be propoſed, to find another like to it; I take a Rational Number at pleaſure, as 8, for the greater Part of the Binomial ſought, then by the Rule of Three as 3 is to $\sqrt{5}$, ſo 8 to a fourth Proportional, to wit $\sqrt{\frac{320}{9}}$, for the leſſer Part ſought, therefore $8+\sqrt{\frac{320}{9}}$ ſhall be a new firſt Binomial, and $8-\frac{320}{9}$ a new firſt Reſidual; and ſo of the reſt.

Sect. XVI. *Concerning the Extraction of the ſquare Root out of Binomials and Reſiduals conſtituted in ſuch manner as has been ſhewn in the preceding* Sect. 15.

Every one of the Binomials and Reſiduals, whoſe Conſtruction has been ſhewn in the preceding *Sect.* 15. has a ſquare Root, that is, ſuch a Binomial or Reſidual that if it be multiplied into itſelf will produce the given Binomial or Reſidual; as may be evidently collected out of *Prop.* 55, 56, 57, 58, 59, and 60; alſo out of *Prop.* 92, 93, 94, 95, 96, and 97. of the tenth Book of *Euclid*'s Elements.

As for Example, a Binomial of the firſt kind, ſuppoſe $7+\sqrt{48}$, has a ſquare Root, to wit $2+\sqrt{3}$, for this being ſquared (or multiplied into itſelf) produces that Binomial $7+48$, whoſe greater Part 7 is compoſed of 4 and 3, the Squares of the Parts of the Root $2+\sqrt{3}$; and the leſſer part $\sqrt{48}$ is the double of the Product made by the Multiplication of 2 into $\sqrt{3}$, the Parts of the Root $2+\sqrt{3}$: all which is evident by the Multiplication of $2+\sqrt{3}$ into itſelf. The like effect will be found in every one of the reſt of the Binomials conſtituted in the preceding *Sect.* 15. Therefore if a Binomial be propoſed, and its ſquare Root deſired, there is given the Sum of the Squares of the Parts of the Root, (which Sum is the greater Part of the Binomial propoſed) and the double of the Product of the Parts of the Root (which double Product is the leſſer Part of the Binomial propoſed) to find out the two Parts of the Root ſeverally. And therefore in order to the Extraction of the ſquare Root of a Binomial, it will be requiſite to ſearch out a Canon for the ſolving of this following

QUESTION.

The Sum (b) of the Squares of two Numbers being given, as alſo (c) the double Product of the Multiplication of the ſame two Numbers, to find the Numbers ſeverally.

RESOLUTION.

1. For one of the two Numbers ſought put . . . . a
2. Then foraſmuch as the double of the Product of their Multiplication is given c, therefore the Product itſelf is . . . $\frac{c}{2}$
3. Which Product divided by the firſt Number a gives the other Number . . . . $\frac{c}{2a}$
4. Therefore the Square of the firſt Number is . . . . aa
5. And the Square of the other Number is . . . . . . $\frac{cc}{4aa}$
6. Therefore the Sum of the ſquares of the two Numbers is $aa+\frac{cc}{4aa}$

7. Which ſum muſt be equal to b the given ſum of the ſquares; hence this Equation, ... $aa+\frac{cc}{4aa}=b$

8. From this Equation after due Reduction, there will ariſe . $baa-aaaa=\frac{1}{4}cc$

9. And from the laſt Equation (*per* Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) there will ariſe this following Canon, to find out the two Numbers ſought, *viz.*

CANON 1.

$$\left\{\begin{array}{l}\sqrt{:\tfrac{1}{2}b+\sqrt{.\tfrac{1}{4}bb-\tfrac{1}{4}cc.}:}=\text{the greater Number.}\\ \sqrt{:\tfrac{1}{2}b-\sqrt{.\tfrac{1}{4}bb-\tfrac{1}{4}cc.}:}=\text{the leſſer Number.}\end{array}\right.$$

That is in words,

From a quarter of the ſquare of the given ſum of the ſquares, ſubtract a quarter of the ſquare of the double Product given, then add and ſubtract the ſquare Root of that Remainder to and from half the given ſum of the Squares, ſo ſhall the ſquare Roots of the Sum and Remainder of that Addition and Subtraction be the two Numbers ſought.

10. Moreover, becauſe $\frac{b+\sqrt{bb-cc}}{2}=\frac{1}{2}b+\sqrt{:\tfrac{1}{4}bb-\tfrac{1}{4}cc:}$

11. Therefore . . . . $\sqrt{:\frac{b+\sqrt{bb-cc}}{2}:}=\sqrt{:\tfrac{1}{2}b+\sqrt{.\tfrac{1}{4}bb-\tfrac{1}{4}cc}:}$

12. Likewiſe, becauſe $\frac{b-\sqrt{bb-cc}}{2}=\frac{1}{2}b-\sqrt{:\tfrac{1}{4}bb-\tfrac{1}{4}cc:}$

13. Therefore . . . . $\sqrt{:\frac{b-\sqrt{bb-cc}}{2}:}=\sqrt{\tfrac{1}{2}:b-\sqrt{:\tfrac{1}{4}bb-\tfrac{1}{4}cc}:}$

14. Therefore from the eleventh and thirteenth ſteps another Canon ariſes to ſolve the Queſtion, *viz.*

CANON 2.

$$\left\{\begin{array}{l}\sqrt{:\frac{b+\sqrt{bb-cc}}{2}:}=\text{the greater Number.}\\ \sqrt{:\frac{b-\sqrt{bb-cc}}{2}:}=\text{the leſſer Number.}\end{array}\right.$$

That is in words,

From the Square of the given Sum of the Squares ſubtract the Square of the double Product given, then add and ſubtract the ſquare Root of the Remainder to and from the given Sum of the Squares; ſo ſhall the ſquare Root of half the Sum and Remainder of that Addition and Subtraction be the two Numbers ſought.

By the help of either of thoſe Canons we may extract the ſquare Root of a Binomial or Reſidual, but I ſhall uſe the latter only, whence ariſes

A general Rule for the Extraction of the Square Root out of Binomials and Reſiduals.

From the Square of a greater part of a given Binomial or Reſidual ſubtract the Square of the leſſer, then add the ſquare Root of the Remainder to the greater part, and ſubtract it alſo from the ſame; laſtly, connect the ſquare Roots of the half of that Sum and Remainder by the Sign + if a Binomial be propoſed, but by — if a Reſidual: ſo you have the deſired ſquare Root of the given Binomial or Reſidual.

The Practice of this Rule will be ſhewn at large in the following Examples.

Example 1.

Let it be required to extract the ſquare Root out of this firſt Binomial $27+\sqrt{704}$.

The Operation.

1 From the Square of the greater part 27, *viz.* from . 729
2 Subtract the Square of the leſſer part $\sqrt{704}$, to wit, . 704
3. The Remainder is 25
4. The ſquare Root of that Remainder is 5

5. To which ſquare Root add the greater part 27
6. The Sum is 32
7. The half of that ſum is 16
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, ... 4
9. Then from the greater part of the given Binomial, *viz.* from . . 27
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, . 5
11. The Remainder is 22
12. The half of which Remainder is 11
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\surd 11$
14. I ſay, the two Names or Parts in the eighth and thirteenth ſteps being connected by + ſhall be the ſquare Root ſought, to wit. $4+\surd 11$

But if — inſtead of + be prefix'd to the leſſer part of the ſaid Root, it will give $4-\surd 11$, which is the ſquare Root of the firſt Reſidual or Apotome $27-\surd 704$.

The former of thoſe two Roots anſwers to the Irrational Line called (in *Prop.* 37. & 55. *lib.* 10. *Elem. Eucl.*) a *Binomial Line*, and the latter anſwers to the Irrational Line called (in *Prop.* 74. & 92.) an *Apotome* or *Reſidual Line.*

The Proof of the Root above extracted out of the firſt Binomial is made by multiplying the Root into itſelf thus:

The Sum of the Squares of the Parts of $4+\surd 11$, the Root found out is . . . . $16+11$, that is, 27

The Product of the ſame Parts multiplied one into the other is . . . . $4\surd 11$, that is, $\surd 176$

The double of the ſaid Product is . . . . $8\surd 11$, that is, $\surd 176$.

The Sum of the ſaid Sum of the Squares of the Parts and the double Product is . . . . $27+\surd 704$

Whence it is manifeſt that $27+\surd 704$ is the Square of $4+\surd 11$, therefore this is the true ſquare Root of that firſt Binomial; which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid Sum of the Squares of the Parts, the Remainder $27-\surd 704$ is the Square of $4-\surd 11$; therefore this is the ſquare Root of that firſt Reſidual.

Example 2.

Let it be required to extract the ſquare Root of this ſecond Binomial . $\surd\frac{147}{4}+6$.

The Operation.

1. From the Square of the greater part $\surd\frac{147}{4}$, *viz.* from . . $\frac{147}{4}$
2. Subtract the Square of the leſſer part 6, to wit 36
3. The Remainder is . . . . $\frac{3}{4}$
4. The ſquare Root of that Remainder is . . . . $\surd\frac{3}{4}$
5. To which ſquare Root add the greater part, (by the Rule in *Sect.* 8. of this *Chap.*) . . . . $\surd\frac{147}{4}$
6. The Sum is . . . . $\surd 48$
7. The half of which Sum is . . . . $\surd 12$
8. The ſquare Root of that half Sum is the greater part of the Root ſought, to wit, ... $\surd(4)12$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\surd\frac{147}{4}$
10. Subtract the ſquare Root before found in the fourth ſtep, (by the ſaid Rule in *Sect.* 8.) *viz.* . . . . . . $\surd\frac{3}{4}$
11. The Remainder is . . . . $\surd 27$
12. The half of which Remainder is . . . . $\surd\frac{27}{4}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit; . . . . . $\surd(4)\frac{27}{4}$
14. I ſay, the two parts in the eighth and thirteenth ſteps being connected by the Sign + ſhall be the Root ſought, to wit, ... $\surd(4)12+\surd(4)\frac{27}{4}$

And if — inſtead of + be prefix'd to the leſſer part of the ſaid Root, it will give $\surd(4)12-\surd(4)\frac{27}{4}$, which is the ſquare Root of the ſecond Reſidual $\frac{147}{4}-6$.

The former of thoſe two Roots anſwers to the Irrational Line called (in *Prop.* 38. *&* 56. *lib.* 10. *Elem. Eucl.*) a *firſt Bimedial*, and the latter anſwers to the Irrational Line called (in *Prop.* 75. *&* 93.) a *firſt Medial Reſidual.*

The Proof of the Root above extracted out of the ſecond Binomial.

The Squares of the parts of $\surd(4)12+\surd(4)\frac{27}{4}$ the Root found out are . . . . $\surd 12$ and $\surd\frac{27}{4}$

Which Squares added together (as in Example 6. *Sect.* 8. of this *Chap.* is manifeſt) makes the Sum . . . . . $7\surd\frac{3}{4}$, that is, $\surd\frac{147}{4}$

The Product of the parts, *viz.* $\surd(4)12$ into $\surd(4)\frac{27}{4}$ is $\surd(4)81$, that is, 3.

The double of the ſaid Product is 6

Therefore the Sum of the Sum of the Squares of the parts and the ſaid double Product is . . . . $\surd\frac{147}{4}+6$

Whence it is manifeſt that $\surd\frac{147}{4}+6$ is the Square of $\surd(4)12+\surd(4)\frac{27}{4}$, therefore this is the true ſquare Root of that ſecond Binomial, which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid Sum of the Squares of the Parts, the Remainder $\surd\frac{147}{4}-6$ is the ſquare of $\surd(4)12-\surd(4)\frac{27}{4}$; therefore this is the ſquare Root of that ſecond Reſidual.

Example 3.

Let it be required to extract the ſquare Root of this third Binomial $\surd\frac{245}{3}+\surd 80$.

The Operation.

1. From the Square of the greater part $\surd\frac{245}{3}$, *viz.* from . . . $\frac{245}{3}$
2. Subtract the ſquare of the leſſer part, to wit, ... 80
3. The Remainder is . . . . $\frac{5}{3}$
4. The ſquare Root of that Remainder is . . . . $\surd\frac{5}{3}$
5. To which ſquare Root add the greater part . . . . . . $\surd\frac{245}{3}$
6. The Sum is . . . . $\surd\frac{320}{3}$
7. The half of which Sum is . . . . $\surd\frac{80}{3}$
8. The ſquare Root of that half Sum is the greater part of the Root ſought, to wit, ... $\surd(4)\frac{80}{3}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\surd\frac{245}{3}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... $\surd\frac{5}{3}$
11. The Remainder is . . . . $\surd 60$
12. The half of which Remainder is . . . . $\surd 15$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\surd(4)15$
14. I ſay, the two parts in the eighth and thirteenth ſteps being connected by + ſhall be the ſquare Root ſought, to wit, ... $\surd(4)\frac{80}{3}+\surd(4)15$

And if — inſtead of + be prefix'd to the leſſer part of the ſaid Root, it gives $\surd(4)\frac{80}{3}-\surd(4)15$, which is the ſquare Root of the third Reſidual $\surd\frac{245}{3}-\surd 80$.

The former of thoſe two Roots anſwers to the Irrational Line called (in *Prop.* 39. *&* 57. *lib.* 10. *Elem. Eucl.*) a *ſecond Bimedial*, and the latter anſwers to the Irrational Line called (in *Prop.* 76. *&* 94.) a *ſecond Medial Reſidual.*

The Proof of the Root above extracted out of the third Binomial.

The Squares of the Parts of $\surd(4)\frac{80}{3}+\surd(4)15$, the Roots found out, are . . . . $\surd\frac{80}{3}$ and $\surd 15$

Which Squares added together make . . . . $7\surd\frac{5}{3}$, that is, $\surd\frac{245}{3}$

The Product of the parts, *viz.* $\surd(4)\frac{80}{3}$ into $\surd(4)15$ is $\surd(4)400$, that is, $\surd 20$

The double of the ſaid Product is . . . . $\surd 80$

Therefore the Sum of the Sum of the Squares of the parts and the ſaid double Product is . . . . $\surd\frac{245}{3}+\surd 80$

Whence it is manifeſt, that $\surd\frac{245}{3}+\surd 80$ is the Square of $\surd(4)\frac{80}{3}+\surd(4)15$; therefore this is the ſquare Root of that third Binomial: which was to be proved

Moreover, if the ſaid double Product be ſubtracted from the ſaid Sum of the Squares of the Parts, the Remainder $\sqrt{\tfrac{245}{3}}-80$ is the Square of $\sqrt{(4)}\tfrac{80}{3}-\sqrt{(4)}15$; therefore this is the ſquare Root of that third Reſidual.

Example 4.

Let it be required to extract the ſquare Root of this fourth Binomial . $7+\sqrt{20}$.

The Operation.

1. From the Square of the greater part 7, *viz.* from 49
2. Subtract the Square of the leſſer part $\sqrt{20}$, to wit, . . . 20
3. The Remainder is 29
4. The ſquare Root of that Remainder is . . . . $\sqrt{29}$
5. To which ſquare Root add the greater part 7
6. The Sum is . . . . $7+\sqrt{29}$
7. The half of which is . . . . $\tfrac{7}{2}+\sqrt{\tfrac{29}{4}}$
8. The ſquare Root of that half Sum is the greater part of the Root ſought, to wit, ... $\sqrt{:\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}:}$
9. Again, from the greater part of the given Binomial, *viz.* from 7
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... $\sqrt{29}$
11. The Remainder is . . . . $7-\sqrt{29}$
12. The half of which Remainder is . . . . $\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{:\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}:}$
14. I ſay, the two parts in the eighth and thirteenth ſteps (the former of which is a Binomial, and the latter a Reſidual) being connected by + ſhall be the ſquare Root ſought, to wit, ... $\sqrt{:\tfrac{7}{2}+\sqrt{\tfrac{29}{4}}:}+\sqrt{:\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}:}$

Which Root anſwers to the Irrational Line called (in *Prop.* 40. & 58. *lib.* 10. *Elem. Eucl.*) a *Major Line.*

And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by interpoſing the Sign —, it gives $\sqrt{:\tfrac{7}{2}+\sqrt{\tfrac{29}{4}}:}-\sqrt{:\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}:}$ which is the Root of the fourth Reſidual $7-\sqrt{20}$, and anſwers to the Irrational Line called (in *Prop.* 77. & 95. *lib.* 10. *Elem. Eucl*) a *Minor Line.*

The Proof of the Root above extracted out of the fourth Binomial.

The Squares of the Parts of the Root found out are . $\tfrac{7}{2}+\sqrt{\tfrac{29}{4}}$ and $\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}$

Therefore the Sum of the Squares of the Parts is . . . $\tfrac{7}{2}+\tfrac{7}{2}$, that is, 7.

The Product of the Parts will be found (by Rule 2. *Sect.* 12. of this *Chap.*) . . . . $\sqrt{:\tfrac{49}{4}-\tfrac{29}{4}:}$ that is, $\sqrt{5}$

The double of the ſaid Product is . . . . $\sqrt{20}$

Therefore the Sum of the ſaid Sum of the Squares of the Parts and the ſaid double Product is . . . . $7+\sqrt{20}$

Whence it is manifeſt that $7+\sqrt{20}$ is the Square of $\sqrt{:\tfrac{7}{2}+\sqrt{\tfrac{29}{4}}:}+\sqrt{:\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Binomial; which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid Sum of the Squares of the Parts, the Remainder $7-\sqrt{20}$ is the Square of $\sqrt{:\tfrac{7}{2}+\sqrt{\tfrac{29}{4}}:}-\sqrt{:\tfrac{7}{2}-\sqrt{\tfrac{29}{4}}:}$ therefore this is the ſquare Root of that fourth Reſidual $7-\sqrt{20}$.

Example 5.

Let it be required to extract the ſquare Root out of this fifth Binomial $\sqrt{20}+4$.

The Operation.

1. From the Square of the greater part $\sqrt{20}$, *viz.* from . . 20
2. Subtract the Square of the leſſer part 4, to wit, . . . 16
3. The Remainder is 4
4. The ſquare Root of that Remainder is 2
5. To which ſquare Root add the greater part . . . . . $\sqrt{20}$

6. The Sum is . . . . $\sqrt{20}+2$
7. The half of that Sum is . . . . $\sqrt{5}+1$
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, ... $\sqrt{:\sqrt{5}+1:}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\sqrt{20}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... 2
11. The Remainder is . . . . $\sqrt{20}-2$
12. The half of which Remainder is . . . . $\sqrt{5}-1$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{:\sqrt{5}-1:}$
14. I ſay, The two Parts in the eighth and thirteenth ſteps, (the former of which Parts is a Binomial, and the latter a Reſidual) being connected by $+$ ſhall be the ſquare Root ſought, to wit, ... $\sqrt{:\sqrt{5}+1:}+\sqrt{:\sqrt{5}-1:}$

Which Root anſwers to the Irrational Line called (in *Prop.* 41. & 59. *lib.* 10. *Elem. Eucl.*) a Line containing in Power a *Rational* and a *Medial Rectangle.* And if the leſſer Name of the ſaid Root be ſubtracted from the greater, by the interpoſition of the Sign $-$, it gives $\sqrt{:\sqrt{5}+1:}-\sqrt{:\sqrt{5}-1:}$ which is the ſquare Root of the fifth Reſidual $\sqrt{20}-4$, and anſwers to the Irrational Line which (in *Prop.* 78 & 96. *lib.* 10.) is called a Line making with a *Rational Space* the whole *Space Medial.*

The Proof of the Root above Extracted out of the fifth Binomial.

The ſquares of the Parts of $\sqrt{:\sqrt{5}+1:}+\sqrt{:\sqrt{5}-1:}$ (the Root found out) are . . . . $\sqrt{5}+1$ and $\sqrt{5}-1$
Therefore the Sum of the ſaid Squares of the Parts is $\sqrt{5}+\sqrt{5}$, that is, $\sqrt{20}$
The Product of the Parts multiplied one into the other (according to Rule 2. *Sect* 12. of this *Chap.*) is $\sqrt{:5-1:}$ that is, 2.
The double of the ſaid Product is . . . . 4.
Therefore the Sum of the ſaid Sum of the Squares of the Parts and double Product is . . . . $\sqrt{20}+4$

Whence it is manifeſt that $\sqrt{20}+4$ is the Square of $\sqrt{:\sqrt{5}+1:}+\sqrt{:\sqrt{5}-1:}$ therefore this is the ſquare Root of that fifth Binomial; which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid Sum of the Squares of the Parts, the Remainder $\sqrt{20}-4$ is the ſquare of $\sqrt{:\sqrt{5}+1:}-\sqrt{:\sqrt{5}-1:}$ therefore this is the ſquare Root of the ſaid fifth Reſidual $\sqrt{20}-4$.

Example 6.

Let it be required to extract the ſquare Root of this ſixth Binomial $\sqrt{20}+\sqrt{8}$.

The Operation.

1. From the ſquare of the greater Part $\sqrt{20}$, *viz.* from 20
2. Subtract the ſquare of the leſſer part $\sqrt{8}$, to wit, ... 8
3. The Remainder is 12
4. The ſquare Root of that Remainder is . . . . $\sqrt{12}$
5. To which ſquare Root add the greater Part . . . . $\sqrt{20}$
6. The Sum is . . . . $\sqrt{20}+\sqrt{12}$
7. The half of which Sum is . . . . $\sqrt{5}+\sqrt{3}$
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, . . . $\sqrt{:\sqrt{5}+\sqrt{3}:}$
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $\sqrt{20}$
10. Subtract the ſquare Root before found in the fourth ſtep, *viz.* . . . . $\sqrt{12}$
11. The Remainder is . . . . $\sqrt{20}-\sqrt{12}$
12. The half of which Remainder . . . . $\sqrt{5}-\sqrt{3}$

13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, } $\sqrt{:\sqrt{5}-\sqrt{3}:}$

14. I ſay, the two Parts in the eighth and thirteenth ſteps (the former of which Parts is a Binomial, and the latter a Reſidual) being connected by + ſhall be the ſquare Root ſought, to wit, . . } $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$

Which Root anſwers to the Irrational Line which (in *Prop.* 42. & 60. *lib.* 10. *Elem. Eucl.*) is called a Line containing in Power two *Medial Rectangles.* And if the leſſer part of the ſaid Root be ſubtracted from the greater, by the interpoſing of the Sign —, it gives $\sqrt{:\sqrt{5}+\sqrt{3}:}-\sqrt{:\sqrt{5}-\sqrt{3}:}$ which is the Root of the ſixth Reſidual $\sqrt{20}-\sqrt{8}$, and anſwers to the Irrational Line which (in *Prop.* 79. & 97. *lib.* 10. *Eucl.*) is called a Line making with a *Medial Rectangle* a whole *Space Medial.*

The Proof of the Root above extracted out of the ſixth Binomial.

The Squares of the Parts of $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$ the Root ſought are . . . . } $\sqrt{5}+\sqrt{3}$ and $\sqrt{5}-\sqrt{3}$

Therefore the Sum of the ſaid Squares of the Parts is $\sqrt{5}+\sqrt{5}$, that is, $\sqrt{20}$

The Product of the Parts multiplied one into the other is $\sqrt{:5-3:}$ that is, $\sqrt{2}$.

The double of the ſaid Product is . . . . $\sqrt{8}$.

Therefore the Sum of the ſaid Sum of the Squares of the Parts and double Product is . . . . } $\sqrt{20}+\sqrt{8}$.

Whence it is manifeſt that $\sqrt{20}+\sqrt{8}$ is the Square of $\sqrt{:\sqrt{5}+\sqrt{3}:}+\sqrt{:\sqrt{5}-\sqrt{3}:}$ therefore this is that ſquare Root of the ſixth Binomial; which was to be proved. Moreover, if the ſaid double Product be ſubtracted from the ſaid ſum of the Squares of the Parts, the Remainder $\sqrt{20}-\sqrt{8}$ is the Square of $\sqrt{:\sqrt{5}+\sqrt{3}:}-\sqrt{:\sqrt{5}-\sqrt{3}:}$ therefore this is the ſquare Root of that ſixth Reſidual.

Note. In every Binomial and Reſidual conſtituted according to the preceding *Sect.* 15. the ſquare Root of the Difference of the Squares of the Names or Parts is equal to the Difference of the Squares of the Parts of the Root of the Binomial or Reſidual.

As in the firſt Binomial $27+\sqrt{704}$, whoſe ſquare Root has before been found $4+\sqrt{11}$, the Square of 27, to wit 729, exceeds 704, the Square of $\sqrt{704}$ by 25, whoſe ſquare Root 5 is equal to the Difference of the Squares of the Parts of the Root of the Binomial propoſed, to wit, the Difference between 16 and 11.

This Property may be demonſtrated thus; let $b+\sqrt{d}$ repreſent a Binomial Root, whoſe greater Part is b; then the Square of that Root is $bb+2b\sqrt{d}+d$, this divided into its Names or Parts makes the Binomial $bb+d$ more $2b\sqrt{d}$; then the Squares of the Parts of this Binomial are $bbbb+2bbd+dd$ and $4bbd$, and the Difference of thoſe Squares is $bbbb-2bbd+dd$, whoſe ſquare Root $bb-d$ is manifeſtly the Difference of the Squares of the Parts of the Root $b+\sqrt{d}$ firſt propoſed; which was to be ſhewn. The like Property may be demonſtrated in a Reſidual.

How to extract the Square Root out of a Binomial deſign'd by Letters, if it has a Binomial Root.

By the ſame general Rule which has before been exercis'd in extracting the ſquare Root out of Binomials expreſs'd by Numbers, we may extract the ſquare Root out of a Binomial deſign'd by Letters, when it has a Binomial Root, as will be evident by the following Examples; where for the more apparent diſtinction of the Parts of the given Binomial, inſtead of + I ſet the Word [*more*] between the Parts, and inſtead of — I ſet the Word [*leſs*] between the Parts of a given Reſidual.

Example 1.

Let it be required to extract the ſquare Root out of this Binomial . . . . } $bb+d$ more $2b\sqrt{d}$

The Operation.

1. From the Square of the greater part, (which ſuppoſe to be $bb+d$) *viz.* from . . . . } $bbbb+2dbb+dd$
2. Subtract the Square of the leſſer part $2b\sqrt{d}$, to wit, . . $+5dbb$

3. The Remainder is . . . . $bbbb-2dbb+dd$
4. The ſquare Root of that Remainder is . . . . $bb-d$
5. To which ſquare Root add the greater part, to wit, . . $bb+d$
6. The Sum is . . . . $2bb$
7. The half of which Sum is . . . . bb
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, ... b
9. Then from the greater part of the given Binomial, *viz.* from $bb+d$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, $bb-d$
11. The Remainder is . . . . $+2d$
12. The half of which Remainder is . . . . $+d$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, ... $\sqrt{d}$
14. I ſay, The two Parts in the eighth and thirteenth ſteps being connected by the Sign +, ſhall be the ſquare Root ſought, to wit, $b+\sqrt{d}$

Which Root being ſquared, or multiplied into it ſelf, will evidently produce the given Binomial $bb+d$ more $2b\sqrt{d}$.

Example 2.

Let it be required to extract the ſquare Root out of this Binomial, ... $mm+\frac{pxx}{m}$ more $x\sqrt{4mp}$

The Operation.

1. From the Square of the greater part $mm+\frac{pxx}{m}$ *viz.* from . . . . $mmmm+2mpxx+\frac{ppxxxx}{mm}$
2. Subtract the ſquare of the leſſer part $x\sqrt{4mp}$, to wit, . $+4mpxx$
3. The Remainder is . . . . $mmmm-2mpxx+\frac{ppxxxx}{mm}$
4. The Square of that Remainder is . . . . . $mm-\frac{pxx}{m}$
5. To which ſquare Root add the greater part, to wit, $mm+\frac{pxx}{m}$
6. The Sum is . . . . $2mm$
7. The half of which Sum is . . . . mm
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, . . . m
9. Again, from the greater part of the given Binomial, *viz.* from . . . . $mm+\frac{pxx}{m}$
10. Subtract the ſquare Root before found in the fourth ſtep, to wit, ... $mm-\frac{pxx}{m}$
11. The Remainder is . . . . $+\frac{2pxx}{m}$
12. The half of which Remainder is . . . . $+\frac{pxx}{m}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit, . . . $\sqrt{\frac{pxx}{m}}$ or $x\sqrt{\frac{p}{m}}$
14. I ſay, the two Parts in the eighth and thirteenth ſteps being connected by + ſhall be the ſquare Root ſought, to wit, ... $m+x\sqrt{\frac{p}{m}}$

Which Binomial Root being ſquared, or multiplied into it ſelf, will produce the given Binomial.

Example 3.

Let it be required to extract the ſquare Root out of this Binomial, ... $\overline{a+b}\sqrt{ab}$ more $2ab$

The Operation.

1. From the ſquare of the greater part, *viz.* from $aaab+2aabb+abbb$
2. Subtract the Square of the leſſer part, to wit, $+4aabb$
3. The Remainder is $aaab-2aabb+abbb$
4. The ſquare Root of that Remainder is $\overline{a-b}\sqrt{ab}$
5. To which ſquare Root add the greater part, to wit, $\overline{a+b}\sqrt{ab}$
6. The Sum is $2a\sqrt{ab}$
7. The half of which Sum is $a\sqrt{ab}$
8. The ſquare Root of the ſaid half Sum is the greater part of the Root ſought, to wit, $\sqrt{:a\sqrt{ab}:}$ or $\sqrt{(4)}aaab$
9. Again, from the greater part of the given Binomial, *viz* from $\overline{a+b}\sqrt{ab}$
10. Subtract the ſquare Root before found in the fourth ſtep, *viz.* $\overline{a-b}\sqrt{ab}$
11. The Remainder is $2b\sqrt{ab}$
12. The half of which Remainder is $b\sqrt{ab}$
13. The ſquare Root of the ſaid half Remainder is the leſſer part of the Root ſought, to wit $\sqrt{:b\sqrt{ab}:}$ or $\sqrt{(4)}abbb$
14. I ſay, the two Parts in the eighth and thirteenth ſteps, being connected by $+$, ſhall be the ſquare Root ſought, to wit, $\sqrt{:a\sqrt{ab}:}+\sqrt{:b\sqrt{ab}:}$
15. Which Binomial Root may be alſo expreſs'd thus $\sqrt{(4)}aaab+\sqrt{(4)}abbb$

The Proof may be made by multiplying the Root found out into it ſelf.

Example 4.

Again, if the ſquare Root of this Reſidual be deſired $\overline{a+d}\sqrt{bc}$ leſs $2\sqrt{abcd}$
The Root being extracted by the precedent Method will be found $\sqrt{:a\sqrt{bc}:}-\sqrt{:d\sqrt{bc}:}$
Which Root may be alſo expreſs'd thus $\sqrt{(4)}aabc-\sqrt{(4)}ddbc$

But if it happen that when the Square of the leſſer part of the given Binomial or Reſidual is ſubtracted from the ſquare of the greater part, the ſquare Root of the Remainder and the greatet part are not commenſurable, (according to the Definition before given in *Sect* 7. of this *Chap.*) there is no more to be done in ſuch a caſe, but to prefix before the given Binomial or Reſidual the Sign $\sqrt{}$, with a Line drawn over both its Parts, to denote the univerſal ſquare Root of the given Binomial or Reſidual. As to extract the ſquare Root out of this Reſidual $\sqrt{:\tfrac{1}{4}aa+bb:}-\tfrac{1}{2}a$, I write $\sqrt{:\sqrt{\tfrac{1}{4}aa+bb}-\tfrac{1}{2}a:}$ which kind of Roots are commonly called Univerſal.

Sect. 17. *Queſtions to exerciſe the foregoing Rules of this Chapter.*

QUESTION. 1.

To divide 100 into two ſuch parts, that if each part be divided by the other part, the Sum of the Quotient may make 3.

RESOLUTION.

1. For one of the parts ſought put a
2. Then conſequently the other part is $100-a$
3. Therefore according to the import of the Queſtion this Equation ariſes, *viz.* $\frac{a}{100-a}+\frac{100-a}{a}=3$
4. Which Equation duly reduced gives $1000a-aa=2000$
5. Wherefore by reſolving the ſaid Equation by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two values of a, which are the deſired parts of 100, will be found theſe, to wit, $a=\begin{cases}50+10\sqrt{5}\\50-10\sqrt{5}\end{cases}$

6. The Sum of the ſaid Parts or Numbers found out is manifeſtly 100, ſo it remains only to prove that,

$$\frac{50+10\sqrt{5}}{50-10\sqrt{5}}+\frac{50-10\sqrt{5}}{50+10\sqrt{5}}=3$$

The Proof.

7. To add thoſe two Surd Fractions in the ſixth ſtep into one Sum, reduce them to a common Denominator, *viz.* multiply $50+10\sqrt{5}$ by $50+10\sqrt{5}$, and the Product (by the firſt of the three compendious Rules in *Sect.* 10. of this *Ch.*) will be found $\}$ $3000+1000\sqrt{5}$

8. Likewiſe multiply $50-10\sqrt{5}$ by $50-10\sqrt{5}$ and the Product (by the ſecond of the ſaid three Rules (will be . . $\}$ $3000-1000\sqrt{5}$

9. Then take the Sum of thoſe two Products for the Numerator of a Fraction or a Dividend, to wit, ... $\}$ 6000

10. Alſo multiply the two Denominators of the Surd Fractions in the ſixth ſtep one by the other, (according to the laſt of the three Rules above cited) and take the Product for a Denominator or Diviſor, *viz.* . . . . $\}$ 2000

11. Laſtly, the Numerator in the ninth ſtep being ſet over the Denominator in the tenth gives the Sum of the two Surd Fractions or Quotients in the ſixth ſtep, *viz.* . . . $\}$ $\frac{6000}{2000}=3$

Which Sum is manifeſtly 3, as was to be proved.

Another Proof.

The Quotient that ariſes by dividing $50+10\sqrt{5}$ by $50-10\sqrt{5}$ (according to the Rule of Diviſion in the ſixth branch of *Sect.* 11. of this *Chap.*) is . . . . $\}$ $\frac{3}{2}+\sqrt{\frac{5}{4}}$

Likewiſe the Quotient that ariſes by dividing $50-10\sqrt{5}$ by $50+10\sqrt{5}$ is . . . . $\}$ $\frac{3}{2}-\sqrt{\frac{5}{4}}$

The Sum of thoſe two Quotients is manifeſtly 3, (as before.)

QUESTION. 2.

To divide a given Number (ſuppoſe 6) into three ſuch unequals Numbers in continual proportion, that the Sum of the Squares of the Extremes may be to the Square of the Mean in a given proportion; but the firſt Term of this proportion muſt exceed the double of the latter Term. Let it therefore be deſired that the Sum of the Squares of the Extremes may be to the ſquare of the Mean as 3 to 1.

RESOLUTION.

1. For the mean Proportional put . . . . a
2. Then becauſe the ſum of all the three Proportionals muſt make 6, and the Mean is a, the ſum of the Extremes ſhall be $\}$ $6-a$
3. Therefore the ſquare of the ſum of the Extremes is . . $36-12a+aa$
4. But (by *Theor.* 3. *Chap.* 6. of this Book) the ſquare of the Sum of the Extremes of three Numbers continually proportional is equal to the ſquares of the Extremes, together with the double ſquare of the Mean; therefore from the ſquare in the third ſtep I ſubtract $2aa$ (the double ſquare of the Mean) and there remains the ſum of the ſquares of the Extremes, to wit, ... $\}$ $36-12a-aa$
5. But (according to the Queſtion) the ſum of the ſquares of the Extremes muſt be equal to the triple ſquare of the Mean; therefore from the fourth and firſt ſtep this Equation ariſes, *viz.* . . . . $\}$ $36-12a-aa=3aa$
6. From which Equation after due Reduction this ariſes, *viz.* $aa+3a=9$
7. Therefore by reſolving the laſt Equation (according to the Canon in *Sect.* 6 *Chap.* 15.) the value of a, that is, the mean Proportional ſought will be diſcovered, *viz.* . $\}$ $\sqrt{\frac{45}{4}}-\frac{3}{2}=$ the Mean

8. And from the ſeventh and ſecond ſteps the Sum of the Extremes will be alſo made known, *viz.* $\left.\right\}\frac{15}{2}-\sqrt{\frac{45}{4}}=$ Sum of the Extremes.

9. Then (as is manifeſt by *Queſt.* 4. *Chap.* 16. *Book* 1.) the Sum of the Extremes of three Numbers continually proportional being given, as alſo the Mean, the Extremes ſhall be given ſeverally by this following

CANON.

From the Square of half the Sum of the Extremes ſubtract the Square of the Mean, and extract the ſquare Root of the Remainder; then this ſquare Root being added to, and ſubtracted from the ſaid half Sum, will give the Extremes ſeverally. Therefore,

10. From the ſquare of the half of $\frac{15}{2}-\sqrt{\frac{45}{4}}$, that is, from . . $\frac{135}{8}-\frac{15}{4}\sqrt{\frac{45}{4}}$
11. Subtract the ſquare of $\frac{45}{4}-\frac{3}{2}$, *viz.* . . . . $\frac{108}{8}-\frac{12}{4}\sqrt{\frac{45}{4}}$
12. The Remainder is . . . . $\frac{27}{8}-\frac{3}{4}\sqrt{\frac{45}{4}}$
13. The ſquare Root of that Remainder being extracted (by the general Rule before delivered in *Sect.* 16. of this *Chap.* for extracting the ſquare Root out of Binomials) will be found . $\left.\right\}$ $\sqrt{\frac{45}{16}}-\frac{3}{4}$
14. Which ſquare Root added to the half of $\frac{15}{2}-\sqrt{\frac{45}{4}}$, gives the greater Extreme ſought, to wit, ... $\left.\right\}$ . . 3
15. But the ſaid ſquare Root ſubtracted from the half of $\frac{15}{2}-\sqrt{\frac{45}{4}}$, leaves the leſſer Extreme, to wit . . . . . . $\left.\right\}$ . . $\frac{9}{2}-\sqrt{\frac{45}{4}}$
16. Wherefore (in the ſeventh, fourteenth and fifteenth ſteps) three Numbers continually proportional are found out, *viz.* 3, $\sqrt{\frac{45}{4}}-\frac{3}{2}$, and $\frac{9}{2}-\sqrt{\frac{45}{4}}$, whoſe Sum is 6; and the Sum of the Squares of the Extremes is equal to the triple of the Square of the Mean, as will appear by

The Proof.

Firſt, The Product made by the Multiplication of the firſt and third Numbers one into the other, that is, of 3 into $\frac{9}{2}-\sqrt{\frac{45}{4}}$, is $\frac{27}{2}-3\sqrt{\frac{45}{4}}$, which is alſo the ſquare of the ſecond Number $\sqrt{\frac{45}{4}}-\frac{3}{2}$, (as will eaſily appear by Multiplication;) therefore the ſaid three Numbers are Proportionals.

Secondly, The Sum of the ſaid three proportional Numbers is 6; for the Mean $\sqrt{\frac{45}{4}}-\frac{9}{2}$ added to $\frac{9}{2}-\sqrt{\frac{45}{4}}$ the leſſer Extreme, makes 3, to which adding the greater Extreme 3, the Sum is 6.

Thirdly, The Sum of the Squares of the Extremes 3 and $\frac{9}{2}-\sqrt{\frac{45}{4}}$ is equal to the triple of the Square of the Mean $\sqrt{\frac{45}{4}}-\frac{3}{2}$; for the ſaid Sum, as alſo the ſaid triple Square will by Multiplication be found $\frac{81}{2}-9\sqrt{\frac{45}{4}}$. Therefore all the Conditions in the Queſtion are ſatisfied.

But that the neceſſity of Determination annexed to the Queſtion may be made manifeſt, it remains to prove, that if three unequal Numbers be in continual proportion, the Sum of the Squares of the Extremes is greater than the double of the Square of the Mean. Therefore,

Let three unequal Numbers in continual proportion be expoſed, ſuppoſe theſe, ... $\left.\right\}$ $a\,,\ \sqrt{ae}\,,\ e \div$

Then their Squares ſhall be alſo Proportionals, (*per* 23 *Prop.* 6 *Elem. Eucl.*) *viz.* . . . . $\left.\right\}$ $aa\,.\,ae :: ae\,.\,ee$

Therefore (by 25 *Prop.* 5. *Elem. Eucl.*) . . . . $aa+ee \sqsubset 2ae$

But $aa+ee$ is the Sum of the Squares of the Extremes of the three Proportionals expoſed, and $2ae$ is equal to the double Square of the Mean proportional; wherefore the Theorem is proved, and conſequently the Determination is manifeſtly neceſſary to be annexed to the Queſtion propoſed, that there may be a poſſibility of finding out what is thereby deſired. The Determination may alſo be eaſily inferr'd from the Canon in the foregoing ninth ſtep.

QUESTION 3.

What is the Product made by the continual Multiplication of theſe four Numbers one into another, which differ by an equal Exceſs, to wit, Unity? . . . . .

$$\left\{\begin{array}{l}\sqrt{:\frac{5}{4}+\sqrt{101}:}-\frac{3}{2}\\ \sqrt{:\frac{5}{4}+\sqrt{101}:}-\frac{1}{2}\\ \sqrt{:\frac{5}{4}+\sqrt{101}:}+\frac{1}{2}\\ \sqrt{:\frac{5}{4}+\sqrt{101}:}+\frac{3}{2}\end{array}\right.$$

Anſw. The deſired Product is exactly 100

For, (by the laſt of the three compendious Rules before delivered in *Sect.* 10. of this *Chap.* for the Multiplication of Binomials and Reſiduals) the Product of the firſt and fourth Number is . $\sqrt{101}-1$

Likewiſe the Product of the ſecond and third Number is . . $\sqrt{101}+1$

Laſtly, the two laſt preceding Products being multiplied one into another (by the ſame Rule) make 100

QUESTION 4.

1. If a, b, c, be ſuch Quantities, that . . . . . . $aa+ca=b$
   What is the value of a?

2. *Anſw.* By the Canon in *Sect.* 6. *Chap.* 15. *Book.* 1. . . . $a=\sqrt{:b+\frac{1}{4}cc:}-\frac{1}{2}c$
   By which value of a the Equation propos'd may be expounded (as is uſual) by the following

Demonſtration.

3. If . . . . $a=\sqrt{:b+\frac{1}{4}cc:}-\frac{1}{2}c$
4. Then conſequently by adding $\frac{1}{2}c$ to each part . . . $a+\frac{1}{2}c=\sqrt{:b+\frac{1}{4}cc:}$
5. And by multiplying each part of the laſt Equation into it ſelf . . . . $aa+ca+\frac{1}{4}cc=b+\frac{1}{4}cc$
6. Wherefore by ſubtracting $\frac{1}{4}cc$ from each part, there remains . . . . $aa+ca=b$
   Which was to be proved.

Note. This Demonſtration is formed in the way of Compoſition by the ſteps of the Reſolution of the ſame Queſtion in *Sect.* 5. *Chap.* 15. *Book* 1. but in a retrograde or backward order; for the firſt ſtep in the Compoſition (or Demonſtration) is the laſt in the Reſolution, the ſecond ſtep in the Compoſition is the laſt but one in the Reſolution; and ſo by returning backwards by the ſteps of the Reſolution, the Demonſtration ends in the Equation propos'd to be reſolved. But this is largely handled in my fourth Book of Algebraical Elements.

QUESTION. 5.

1. If a, b, k, be ſuch Quantities that . . . . $aa-ba=k$
   What is the value of a?

2. *Anſw.* By the Canon in *Sect.* 8. *Chap.* 15. *Book* 1. . . $a=\frac{1}{2}b+\sqrt{:k+\frac{1}{4}bb:}$
   By which value of a the Equation propos'd may be expounded, as appears by the following

Demonſtration.

3. If . . . . $a=\frac{1}{2}b+\sqrt{:k+\frac{1}{4}bb:}$
4. Then by ſubtracting $\frac{1}{2}b$ from each part . . . . . $a-\frac{1}{2}b=\sqrt{:k+\frac{1}{4}bb:}$
5. And by multiplying each part of the laſt Equation into it ſelf, ... $aa-ba+\frac{1}{4}bb=k+\frac{1}{4}bb$
6. Wherefore by ſubtracting $\frac{1}{4}bb$ from each part, . $aa-ba=k$
   Which was to be proved.

QUESTION 6.

1. If c and n be put for ſuch known Quantities, that, ... n not $=\frac{1}{4}cc$
2. And if a be put for a Quantity unknown, and . . $ca-aa=n$
   What is the value of a?
3. *Anſw.* By the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. theſe two values of a will be found out, *viz.* . . . $a=\begin{cases}\frac{1}{2}c+\sqrt{:\frac{1}{4}cc-n:}\\ \frac{1}{2}c-\sqrt{:\frac{1}{4}cc-n:}\end{cases}$

By each of which values of a the Equation propoſed in the ſecond ſtep may be expounded, *viz.* if either $\frac{1}{2}c+\sqrt{:\frac{1}{2}cc-n:}$ or $\frac{1}{2}c-\sqrt{:\frac{1}{4}cc-n:}$ be put equal to a, then $ca-aa=n$.

De-

DEMONSTRATION.

4. Firſt, if . . . . $a=\frac{1}{2}c+\sqrt{:\frac{1}{4}cc-n:}$
5. Then by ſubtracting $\frac{1}{2}c$ from each part . . . . . . $a-\frac{1}{2}c=\sqrt{:\frac{1}{4}cc-n:}$
6. And by multiplying each part of the laſt Equation into it ſelf . . . . $aa-ca+\frac{1}{4}cc=\frac{1}{4}cc-n$
7. And by adding ca to each part . . . . $aa+\frac{1}{4}cc=\frac{1}{4}cc+ca-n$
8. And by ſubtracting $\frac{1}{4}cc$ from each part . . . . $aa=ca-n$
9. And by adding n to each part . . . . $aa+n=ca$
10. Wherefore by ſubtracting aa from each part . . . . . $n=ca-aa$
11. That is, ... $ca-aa=n$

Which was to be proved.

Again, if . . . . $a=\frac{1}{2}c-\sqrt{:\frac{1}{4}cc-n:}$

12. Then by adding $\sqrt{:\frac{1}{4}cc-n:}$ to each part . $a+\sqrt{:\frac{1}{4}cc-n:}=\frac{1}{2}c$
13. And by ſubtracting a from each part . . . . $\sqrt{:\frac{1}{4}cc-n:}=\frac{1}{2}c-a$
14. And by multiplying each part of the laſt Equation into it ſelf . . . . $\frac{1}{4}cc-n=\frac{1}{4}cc-ca+aa$
15. And by adding ca to each part . . . . $ca+\frac{1}{4}cc-n=\frac{1}{4}cc+aa$
16. And ſubtracting $\frac{1}{4}cc$ from each part . . . . . . $ca-n=aa$
17. And by adding n to each part . . . . $ca=aa+n$
18. Wherefore by ſubtracting aa from each part . . . . $ca-aa=n$

Which was to be proved.

QUESTION 7.

1. If b and c be put for ſuch known Quantities, that c is greater than b, but leſs than $2b$; and if a be put for a Quantity unknown;
2. And if . . . . $\sqrt{\frac{aa+3bb}{4}}+\sqrt{\frac{aa-3bb}{4}}=\sqrt{\frac{baa}{c}}$;

What is the value of a?

RESOLUTION.

3. Becauſe the Squares of equal Quantities are alſo equal, by multiplying each part of the Equation in the ſecond ſtep into it ſelf, this is produced, *viz.*

$$\frac{aa}{2}+\sqrt{\frac{a^4-9b^4}{4}}=\frac{baa}{c}$$

4. Then to the end the Surd Quantity in the Equation in the third ſtep may ſolely make one part of an Equation, let $\frac{aa}{2}$ be ſubtracted from each part of that Equation, and this will remain, *viz.*

$$\sqrt{\frac{a^4-9b^4}{4}}=\frac{baa}{c}-\frac{aa}{2}=\frac{2baa-caa}{2c}$$

5. And to the end the Radical Sign in the firſt part of the laſt Equation may vaniſh, let each part be multiplied by it ſelf, ſo an Equation in Rational Quantities will be produced, *viz.* $\frac{a^4-9b^4}{4}=\frac{4bba^4-4bca^4+ccaa^4}{4cc}$
6. And by reducing the laſt Equation to a common Denominator $4cc$, and then by multiplying each part by the ſame $4cc$, this Equation in Integers will be produced, *viz.*

$$cca^4-9b^4cc=4bba^4-4bca^4+cca^4$$

7. And from the Equation in the laſt preceding ſtep, after due Reduction is made, to make thoſe Quantities wherein a^4 is found to poſſeſs one part, this following Equation ariſes, *viz.* $4bca^4-4bba^4=9b^4cc$
8. Then by dividing each part of the laſt Equation by $4bc-4bb$, to the end that a^4 may ſtand alone, this Equation ariſes, *viz.*

$$a^4=\frac{9b^4cc}{4bc-4bb}=\frac{9b^3cc}{4c-4b}$$

9. But . . . . $\frac{9bbcc}{4}$ into $\frac{b}{c-b}=\frac{9b^3cc}{4c-4b}$

10. Therefore from the two laſt preceding Equations, by exchanging equal Quantities, this Equation ariſes, *viz.*

$$a^4 = \frac{9bbcc}{4} \text{ into } \frac{b}{c-b}$$

11. And by extracting the ſquare Root out of each part of the Equation in the tenth ſtep, this ariſes;

$$aa = \frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}$$

12. Wherefore by extracting the ſquare Root out of each part of the Equation in the eleventh ſtep, the deſired value of a is diſcovered, *viz.*

$$a = \sqrt{:\frac{3bc}{2} \text{ into } \sqrt{\frac{b}{c-b}}:}$$

An Example of Queſt. 7. *in Numbers.*

13. If . . . . $b = 16$

14. And . . . . $c = 25$

15. And . . . . a = a Number unknown

16. And if . . . . $\sqrt{\frac{aa+3bb}{4}} + \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{baa}{c}}$

What is the Number a?

17. *Anſw.* From the thirteenth, fourteenth, and twelfth ſteps, $a=\sqrt{800}$, or $20\sqrt{2}$. By which value of a the Equation propos'd may be expounded, as will appear by

The Proof.

18. If $b = 16$, $c = 25$, and $a = \sqrt{800}$; then it will follow that

$$\sqrt{\frac{aa+3bb}{4}} + \sqrt{\frac{aa-3bb}{4}} = \sqrt{\frac{baa}{c}} \quad (= 8\sqrt{8}, \text{ or } \sqrt{512})$$

Note, The Numbers to expreſs the values of b and c muſt not be taken at pleaſure, but ſuch that the Number c may exceed the Number b, and be leſs than $2b$, as is preſcribed in the Queſtion; the former part of which Determination is diſcovered by the Denominator $c-b$ of the Surd Fraction in the twelfth ſtep, and the latter part of the Determination is manifeſt by the latter part of the Equation in the fourth ſtep, where caa is to be ſubtracted from $2baa$, which cannot be done ſo as to leave a Remainder greater than nothing, unleſs c be leſs than $2b$.

Sect. XVIII. *An Explanation of* Fran. van Schooten's *General Rule to extract what Root you pleaſe out of any Binomial in Numbers, having ſuch a Binomial Root as is deſired.*

Preparation.

Firſt, if the given Binomial has Fractions in it, muſt be freed from them by multiplying the Binomial by their Denominator. As for Example, to extract $\sqrt{(3,)}$ that is, the Cubic Root out of $\sqrt{242}+12\frac{1}{2}$, I multiply the Binomial by 2, and it makes $\sqrt{968}+25$; for $\sqrt{242}$ multiplied by $\sqrt{4}$, (that is, by 2) produces $\sqrt{968}$; and $12\frac{1}{2}$ into 2 makes 25. Likewiſe, if there be propoſed $\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$, I firſt multiply it by $\sqrt{5}$, and it makes $\sqrt{242}+\frac{25}{2}$, then this Binomial multiplied by 2 produces (as before) $\sqrt{968}+25$; and ſo of others.

Secondly, if neither of the two Parts of the given Binomial be Rational, it muſt be reduced by Multiplication or Diviſion to another Binomial that ſhall have one of its Parts Rational; which Reduction may always be done by the Multiplication of either Part, but oftentimes more briefly by the Multiplication or Diviſion of the leſſer Number. As for Example, $\sqrt{242}+\sqrt{243}$ may be multiplied by $\sqrt{242}$, and it makes $242+\sqrt{58806}$; but more compendiouſly by $\sqrt{2}$, and there comes forth $22+\sqrt{486}$. After the ſame manner: $\sqrt{(3)}3993+\sqrt{(6)}17578125$ may be firſt multiplied by $\sqrt{(3)}3993$, and the Product again by $\sqrt{(3)}3993$, ſo there will be produced another Binomial, whoſe Rational Part is the abſolute Number 3993; but more briefly by $\sqrt{(3)}9$, and there will be produced another Binomial whoſe Rational Part is 33; and yet more compendiouſly if the Binomial propos'd be divided by $\surd(3)3$, there will ariſe $11+\surd 125$.

But here is to be noted, that when one part of a Binomial is Rational, whither it be of a Binomial firſt given, or of another deduced (as above) from that given, then alſo the Square of the other part ought to be rational, otherwiſe no Root can be extracted out of the Binomial, or the other deduced from it.

Thirdly, to extract $\surd(6)$ out of a given Binomial qualified as above is ſuppoſed, we muſt firſt extract the ſquare Root, and then out of this the Cubic Root; and to extract $\surd(9)$ we muſt firſt extract $\surd(3)$, and then out of the Cubic Root found out we muſt again extract $\surd(3)$; and ſo of any other Root whoſe Index is a Compoſit Number. But as to the Extraction of the ſquare Root out of a Binomial, a Rule has been already given and exemplified in the preceding *Sect.* 16, ſo that here there is need only that I ſhew how to extract $\surd(3)$, $\surd(5)$, $\surd(7)$, $\surd(11)$, and ſuch like whoſe Indices are Prime Numbers.

Fourthly, to extract $\surd(3)$, $\surd(5)$, $\surd(7)$, or the like Root, whoſe Index is a Prime Number, we muſt firſt of all try whether out of the given Binomial there can be extracted a Binomial Root which has one part Rational, but that may be diſcovered by ſubtracting the Square of the leſſer part of the given Binomial from the ſquare of the greater, and extracting the Root out of the Remainder, to wit, the Cubic Root of $\surd(3)$ be to be extracted out of the given Binomial, or the Root of the fifth Power, if $\surd(5)$ be to be extracted: and ſo of others. For if the Root of the ſaid Remainder be not a Rational Number, then the Binomial Root ſought will certainly want a Rational part, *viz.* each of its parts will be Surd; in which caſe, in order to extract the Root, the given Binomial muſt be multiplied by the Difference of the Squares of the Parts, if the Queſtion be concerning the Extraction of the Cubic Root; or by the Square of the ſaid Difference, if $\surd(5)$ be ſought; or by the Cube of the ſame Difference, if $\surd(7)$ be required; or by the fifth Power of the ſaid Difference, if $\surd(11)$ be ſought; and ſo of the reſt By which Multiplication another Binomial will always be produced, wherein the Root of the Difference of the Squares of the Parts will be the ſame with the Difference of the Squares of the Parts of the former Binomial.

As to extract the Cubic Root out of $25+\surd 968$, I firſt ſubtract 625 the Square of 25, from 968 the Square of $\surd 968$, and there remains 343, whoſe Cubic Root 7 is a Rational Number; which argues that the Root of the given Binomial, if there can be a Root extracted out of it, is a Binomial which has one of its Parts Rational.

Likewiſe, to extract the Cubic Root out of $22+\surd 486$, we muſt ſubtract 484, the Square of 22, from 486, and extract the Cubic Root out of the Remainder 2; but becauſe that cannot be done exactly, it ſhews that the Cubic Root of $22+\surd 486$ wants a Rational Part; and therefore $22+\surd 486$ muſt be multiplied by the ſaid Remainder 2, that there may be a Binomial $44+\surd 1944$, wherein the Cubic Root of the Difference of the Squares of the Parts in 2.

So to extract $\surd(5)$ out of $11+\surd 125$, becauſe 121 the Square of 11 ſubtracted from 125 leaves 4, which conſidered as a fifth Power has not an exact Rational Root, we muſt multiply $11+\surd 125$ by 16 the Square of 4, that there may come forth $176+\surd 32000$, where $\surd(5)$ of the Difference of the Squares of the Parts is 4.

Again, to extract $\surd(7)$ out of $338+\surd 114242$, wherein the Difference of the Squares of the parts is 2; becauſe this 2 is not the ſeventh Power of any Rational Number, the given Binomial may be multiplied by 8, that is, by the Cube of 2, and it make $2704+\surd 7311488$, wherein the $\surd(7)$ of the Difference of the Squares of the Parts in 2.

The RULE.

When a Binomial given, or another deduced from it, (if need be) by the Precedent Preparation is ſuch, that one of its parts and the Square of the other part, as alſo the Root of the Difference of the Squares of the Parts, (to wit, the Cubic Root when $\surd(3)$, or $\surd(5)$ when $\surd(5)$ is ſought) are Rational whole Numbers; then out of a Binomial ſo qualified $\surd(3)$, or $\surd(5)$, or $\surd(7)$, *&c.* may be extracted, if it has ſuch a Root, in manner following, *viz.*

Firſt, extract the Root of the Difference of the Squares of the parts of the Binomial qualified as aforeſaid, *viz.* the Cubic Root when $\surd(3)$ is ſought, but $\surd(5)$ when $\surd(5)$, or $\surd(7)$ when $\surd(7)$, *&c.* which Root ſo extracted is to be reſerved for a Dividend.

Secondly, find out a Rational Number a little greater than the Root ſought with this caution, that the Rational Number found out may not exceed the ſaid Root above $\frac{1}{2}$; which may eaſily be done by Vulgar Arithmetick, and take the ſaid Rational Mumber for a Diviſor.

Thirdly, divided the ſaid Dividend by the ſaid Diviſor, and if the Rational part of the given Binominal be greater than the other part, add the Quotient to the ſaid Rational Diviſor, and the half of the greateſt whole Number contained in the Sum ſhall be the Rational part of the Root ſought; then from the ſquare of that Rational part ſubtract the Root of the Difference of the ſquares of the parts, (to wit, the Dividend firſt found out as above) ſo the Remainder ſhall be the Square of the other part, when ſuch a Root as was required can be extracted out of the given Binomial; which you may eaſily try by multiplying this Root found out into itſelf, according to the degree of the Power repreſented by the given Binomial: for the Root found out being multiplied into itſelf cubically, if $\sqrt{(3)}$ was ſought, or five times into itſelf if $\sqrt{(5)}$ was ſought, ought to produce the given Binomial.

But if the Rational part of the given Binomial be leſs than the other part, then after you have found out the Quotient as above, ſubtract it from the Rational Diviſor, and the half of the greateſt whole Number contained in the Remainder ſhall be the Rational Part of the Root ſought; to the ſquare of which part if there be added the Dividend firſt found out as above, the Sum will be the Square of the other part, when the Binomial propoſed has a Root; but by multiplying the Root found out into itſelf as before, you may eaſily try whether it be a true Root or not.

Example 1. *To extract the Cubic Root out of* $20+\sqrt{392}$.

Firſt, the Difference of the Squares of the Parts of the given Binomial, *viz* the Exceſs of 400, the Square of 20, above 392, the Square of $\sqrt{392}$ is 8, whoſe Cubic Root I reſerve for a Dividend.

Secondly, I ſeek a Rational Number that may be greater than the Cubic Root of $20+\sqrt{392}$, the given Binomial, yet ſo that the Exceſs may not be greater than $\frac{1}{2}$; to which end I extract the Square Root of 392, and find it to be greater than 19, but leſs than 20; then to 20 the Rarional part of the given Binomial I add 19 and 20 ſeverally and it makes 39 and 40, which are the neareſt Rational whole Numbers that can expreſs the true value of the given Binomial; whence the Cubic Root thereof will be found greater than 3, but leſs than $3\frac{1}{2}$: this $3\frac{1}{2}$ which (according to the Caution before given) exceeds the true Cubic Root of the given Binomial by an Exceſs not greater than $\frac{1}{2}$, I reſerve for a Diviſor.

Thirdly, I divide 2 (the Dividend before reſerved) by the ſaid Diviſor $3\frac{1}{2}$, and the Quotient is $\frac{4}{7}$. Now becauſe 20 the Rational part of the given Binomial is greater than the other part $\sqrt{392}$, I add the ſaid Quotient $\frac{4}{7}$ to the ſaid Diviſor $3\frac{1}{7}$, and it makes the Sum $4\frac{1}{14}$, wherein the greateſt whole Number is 4, whoſe half is 2 the Rational part of the Root ſought: by the help of which Rational part the other part is eaſily diſcovered, for if from 4 the Square of the ſaid 2 you ſubtract 2, the Cubic Root of the Difference of the Squares of the parts of the given Binomial, there will remain 2 the Square of the other part. So that $2+\sqrt{2}$ is the Cubic Root of $20+\sqrt{392}$ the Binomial propoſed, as will appear by the Proof; for $2+\sqrt{2}$ being multiplied into it ſelf cubically produces $20+\sqrt{392}$, and for the ſame reaſon $2-\sqrt{2}$ is the Cubic Root of $20-\sqrt{392}$.

Example 2. *To extract the Cubic Root out of* $44+\sqrt{1944}$.

Firſt, the Cubic Root of the Difference of the Squares of the Parts is 2 for a Dividend. Secondly, the ſquare Root of 1944 is greater than 44, but leſs than 45; theſe added ſeverally to 44 the rational part of the given Binomial, make 88 and 89, whoſe Cubic Roots being extracted, do ſhew that the Cubic Root of the given Binomial is greater than 4, but leſs than $4\frac{1}{2}$; this Rational Number $4\frac{1}{2}$, which according to the Caution before given exceeds the true Root ſought by an exceſs not greater than $\frac{1}{2}$, I take for a Diviſor. Thirdly, I divide the ſaid Dividend 2 by the ſaid Diviſor $4\frac{1}{2}$, and the Quotient is $\frac{4}{9}$, which I ſubtract from the ſaid $4\frac{1}{2}$, (I ſubtract, becauſe 44 the Rational part of the given Binomial is leſs than the other Part $\sqrt{1944}$) and there remains $4\frac{1}{18}$; then the half of 4, the greateſt whole number contained in $4\frac{1}{18}$, is 2, which is the Rational Part of the Root ſought. Laſtly, to 4 the Square of the ſaid 2 I add 2, the Cubic Root of the Difference of the Squares of the Parts, and it makes 6 the Square of the other part. So that $2+\sqrt{6}$ is the Cubic Root ſought, as will appear by the Proof;

for

for if it be multiplied into itſelf cubically, it produces $44+\sqrt{1944}$ the Binomial propoſed; and for the ſame Reaſon $\sqrt{6}-2$ is the Cubic Root of $\sqrt{1944}-44$.

Example 3. *To Extract* $\surd(5)$ *out of* $176+\sqrt{32000}$.

Firſt, the Difference of the Squares of the Parts will be found 1024, whoſe $\surd(5)$ is 4 for a Dividend: Secondly, the Sum of the Parts will be found greater than 354, but leſs than 355; and conſequently $\surd(5)$ of the ſum of the Parts is greater than 3, but leſs than $3\frac{1}{2}$. Thirdly, by the ſaid $3\frac{1}{2}$ I divide the ſaid 4, and the Quotient is $1\frac{1}{7}$, which I ſubtract from the ſaid Diviſor $3\frac{1}{2}$, (becauſe the rational Part of the given Binomial is leſs than the other Part) and there remains $2\frac{5}{14}$; then the half of 2 (the greateſt whole Number contained in $2\frac{5}{14}$) is 1, the Rational Part of the Root ſought. Laſtly, the Square of the ſaid 1, to wit 1, added to 4 (the $\surd(5)$ of the Difference of the ſquares of the Parts of the given Binomials) makes 5 the ſquare of the other Part. So that $1+\sqrt{5}$ is the $\surd(5)$ of the given Binomial $176+\sqrt{32000}$, at leaſt if any $\surd(5)$ can be extracted out of the ſame; but $1+\sqrt{5}$ multiplied into itſelf five times makes $176+\sqrt{32000}$; therefore $1+\sqrt{5}$ is manifeſtly the deſired $\surd(5)$ of $176+\sqrt{32000}$.

Example 4. *To Extract* $\surd(7)$ *out of* $2704+\sqrt{7311488}$.

Firſt, the $\surd(7)$ of the Difference of the ſquares of the Parts in 2 for a Dividend. Secondly, the value of the given Binomial will be found greater than 5407, but leſs than 5408; whence the $\surd(7)$ thereof will be diſcovered to be greater than 3, but leſs than $3\frac{1}{2}$. Thirdly, by the ſaid $3\frac{1}{2}$ I divide the Dividend before found 2, and the Quotient is $\frac{4}{7}$, which I add to the Diviſor $3\frac{1}{2}$, (becauſe the Rational Part 2704 is greater than the other Part) and it makes the Sum $4\frac{1}{14}$; and therefore 2 the half of the greateſt whole Number contained in $4\frac{1}{14}$, is the Rational part of the Root ſought. Laſtly, from 4 the ſquare of the ſaid 2 I ſubtract 2, to wit $\surd(7)$, of the Difference of the Squares of the Parts of the given Binomial, and there remains 2 the ſquare of the other Part. So that $2+\sqrt{2}$ is the deſired $\surd(7)$ of the given Binomial $2704+\sqrt{7311488}$; for this is the ſeventh Power of $2+\sqrt{2}$, as will appear by Multiplication.

But here is to be noted, that when the given Binomial has been multiplied or divided by ſome Number, and thereby reduced to another Binomial, and the Root of this latter is found out, we muſt divide or multiply the Root found out by the Root of the Number by which the Binomial was multiplied or divided; ſo there will come forth the Root of the given Binomial.

As for Example, becauſe to extract the Cubic Root out of $\sqrt{242}+12\frac{1}{2}$, we firſt multiplied this Binomial by 2, and found $25+\sqrt{968}$, whoſe Cubic Root by the Rule before given will be found $1+\sqrt{8}$; this muſt be divided by $\surd(3)2$, and the Quotient $\surd(3)\frac{1}{2}+\surd(6)128$ ſhall be the Cubic Root of $\sqrt{242}+12\frac{1}{2}$ the Binomial propoſed.

But that the reaſon of the ſaid Diviſion by $\surd(3)2$ may the more clearly appear, let there be put $d=1+\sqrt{8}$, then it follows that $ddd=25+\sqrt{968}$, and $\frac{ddd}{2}=\sqrt{242}+12\frac{1}{2}$ (the Binomial propoſed.) Therefore by extracting the Cubic Root out of each part of the laſt Equation there ariſes $\surd(3)\frac{ddd}{2}$, that is, $\frac{d}{\surd(3)2}, =\surd(3)\!:\overline{\sqrt{242}+12\frac{1}{2}}$: But by ſuppoſition $d=1+\sqrt{8}$; therefore $1+\sqrt{8}$ divided by $\surd(3)2$, that is to ſay, $\surd(3)\frac{1}{2}+\surd(6)128$ ſhall be the Cubic Root of $\sqrt{242}+12\frac{1}{2}$; which was to be ſhewn.

Example 2. *To extract* $\surd(3)$ *out of* $\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$.

Firſt, to prepare it for Extraction we multiplied by $\sqrt{5}$, and found $\sqrt{242}+12\frac{1}{2}$, whoſe $\surd(3)$ (as appears in the laſt preceding Example) is $\surd(3)\frac{1}{2}+\surd(6)128$, which by dividing by $\surd(6)5$ gives the Quotient $\surd(6)\frac{1}{20}+\surd(6)\frac{128}{5}$ for the deſired Cubic Root of $\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$. The reaſon of which Diviſion by $\surd(6)5$ may be thus manifeſted, let there be put $d=\surd(3)\frac{1}{2}+\surd(6)128$; then it follows that $ddd=\sqrt{242}+12\frac{1}{2}=\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$ into $\sqrt{5}$, whence $\frac{ddd}{\sqrt{5}}=\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}$; therefore the Cubic Root of each part of the laſt Equation being extracted there ariſes $\surd(3)\frac{ddd}{\sqrt{5}}$, that is, $\frac{d}{\surd(6)5}$ (for $\surd(3)$ of $\sqrt{5}$ is $\surd(6)5$) $=\surd(3)\!:\overline{\sqrt{\frac{242}{5}}+\sqrt{\frac{125}{4}}}$: But by ſuppoſition $d=\surd(3)\frac{1}{2}+\surd(6)128$;

$\surd(6)128$; therefore $\surd(3)\frac{1}{2}+\surd(6)128$ divided by $\surd(6)5$ gives the true Cubic Root of $\surd\frac{242}{5}+\surd\frac{125}{4}$; which was to be ſhewn.

Example 3. *To extract* $\surd(3)$ *out of* $\surd 242+\surd 243$.

Firſt, (according to the ſecond Rule of the precedent Preparation) I multiply it by $\surd 2$, and there comes forth $22+\surd 486$; this multiplied by 2 (according to the fourth preparatory Rule) makes $44+\surd 1944$, whoſe Cubic Root (as before has been ſhewn) is $2+\surd 6$, which muſt be divided by $\surd 2$, and there will come forth $\surd 2+\surd 3$ for the Cubic Root ſought of $\surd 242+\surd 243$. But to manifeſt the Reaſon of dividing $2+\surd 6$ by $\surd 2$, let there be put $d=2+\surd 6$, then it follows that $ddd=44+\surd 1944=\overline{22+\surd 486}$ into 2, whence $\frac{ddd}{2}=22+\surd 486$, and this Equation divided by $\surd 2$ (becauſe in the Preparation we multiplied by $\surd 2$) gives $\frac{ddd}{\surd 8}=\surd 242+\surd 243$; therefore $\surd(3)$ being extracted out of each Part of the laſt Equation, there ariſes $\surd(3)\frac{ddd}{\surd 8}$, that is, $\frac{d}{\surd(6)8}$, or $\frac{d}{\surd 2}$, $=\surd(3)\,\overline{:\surd 242+243}$: But by ſuppoſition $d=2+\surd 6$; therefore $2+\surd 6$ divided by $\surd 2$, *viz.* the Quotient $\surd 2+\surd 3$ ſhall be the Cubic Root of $\surd 242+\surd 243$; which was to be ſhewn.

Example 4. *To extract* $\surd(5)$ *out of* $\surd(3)3993+\surd(6)17578125$.

Firſt, (according to the ſecond Preparatory Rule) I divide the given Binomial by $\surd(3)3$, and then (according to the fourth Preparatory Rule) I multiply the Quotient $\surd(3)1331+\surd 61953125$ by 16, and there comes forth $176+\surd 32000$, whoſe $\surd(5)$ (as has before been ſhewn) is $1+\surd 5$. Now this Root $1+\surd 5$ divided by $\surd(5)16$, and the Quotient multiplied by $\surd(15)3$ will diſcover the true $\surd(5)$ of $\surd(3)3993+\surd(3)17578125$; the reaſon of which Diviſion and Multiplication may be made manifeſt thus; let there be put $d=1+\surd 5$, then it follows that $ddddd=176+\surd 32000$; and by dividing each part of the laſt Equation by 16, (becauſe in the preparatory work we multiplied by 16) there ariſes $\frac{ddddd}{16}=\surd(3)1331+\surd(6)1953125$; and by multiplying each part of this Equation by $\surd(3)3$, there will be produced $\frac{ddddd\times\surd(3)3}{16}=\surd(3)3993+\surd(6)17578125$. Therefore $\surd(5)$ being extracted out of each part of the laſt Equation there will ariſe $\surd(5)\frac{ddddd\times\surd(3)3}{16}$, that is, $\frac{d\surd(15)3}{\surd(5)16}$ equal to $\surd(5)$ of $\surd(5)1331+\surd(6)17578125$. But by ſuppoſition $d=1+\surd 5$, therefore $1+\surd 5$ multiplied into $\surd(15)3$, and the Product divided by $\surd(5)16$; or $1+\surd 5$ divided by $\surd(5)16$, and the Quotient multiplied $\surd(15)3$ produces the true $\surd(5)$ of $\surd(3)3993+\surd(6)17578125$; which was to be ſhewn.

The Demonſtration follows.

The certainty of the preceding Rule will be made manifeſt by the three following Propoſitions.

PROP. 1.

If a Binomial, whereof one part and the Square of the other are rational Numbers, be multiplied into itſelf cubically, there will be produced another Binomial, the Square of whoſe leſſer Part being ſubtracted from the Square of the greater Part, leaves a Cubic Number, to wit, the Cube of the Difference of the Squares of the Parts of the Root or firſt Binomial.

To make this manifeſt, let there be propoſed the Binomial $b+\surd d$, this multiplied into itſelf cubically produces $bbb+3bb\surd d+3bd+d\surd d$, to wit, the Cube of $b+\surd d$. Here you are to note well, that although in that Cube there be four Parts or Members, yet they are to be eſteemed but as two, one of which, to wit, $bbb+3bd$, may deſign a Rational Number, and the other $\overline{3bb\surd d+d\surd d}$ (or $\overline{3bb+d}\times\surd d$) an Irrational or Surd Number, whoſe Square is Rational; whence it is manifeſt, firſt, that the Cube of a Binomial is alſo a Binomial, *viz.* $b+\surd d$ multiplied into itſelf cubically produces this

Binomial $bbb+3bd$ more $3bb\sqrt{d}+d\sqrt{d}$ (or $\overline{3bb+d}\times\sqrt{d}$.) Secondly, the Rational part $bbb+3bd$ is manifeſtly compoſed of the Cube of the Rational part of the Root, and of the triple Product made by the Multiplication of the ſame Root into the Square of its other part. And laſtly, the Difference of the Squares of the ſaid Parts $bbb+3bd$ and $3bb\sqrt{d}+d\sqrt{d}$ is equal to the Cube of $bb-d$, or of $d-bb$, *viz.* to the Cube of the Difference of the Squares of the Parts of the Root $b+\sqrt{d}$. For the Squares of $bbb+3bd$ and $3bb\sqrt{d}+d\sqrt{d}$ are $bbbbbb+6bbbbd+9bbdd$ and $9bbbbd+6bbdd+ddd$; and if theſe Squares be ſubtracted one from the other, the Remainder is either $bbbbbb-3bbbbd+3bbdd-ddd$, which is the Cube of $bb-d$; or elſe the Remainder is $ddd-3bbdd+3bbbbd-bbbbbb$, which is the Cube of $d-bb$.

To illuſtrate this Propoſition by Numbers, let there be put $b=2$ and $\sqrt{d}=6$; hence the Binomial $2+\sqrt{6}$ multiplied into it ſelf cubically produces the Binomial $44+\sqrt{1944}$, wherein the Difference of the Squares of the Parts (*viz.* the Remainder when 1936 the Square of 44 is ſubtracted from 1944 the Square of $\sqrt{1944}$) is 8, to wit, the Cube of the Difference of the Squares of the Parts of the Binomial Root $2+\sqrt{6}$.

Likewiſe this Binomial $2+\sqrt{2}$ multiplied into itſelf cubically produces the Binomial $20+\sqrt{392}$, wherein the Differences of the Squares of the Parts, to wit 8, is the Cube of the Difference of the Squares of the Parts of the Root $2+\sqrt{2}$.

The ſame Properties adhere alſo to a Reſidual Root, *viz.* the Cube of the Reſidual Root $b \backsim \sqrt{d}$ is alſo a Reſidual, to wit, $\overline{bbb+3bd} \backsim \overline{3bb\sqrt{d}+d\sqrt{d}}$, (or $\overline{3bb+d}\times\sqrt{d}$;) and the Difference of the Squares of the Parts of the later Reſidual is equal to the Cube of the Difference of the Squares of the Parts of the Roots or firſt Reſidual.

PROP. 2.

If a Binomial, whereof one Part and the Square of the other are the Rational Numbers, be multiplied by the Difference of the Squares of the Parts, the Product will be another, Binomial, wherein the difference of the Squares of the Parts is a Cubic Number, to wit, the Cube of the Difference of the Squares of the Parts of the Root multiplied

To make this manifeſt, let there be propoſed the Binomial $b+\sqrt{d}$, and ſuppoſe b greater than $\sqrt{d}$, then $b+\sqrt{d}$ multiplied by $bb-d$, the Difference of the Squares of the Parts, will produce this Binomial, to wit, $\overline{bbb-bd}$ more $\overline{bb\sqrt{d}-d\sqrt{d}}$, the Squares of whoſe Parts are $bbbbbb-2bbbbd+bbd$ and $bbbbd-2bbdd+ddd$; then this later Square ſubtracted from the former leaves $bbbbbb-3bbbbd+3bbdd-ddd$, which is the Cube of $bb-d$, the Difference of the Squares of the Parts of the firſt Binomial $b+\sqrt{d}$. The ſame Property would appear if we ſuppoſed b leſs than $\sqrt{d}$.

To illuſtrate this Propoſition by Numbers, ſuppoſe $b=22$, and $\sqrt{d}=486$; whence the Binomial $22+\sqrt{486}$ multiplied by 2, the difference of the Squares of the Parts, produces the Binomial $44+\sqrt{1944}$, wherein the difference of the Squares of the Parts is 8, which is the Cube of 2, the Difference of the Squares of the Parts of the former Binomial $22+\sqrt{486}$.

PROP. 3.

If the Difference of the Squares of any two Numbers be divided by a Number which doth not exceed the Sum of thoſe two Numbers above $\frac{1}{2}$; then the Quotient added to the ſaid Diviſor will give a Number greater than the double of the greater of the ſaid two Numbers, but the Exceſs will be leſs than Unity. And if the ſaid Quotient be ſubtracted from the ſaid Diviſor, the Remainder ſhall be greater than the double of the leſſer of the two Numbers, but this Exceſs alſo ſhall be leſs than Unity.

To manifeſt this, let a repreſent the greater of two Numbers, and e the leſſer; alſo let b repreſent ſome Fraction not greater than $\frac{1}{2}$; then I ſay, firſt, $a+e+b+\frac{aa-ee}{a+e+b}$ is greater than $2a$, but the Exceſs is leſs than 1, which I prove thus:

It is evident that $aa+ee+bb+2ae+2be+2ba+aa-ee$ is greater than $2aa+2ae+2ba$; therefore by dividing each of thoſe two Compound Quantities by $a+e+b$, it follows, that the firſt Quotient $a+e+b+\frac{a+e+b}{aa-ee}$ ſhall be greater than the later Quotient $2a$; and if this Quantity be ſubtracted from that, the Remainder $\frac{2bc+bb}{a+e+b}$ will be leſs than 1. For by ſuppoſition b is not greater than $\frac{1}{2}$; therefore $2be$ is leſs than

$a+e$, and bb leſs than b; and conſequently the Numerator $2be+bb$ is leſs than the Denominator $a+e+b$: wherefore $\frac{a+e+b}{2be+bb}$ is leſs than 1.

After the ſame manner it may be proved that $a+e+b-\frac{aa-ee}{a+e+b}$ is greater than $2e$; but this Exceſs alſo ſhall be leſs than 1; which was to be ſhewn.

Now to apply the preceding three Propoſitions to the Demonſtration of the Rule before given, let it be required to extract the Cubic Root out of the Binomial $100+\sqrt{7803}$, whoſe Rational part 100 is greater than the other part $\sqrt{7803}$; Here we may ſuppoſe $bbb+3bd$ to be 100, and $3bb\sqrt{d}+d\sqrt{d}$ (or $\overline{3bb+d}\times\sqrt{d}$) to be $\sqrt{7803}$; ſo that $\overline{bbb+3bd}$ more $\overline{3bb+d}\times\sqrt{d}$ may deſign the given Binomial $100+\sqrt{7803}$; and its Cubic Root $b+\sqrt{d}$ the Root ſought, whoſe greater part may be b, and the leſſer $\sqrt{d}$. Then according to the Rule:

To extract $\sqrt{(3)}$ *out of* $100+\sqrt{7803}$.

Firſt, from the Square of 100, that is, from . . .	10000
Subtract the Square of $\sqrt{7803}$, that is	7803
The Remainder is	2197
The Cubic Root of that Remainder is	13 $(=bb-d)$

Which Root 13 is (by *Prop.* 1.) equal to the Difference of the Squares of the Parts of the Binomial Root ſought.

Secondly, find out a Rational Number greater than the Sum of the Parts of the Cubic Root ſought, with this caution, that the Exceſs may not be above $\frac{1}{2}$, *viz.*

To the greater part of the given Binomial, that is, to .	100
Add the neareſt value in whole Numbers of the other part $\sqrt{7803}$, that is, ...	88 or 89
So the Sum ſhews that the value in whole Numbers of the given Binomial falls between	188 and 189

Whence the Cubic Root of the given Binomial is greater than $5\frac{1}{2}$, but leſs than 6; ſo that the Exceſs of 6 above the true Root ſought in leſs than $\frac{1}{2}$.

Thirdly, having found out (as above) 13, the true Difference of the Squares of the Parts of the Cubic Root ſought; and 6 a Rational Number, which exceeds not the true Sum of the ſame Parts above $\frac{1}{2}$, we may by the help of *Prop.* 3. and 1 find out the Parts ſeverally in this manner, *viz.*

Divide the ſaid	13
By the ſaid	6
And the Quotient is	$2\frac{1}{6}$
Which added to the ſaid Diviſor 6, makes the Sum . .	$8\frac{1}{6}$

Which Sum $8\frac{1}{6}$ does by (*Prop.* 3) exceed the double of the greater (to wit, the Rational) Part of the Cubic Root ſought, but the Exceſs is leſs than 1: therefore $7\frac{1}{6}$ is leſs than the ſaid double, but $8\frac{1}{6}$ is greater than the ſame; and conſequently becauſe the ſaid greater Part is ſuppoſed to be a Rational whole Number, the double thereof muſt neceſſarily be 8, to wit, the greateſt whole Number between $7\frac{1}{6}$ and $8\frac{1}{6}$, and therefore the ſaid Part it ſelf is 4, which being found out, it is eaſie to find the other Part; for (by *Prop.* 1.) if from 16 the Square of the ſaid greater Part 4, there be ſubtracted 13 the Cubic Root of the Difference of the Squares of the Parts of the given Binomial, there will remain 3 the Square of the other part; ſo that the Cube Root found out is $4+\sqrt{3}$, which will appear by the Proof to be the true Cubic Root ſought; for $4+\sqrt{3}$ being multiplied into it ſelf cubically produces the given Binomial $100+\sqrt{7803}$. And for the ſame reaſon $4-\sqrt{3}$ is the Cubic Root of $100-\sqrt{7803}$.

Or more briefly the Proof may be made thus:

To the Cube of 4 the Rational Part of the Root found out, *viz.* to	64, that is, bbb
Add the Product of thrice that Part multiplied into the Square of the Surd Part found out, *viz.* the Product .	36, that is, $3bd$
And it makes the Sum	100, that is, $bbb+3bd$

Which Sum is the ſame with the Rational part of the given Binomial, and therefore it proves that $4+\sqrt{3}$ is the Cubic Root ſought.

In like manner to extract $\sqrt{(3)}$ out of $44+\sqrt{1944}$, where the Rational Part 44 is leſs than the other Part $\sqrt{1944}$, we may ſuppoſe (as before) $bbb+3bd$ to be 44, and $\overline{3bb+d}\times\sqrt{d}$ (that is, $3bb\sqrt{d}+d\sqrt{d}$) to be $\sqrt{1944}$; ſo that $\overline{bbb+3bd}$ more $\overline{3bb+d}\times\sqrt{d}$ may deſign the given Binomial $44+\sqrt{1944}$, and its Cubic Root $b+\sqrt{d}$ the Root ſought, whoſe leſſer part may be b, and the greater $\sqrt{d}$; then according to the Rule,

To extract $\sqrt{(3)}$ *out of* $44+\sqrt{1944}$.

First, from the Square of $\sqrt{1944}$, *viz.* from . . . 1944

Subtract the Square of 44, ... 1936

The Remainder is 8

The Cubic Root of that Remainder is . . . . . 2 $(=d-bb)$

Which Root 2 is (by *Prop.* 1.) equal to the Difference of the Squares of the Parts of the Binomial Root ſought.

Secondly, find out a Rational Number greater than the Sum of the Parts of the Cubic Root ſought, with this caution, that the Exceſs may not be above $\frac{1}{2}$, which may be done thus, *viz.*

To the leſſer part of the given Binomial, *viz.* to . . 44

Add the neareſt value in whole Numbers of the other part $\sqrt{1944}$, that is, ... 44 or 45

So the Sum ſhews that the value in whole Numbers of the given Binomials falls between . . . . 88 and 89

Whence the Cubic Root of the given Binomial is greater than 4, but leſs than $4\frac{1}{2}$; ſo that the Exceſs of $4\frac{1}{2}$ above the true Root ſought is leſs than $\frac{1}{2}$.

Thirdly, having found out 2, the true Difference of the Squares of the Parts of the Cubic Root ſought; and $4\frac{1}{2}$ a Rational Number, which does not exceed the true Sum of the ſame Parts above $\frac{1}{2}$, we may by the help of *Prop.* 3. and 1. find out the Parts ſeverally in this manner, *viz.*

Divide the ſaid 2

By the ſaid . . . . $4\frac{1}{2}$

And it gives the Quotient . . . . $\frac{4}{9}$

Which ſubtracted from the ſaid Diviſor $4\frac{1}{2}$, there remains . $4\frac{1}{18}$

Which Remainder $4\frac{1}{18}$ does (by *Prop.* 3.) exceed the double of the leſſer Part (which in this Example is the Rational Part of the Cubic Root ſought, but the Exceſs is leſs than 1: therefore $3\frac{1}{18}$ is leſs than the ſaid double, but $4\frac{1}{18}$ is greater than the ſame, and conſequently becauſe the ſaid leſſer Part is a Rational whole Number, the double thereof muſt neceſſarily be 4, to wit, the greateſt whole Number between $3\frac{1}{18}$ and $4\frac{1}{18}$, and therefore the ſaid Part it ſelf is 2, which being found, it is eaſie to find the other Part; for if to 4 the Square of the ſaid leſſer Part 2, there be added 2 the Cubic Root of the Difference of the Squares of the Parts of the given Binomial, the Sum 6 ſhall be the Square of the other Part; ſo that the Cube Root found out is $2+\sqrt{6}$, which will appear to be the true Cubic Root ſought; for $2+\sqrt{6}$ multiplied into it ſelf cubically produces the given Binomial $44+\sqrt{1944}$. And for the ſame Reaſon $\sqrt{6}-2$ is the Cubic Root of $\sqrt{1944}-44$.

Or more briefly the Proof may be made thus:

To the Cube of 2 the Rational Part of the Root found out, *viz.* to . . . . 8, that is, bbb

Add the Product of thrice that Part multiplied into the Square of the Surd Part found out, *viz.* the Product . . 36, that is, $3bd$

And the Sum is . . . . 44, that is, $bbb+3bd$.

Which Sum is the ſame with the Rational Part of the given Binomial, and therefore it proves that $2+\sqrt{6}$ is the Cubic Root ſought.

Laſtly, what has here been ſhewn concerning the Demonſtration of the Extraction of the Cubic Root, may eaſily be applied to the Extraction of the other Roots before mentioned, ſo that there is no need of further Diſcourſe in this Matter.

CHAP. X.

An Explication of Simon Stevin's *General Rule, to extract one Root out of any poſſible Equation in Numbers, either exactly or very nearly true.*

I. Equations falling under any of the Forms in the fourteenth and fifteenth Chapters of the Firſt Book of theſe Elements, are capable (as has there been ſhewn) of perfect Reſolutions in Numbers, *viz.* the value of the Root or Roots ſought in any of thoſe Equations may be found out and expreſs'd exactly, either by ſome rational or irrational Number or Numbers; but the perfect Reſolution of all manner of Compound Equations in Numbers I have not found in any Author. And ſince an Expoſition of the General Method of *Vieta*, the Rules of *Huddenius* and others to that purpoſe, would make a large Treatiſe, and after all leave the curious Analyſt diſſatisfied, I ſhall not clog theſe Elements with a tedious Diſcourſe upon thoſe difficult Rules, which at the beſt are exceeding tedious in Operation, and ſome of them uncertain too; but rather purſue my firſt deſign, which was to explain Fundamentals, and ſuch Rules as are certain and moſt important in this profound Art. However, I ſhall lead the induſtrious Learner to a few ſteps further, in order to his underſtanding the Reſolution of all manner of Compound Equations in Numbers, and in this Chapter explain *Simon Stevin*'s General Rule, which with the help of the Rules in the following eleventh Chapter will diſcover all the Roots of any poſſible Equation in Numbers, either exactly if they be Rational, or very nearly true if Irrational.

QUESTION. 1.

If . . . . $aaa+26a=40188$, what is the Number a?

RESOLUTION.

This Equation not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. cannot be reſolved by any of the Canons in that Chapter, and therefore according to *Simon Stevin*'s General Method I ſearch out the Number a by tryals thus, *viz.*

1. I ſuppoſe . . . . $a = 1$
Thence it follows that . . . . $aaa = 1$
And . . . . $26a = 26$
Therefore . . . . $aaa+26a = 27$

Which 27 ought to have been 40188, but it's too little; whereby I find that by ſuppoſing a to be 1 I did not hit upon the true Number a, and therefore I make another tryal in like manner as before, *viz.*

2. I ſuppoſe . . . . $a = 10$
Thence it follows that . . . . $aaa = 1000$
And . . . . $26a = 260$
Therefore . . . . $aaa+26a = 1260$

Which 1260 being yet too little, I make a third tryal, *viz.*

3. I ſuppoſe . . . . $a = 100$
Thence it follows that . . . . $aaa+26a = 1002600$

Which 1002600 exceeds the juſt Reſult or abſolute Number 40188 in the latter part of the Equation firſt propos'd, and therefore the true Number a is leſs than 100; but the ſecond tryal ſhews it to be greater than 10, and therefore the whole Number which expreſſes the exact, or at leaſt part of the value of a, muſt neceſſarily conſiſt of two Characters, and conſequently the firſt (towards the left Hand) muſt be one of theſe nine, 1, 2, 3, 4, 5, 6, 7, 8, 9; but becauſe by the ſecond Inquiry 10 was found too little, I now make tryal with 2 for the firſt Figure of the Root a, *viz.*

4. I ſuppoſe . . . . $a = 20$
Thence . . . . $aaa+26a = 8520$

Which Reſult 8520 being yet leſs than the juſt Reſult 40188, I make tryal again, *viz.*

5. I ſuppoſe . . . . $a = 30$
Thence . . . . $aaa+26a = 27780$

Which is yet too little ; therefore,

6. I ſuppoſe . . . . $a = 40$

Thence . . . . $aaa+26a = 65040$

Which 65040 being greater than 40188, it ſhews me that the true Root or value of a is leſs than 40; but by the fifth Tryal it's greater than 30, and conſequently the laſt Figure of the Root is 3.

Now the ſecond Character of the Root muſt neceſſarily be one of theſe, *viz.* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; and becauſe it has been diſcovered, that the true value of the Root a is greater than 30, the ſecond Character cannot be 0, I therefore make tryal with 1, and ſuppoſe $a=31$; which proving too little, I make tryal with 32, 33, 34, &c. ſeverally in like manner as before, and at length I find 34 to be the true Number a ſought, by which the Equation propos'd may be expounded; for if $a=34$, then conſequently $aaa+26a=40188$.

II. But if after tryals made (as before) the value of a the Root ſought happens to fall between two whole Numbers that differ by Unity; then tryals are to be made with the leſſer whole Number increaſed with $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, &c. until you have found the value of a in ſome mixt Number conſiſting of a whole Number and ſome certain tenth parts of an Unit. But if the ſaid value of a happens not to be expreſs'd exactly by the ſaid leſſer whole Number increaſed with certain tenth parts, then you are to make tryals with the ſaid leſſer whole Number increaſed with a Decimal Fraction, having for its Numerator a Number greater than 10, but leſs than 100; and for its Denominator 100, as with $\frac{11}{100}$, $\frac{12}{100}$, &c. and by proceeding in that manner you may find the exact value of the Root a, when its fractional part is exactly equal to ſome Decimal Fraction: or elſe approach infinitely near to the ſaid exact value when 'tis Irrational or Surd, as in this following

QUESTION. 2.

If . . . . . $aaaa+50a=184638.6801$; (or $184638\frac{6801}{10000}$;) what is the Number a?

RESOLUTION.

Firſt, I ſuppoſe $a=1$, but this proving too little I put $a=10$, this alſo proving too little, I aſſume $a=100$, which after tryal I find to be greater than the true Number a, and conſequently the Number a falls between 10 and 100; then making tryal with 20 I find it too little, but making tryal with 30 I find this too great, and therefore the true Root a falls between 20 and 30. Again, making tryal with 21 I find it too great, but 20 was before found too little; therefore the true Root a is between 20 and 21; then I make tryal with 20.1, (that is, $20\frac{1}{10}$) 20.2, 20.3, &c. and at length find 20.7 to be the true Number a ſought; for if $a=20.7$ (that is, $20\frac{7}{10}$) it will make $aaaa+50a=184638.6801$ the Equation propoſed.

But if 20.7 had proved too little, and 20.8 too great, then tryals muſt have been made with 20.71, (that is, $20\frac{71}{100}$) 20.72, 20.73, &c. In like manner, if 20.7 had been too little, but 20.71 (that is, $20\frac{71}{100}$) too great, then tryals muſt have been made with 20.701, (that is, $20\frac{701}{1000}$) 20.702, 20.703, &c. This will be partly exercis'd in reſolving the Equation in this following

QUESTION. 3.

If . . . . $aaa+20aa=1954$, what is the Number a?

Anſw. . . . . . $a=8.308$, &c. found out by tryals as before.

III. When the value of (a) the required Root of an Equation happens to be leſs than Unity, then trial is to be made with $\frac{1}{10}$; but if this prove too great, then with $\frac{1}{100}$, &c. Now ſuppoſe .1 (that is $\frac{1}{10}$) be too great, .01 (that is, $\frac{1}{100}$) too little, then tryal muſt be made with .02 | .03 | .04 |, &c. until you have found out the greateſt Figure that muſt ſtand in the ſecond place of the Decimal Fraction expreſſing the Root ſought; ſuppoſing then ſuch Figure to be found 8, *viz.* that .08 (or $\frac{8}{100}$) is leſs, but .09 (or $\frac{9}{100}$) is greater than the Root, tryal muſt be made with .081, (that is, $\frac{81}{1000}$) .082 | .083 | &c. as in this following

QUESTION 4.

If . . . . $aaa+3240a=269$, what is the Number a?

Anſw. . . . . $a=.083$, &c. that is, $\frac{83}{1000}$, &c.

IV. The preceding Examples may ſuffice to ſhew the uſe of this general Method, when all the Terms of the unknown part of an Equation are Affirmative, (*viz.* when + is prefix'd to each Term) in which caſe there is but one Affirmative Root; in the ſearch whereof by Tryals (as before) if the Numbers aſſumed ſeverally for the value of the Root ſought do aſcend greater and greater, then the abſolute Numbers reſulting from thoſe aſſumed Values will likewiſe aſcend; and contrarily, if the aſſumed Roots do deſcend from a greater to a leſs, the Reſults will likewiſe grow leſs and leſs: whence by comparing an abſolute Number reſulting from an aſſumed Root with the juſt abſolute Number of the Equation propos'd, you may certainly know (if the ſaid Reſult and juſt Abſolute be not equal to one another) whether you are to take a Number greater or leſs than that laſt before aſſumed.

But when the unknown part of an Equation conſiſts of affirmative and negative Terms mingled one with another, then the ſearch by Tryals will be more intricate and doubtful than before; for ſometimes it will be hard to diſcern whether a following aſſumed Root muſt be taken greater or leſs than that which was taken next before. Moreover, a compound Equation of this latter kind may happen to be ſuch, that it may be expounded by as many ſeveral affirmative Roots, as there be Unities in the Index of the higheſt unknown Power, *viz.* a Cubical Equation may be ſo conſtituted, that it ſhall have three different affirmative Roots, a Biquadratic Equation four ſeveral Roots; and ſo of higher Equations, as will be ſhewn in the following *Chap* 11. But in what manner ſoever any poſſible Equation is conſtituted in Rational Numbers, this general Method will always find out one affirmative Root, either exactly true, or at leaſt very near the truth, as will further appear by the following Queſtions.

QUESTION 5.

If . . . . $aaa-22aa+157a=360$, what is the Number a?

RESOLUTION.

1. I ſuppoſe . . . . $a=1$
Thence it follows that . . . . . $aaa-22aa+157a=136$

Which 136 is leſs than the juſt abſolute Number 360, and therefore I make another Tryal, *viz.*

2. I ſuppoſe . . . . $a=10$
Thence it follows that . . . . . $aaa-22aa+157a=370$

Which 370 exceeds the juſt abſolute Number 360, and therefore I conclude there is one affirmative value of a, (either Rational or Irrational) between 1 and 10; which value, after Tryals made with 2,3,4,5, I find to be 5; this will conſtitute the Equation propoſed, for if $a=5$, then $aaa-22aa+157a$ will exactly make 360.

But there are two other Roots or Values of a, to wit 8 and 9, each of which will likewiſe conſtitute the Equation firſt propoſed, but how they are found out will be ſhewn in *Sect.* 9. of the following *Chap.* 11.

QUESTION 6.

If . . . . $3200a-aaa=46577$ (juſt,) what is the Number a?

RESOLUTION.

1. I ſuppoſe . . . . $a=1$
Thence . . . . $3200a-aaa=3199$ (leſs than juſt)
2. I ſuppoſe . . . . $a=10$
Thence . . . . $3200a-aaa=31000$ (leſs than juſt)
3. I ſuppoſe . . . . $a=100$
Thence . . . . $3200a-aaa=-680000$ (leſs than juſt)

Now becauſe the ſecond Reſult (or abſolute Number) +31000 is Affirmative, and the laſt Reſult 680000 is Negative, I make tryals with Numbers between 10 and 100 for the value of a; for if the Equation propoſed be poſſible, before the affirmative Reſults fall off to negatives, there will be a Root or Value of a producing an Affirmative Reſult either exactly equal, or very near to the juſt Reſult 46577; therefore,

4. I ſuppoſe . . . . $a=20$
Thence . . . . $3200a-aaa=56000$ (greater than juſt)

Now becauſe by taking 20 for the value of a, the Reſult 56000 exceeds the juſt Reſult 46577; but by taking 10 for a, the Reſult 31000 happened to be leſs than the ſaid 46577, it ſhews there is one affirmative Root or value of a between 10 and 20, which Root, after tryals made with intermediate Numbers (as in former Examples) will be found 15, 7, *&c.* Moreover, becauſe by ſuppoſing $a=20$ the Reſult 56000 happened to exceed the juſt Reſult 46577, but by putting $a=100$ the Reſult —680000 proved to be leſs than the ſame 46577, it ſhews there is an Affirmative value of a between 20 and 100, which value after tryals made will be found 47; ſo that there are two affirmative Roots or values of a found out, to wit, 15, 7. *&c.* (or $15\frac{7}{10}$, *&c.*) and 47; the former of which will nearly, and the latter exactly conſtitute the Equation propoſed.

V. *Florimond de Beaune* in the latter of two ſmall Treatiſes printed in 1659, concerning the Nature, Conſtitution, and Limits of Equations, ſhews how to find out Limits within which the Roots of all compound Equations not aſcending above the Biquadratic kind are confined; which Limits when they may be diſcovered without much trouble, and are not very wide aſunder, will help to leſſen the tryals in the general Method before delivered. As in the laſt Example, where

The Equation propoſed was . . . . $3200a—aaa = 46577$
Firſt, becauſe aaa muſt be ſubtracted from $3200a$, and leave a Remainder equal to 46577, it preſuppoſes . . . $aaa \sqsupset 3200a$
Therefore by dividing each part by a . . . . $aa \sqsupset 3200$
And by extracting the ſquare Root out of each part, it follows that . . . . $a \sqsupset 56.5$, *&c.*
Again, from the Equation propos'd by Tranſpoſition 'tis evident that . . . . $3200a—46577 = aaa$
Whence 'tis alſo manifeſt that . . . . $3200a \sqsubset 46577$
And conſequently by dividing each part by 3200, ... $a \sqsubset 14.5$, *&c.*

Thus it is found that the value of a the Root ſought is greater than 14.5, *&c.* but leſs than 56.5, *&c.* and therefore tryals according to the general Method aforeſaid need not be made with any Numbers that are not within thoſe Limits.

From the Premiſes 'tis evident that this general Method finds not a perfect Root of an Equation, unleſs ſuch Root be a whole Number, or elſe a Fraction exactly equal to ſome Decimal Fraction; or laſtly, a mixt Number compos'd of a whole Number and a perfect Decimal Fraction.

Note. When the Coefficients or known Numbers multiplied into any of the unknown Powers under the higheſt, (which muſt have no Coefficient but Unity) are Vulgar (not Decimal) Fractions, or mixt Numbers whoſe fractional parts are Vulgar Fractions; likewiſe, when the abſolute Number that ſolely poſſeſſes the latter part of the Equation propos'd is a Vulgar Fraction, or mixt Number whoſe Fractional part is a Vulgar Fraction; all thoſe Vulgar Fractions muſt be reduced to Decimal Fractions, or elſe the Equation muſt be reduced to another Equation in Integers (by *Sect.* 7. in the following *Chap.* 11.) before you enter upon the Reſolution by tryals as aforeſaid.

CHAP. XI.

Extractions out of the Algebraical Treatiſes of Vieta *and* Renates des Cartes; *concerning the Conſtitution and Reſolution of Compound Equations in Numbers, eſpecially thoſe which have many Roots.*

I. THE Scope of this Chapter is, firſt, to ſhew how to form an Equation that ſhall have as many different Roots or values of the Quantity ſought as ſhall be deſired; then how to free an Equation from Fractions, and to caſt away the ſecond Term; and laſtly, how to find out the Roots of all manner of Compound Equations in Numbers, either exactly if they be Rational, or very near the truth if Irrational.

But that the Learner may the more eaſily perceive my meaning, I ſhall premiſe a few Definitions in three Sections next following.

II. When the known abſolute Number in an Equation ſolely poſſeſſes one part thereof, let it be transfer'd to the other part by the Sign —, and then there will be an Equation which has 0 or nothing for one part, and the other part is by *Carteſius* called the Sum of the Equation propoſed. As for Example, if this Equation be propoſed, *viz.* $aaa-9aa+26a=24$, by tranſpoſition of 24 it makes $aaa-9aa+26a-24=0$, whoſe firſt part is called the Sum of the Equation propoſed.

III. In the Equations handled in this Chapter I put a, e, or y, to ſignifie an unknown Quantity; and by the firſt Term of an Equation is meant the higheſt unknown Power, to wit, that which has moſt Dimenſions or Degrees of a; by the ſecond Term that which has fewer Dimenſions by one than the firſt, and ſo downwards. As in this Equation, $aaa-9aa+26a-24=0$, the firſt Term is aaa, whoſe Index is 3; the ſecond Term is $-9aa$, where the Index of aa is 2; the third Term is $+26a$, where the Index of a is 1; and the laſt Term is -24, the known abſolute Number, whoſe Index is 0.

IV. The Roots of an Equation are of three kinds, *viz.* either Affirmative, or Negative, or Impoſſible. An Affirmative Root is a Quantity greater than nothing, as $+5$ or $+20$. A negative Root (which *Carteſius* calls a falſe Root) expreſſes a Quantity whoſe Denomination is oppoſite to an affirmative, as -5 or -20; the former of which wants 5, and the latter 20, of being equal to nothing. Laſtly, impoſſible Roots are ſuch whoſe values cannot be conceived or comprehended either Arithmetically or Geometrically; as in this Equation. $a=2-\sqrt{-1}$, where $\sqrt{-1}$, that is, the ſquare Root of -1, is no manner of way intelligible, for no Number can be imagined, which being multiplied by itſelf according to any Rule of Multiplication will produce -1.

V. Theſe things premiſed, I ſhall proceed to the forming of Equations which ſhall have many Roots.

PROP. I.

To form an Equation which ſhall have two Affirmative Roots.

1. Suppoſe . . . . $\begin{cases} a = 2, \text{ that is, } a-2 = 0 \\ a = 3, \text{ that is, } a-3 = 0 \end{cases}$

2. Then by multiplying the ſaid $a-2=0$ by $a-3=0$, this Equation is produced, *viz.* . . $aa-5a+6 = 0$

3. That is, by tranſpoſition, ... $5a-aa = 6$

Which laſt Equation falls under the laſt of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. and may be expounded by either of the two Roots or values of a, which by the Canon in *Sect.* 10. of the ſame *Chap.* will be found 2, and 3, to wit, thoſe from which the ſaid Equation was produced by Multiplication, as above.

Again, if this Equation $aa+6a-55=0$, (that is, $aa+6a=55$) which has one affirmative Root, to wit 5, be multiplied by $a-6=0$, there will be produced $aaa-91a+330=0$, (that is, $91a-aaa=330$) which has two affirmative Roots or values of a, to wit 5 and 6, which may be found out by the Rule hereafter delivered in *Sect.* 9. of this *Chap.*

PROP. II.

To form an Equation which ſhall have one Affirmative and one Negative Root.

1. Suppoſe . . . . $\begin{cases} a = 3, \text{ that is, } a-3 = 0 \\ a = -2, \text{ that is, } a+2 = 0 \end{cases}$

2. Then by multiplying the ſaid $a-3=0$ by $a+2=0$, this Equation is produced, *viz.* . . $aa-a-6 = 0$

3. That is, ... $aa-a = 6$

Which laſt Equation falls under the ſecond of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. and may be expounded by either of theſe two Roots or values of a, whereof one is Affirmative and the other Negative; which after the manner of reſolving *Queſt.* 1. in *Sect.* 7. of the ſame *Chap.* will be found $+3$ and -2, to wit, thoſe from which the ſaid Equation was produced by Multiplication, as before.

PROP. III.

To form an Equation which ſhall have three Affirmative Roots.

1. Suppoſe : . . . . $\left\{\begin{array}{l} a = 2, \text{ that is, } a-2 = 0 \\ a = 3, \text{ that is, } a-3 = 0 \\ a = 4, \text{ that is, } a-4 = 0 \end{array}\right.$

2. Then by multiplying the three laſt Equations (in each of which the latter part is 0) one into another, this Equation will be produced, $\left.\begin{array}{l}\\ \\ \\ \end{array}\right\} aaa-9aa+26a-24 = 0$

3. That is, by tranſpoſition of —24, . . . $aaa-9aa+26a = 24$

Which Equation may be expounded by every one of theſe three affirmative Roots or values of a, to wit, 2, 3, and 4; which may be found out by the Rule in the following *Sect.* 9. of this *Chap.*

The ſame Equation may likewiſe be formed altogether by Letters thus, *viz.* let the ſaid known Roots 2, 3, and 4, be repreſented by b, c, d; and then,

4. Suppoſe . . . . $\left\{\begin{array}{l} a = b, \text{ that is, } a-b = 0 \\ a = c, \text{ that is, } a-c = 0 \\ a = d, \text{ that is, } a-d = 0 \end{array}\right.$

5. Then by multiplying thoſe three laſt Equations, in each of which the latter part is nothing, one into another, this Equation will be produced, *viz.*

$$aaa \left.\begin{array}{l} -b \\ -c \\ -d \end{array}\right\} aa \quad \left.\begin{array}{l} +bc \\ +bd \\ +cd \end{array}\right\} a - bcd = 0$$

That is, ... $aaa - 9aa + 26a - 24 = 0$

PROP. IV.

To form an Equation which ſhall have three Affirmative Roots, and one Negative Root.

1. Suppoſe . . . . $\left\{\begin{array}{l} a = 2, \text{ that is, } a-2 = 0 \\ a = 3, \text{ that is, } a-3 = 0 \\ a = 4, \text{ that is, } a-4 = 0 \\ a = -5, \text{ that is, } a+5 = 0 \end{array}\right.$

2. Then by multiplying the four laſt Equations, in each of which the latter part is 0, one into another, this following Equation will be produced, *viz.*

$$aaaa-4aaa-19aa+106a-120 = 0$$

That is, ... $aaaa-4aaa-19aa+106a = 120$

Which laſt Equation may be expounded by every one of theſe three Affirmative Roots or values of a, *viz* 2, 3, and 4, and by one Negative Root —5, every one of which may be found out by the Rule in the following *Sect.* 9 of this *Chap.*

The ſame Equation may likewiſe be formed altogether by Letters thus, *viz* let the ſaid known Roots 2,3,4, and —5, be repreſented by b, c, d, and $-f$; then,

3: Suppoſe . . . . $\left\{\begin{array}{l} a = b, \text{ that is, } a-b = 0 \\ a = c, \text{ that is, } a-c = 0 \\ a = d, \text{ that is, } a-d = 0 \\ a = -f, \text{ that is, } a+f = 0 \end{array}\right.$

4. Then by multiplying the four laſt Equations, in each of which the latter part is 0, one into another, this following Equation will be produced, *viz.*

$$aaaa \left.\begin{array}{l} -b \\ -c \\ -d \\ +f \end{array}\right\} aaa \quad \left.\begin{array}{l} +bc \\ +bd \\ +cd \\ -bf \\ -cf \\ -df \end{array}\right\} aa \quad \left.\begin{array}{l} -bcd \\ +bcf \\ +bdf \\ +cdf \end{array}\right\} a - bcdf = 0$$

That is, ... $aaaa - 4aaa - 19aa + 106a - 120 = 0$

After the ſame manner you may form an Equation, which ſhall have as many Roots as you pleaſe, either all Affirmative, or ſome of them Affirmative and ſome Negative.

VI. *Ob-*

VI. *Obſervations upon the preceding four Propoſitions.*

1. By what has been ſaid 'tis evident, that ſometimes an Equation may have as many Roots as there be Unities in the Index of the higheſt unknown Term; I ſay, ſometimes, not always: for altho this Equation $aaa-6aa+13a-10=0$, as to its number of Terms and Signs, be like to the Equation formed in the preceding *Prop.* 3, ſo that one may think it has three Roots, yet it has only one affirmative Root, to wit 2, and no other Root either affirmative or negative can conſtitute the ſaid Equation, for 'tis produced by the Multiplication of this impoſſible Equation $aa-4a+5=0$ by $a-2=0$; but that $aa-4a+5=0$, that is, $4a-aa=5$ is an impoſſible Equation, the Determination in *Sect.* 9. *Queſt.* 1. *Chap.* 15. *Book* 1. makes manifeſt.

In like manner, altho this Equation $aaaa-60aaa+1650aa-22500a+115344=0$; as to its Number of Terms and Signs be like to an Equation that has four affirmative Roots, yet that Equation can be expounded only by two affirmative Roots, to wit, 12 and 18, and by no other Root either affirmative or negative; for 'tis made by the Multiplication of $aa-30a+216=0$, which has two affirmative Roots, 12 and 18, into this impoſſible Equation $aa-30a+534=0$.

2. Foraſmuch as Diviſion reſolves or undoes that which is compos'd or done by Multiplication, if the Sum of an Equation which is produced by the Multiplication of two or more Equations one into another, (according to the Method in the preceding four Propoſitions) be divided by a Binomial compos'd of the unknown Quantity *(a)* leſs by the value of any one of the affirmative Roots, or more by the value of one of the negative Roots, the Quotient ſhall be an Equation in which the firſt Term has fewer Dimenſions by one than the firſt Term of the Equation ſo divided. And if the Quotient be divided in like manner, there will come forth an Equation whoſe firſt Term has fewer Dimenſions by one than the former Quotient. As for Example, let there be propoſed the Equation in the preceding *Prop.* 4. to wit, $aaaa-4aaa-19aa+106a-120=0$, which was made by the continual Multiplication of $a-2=0$, $a-3=0$, $a+5=0$; I ſay, if the Equation propoſed be divided by any one of the Binomials $a-2$, $a-3$, $a-4$, $a+5$, the Quotient will be an Equation wherein the firſt Term has only three Dimenſions, which are fewer by one than thoſe in $aaaa$ the firſt Term of the Equation propoſed. So if the ſaid $aaaa-4aaa-19aa+106a-120=0$ be divided by $a-2=0$, there will ariſe $aaa-2aa-23a+60=0$, as you ſee by the ſubſequent Diviſion.

$$\begin{array}{rl}
a-2\;) & aaaa-4aaa-19aa+106a-120\;\;(\;aaa-2aa-23a+60 \\
 & aaaa-2aaa \\
\hline
 & \quad -2aaa-19aa \\
 & \quad -2aaa+4aa \\
\hline
 & \qquad -23aa+106a \\
 & \qquad -23aa+46a \\
\hline
 & \qquad\qquad +60a-120 \\
 & \qquad\qquad +60a-120 \\
\hline
 & \qquad\qquad\quad 0 \quad\; 0
\end{array}$$

Likewiſe if the Quotient, to wit, the Equation $aaa-2aa-23a+60=0$, where the firſt Term aaa has three Dimenſions, be divided by $a-3=0$, there will ariſe $aa+a-20$, whoſe firſt Term aa has but two Dimenſions. And laſtly, if the ſaid latter Quotient $aa+a-20$ be divided by $a-4=0$; there will come forth a ſimple Equation, to wit, $a+5=0$, that is the negative Root $a=-5$.

The like Diviſion may be practiſed with the literal Equations at the latter end of *Prop.* 3. and 4. in the preceding *Sect.* 5.

3. If a compleat Equation, that is, ſuch in which all the Terms are extant, be produced by the Multiplication of poſſible Equations one into another, you may diſcover how many affirmative, and how many negative Roots that Equation has, by this Rule, *viz.* As often as — follows next after +, or + next after —, ſo often there is an affirmative Root; and as often as two Signs — or two Signs + ſtand next to one another, ſo often there is a negative Root. As for Example, in this Equation, (before formed in *Prop.* 4.)

to wit, $aaaa - 4aaa - 19aa + 106a - 120 = 0$, becauſe next after the firſt Term $+aaaa$ there follows $-4aaa$, it ſhews there is one Affirmative Root; and becauſe next after $-4aaa$ there comes $-19aa$, it ſhews that the Equation has one negative Root. Again, becauſe next after $-19aa$ there follows $+106a$, it hints there is another Affirmative Root; and becauſe next after $+106a$ there follows -120, it ſhews there is a third Affirmative Root: ſo that the ſaid Rule diſcovers the Equation propos'd to have three Affirmative Roots, and one negative Root.

4. It is alſo manifeſt from the manner of forming Equations according to the Propoſitions in the preceding *Sect.* 5. that in every Equation which has as many Affirmative Roots as there be Dimenſions in the firſt Term, the Co-efficient or known Quantity in the ſecond Term is equal to the ſum of all the Affirmative Roots; and the known Quantity in the third Term is equal to the ſum of the Products of every two of the ſaid Roots multiplied one by the other, and the known Quantity in the fourth Term is equal to the ſum of the Products of every three of the ſaid Roots; and ſo forward when there be more Terms: but the laſt Term, to wit, the abſolute Quantity given is equal to the Product of all the Roots multiplied one into another. As in the following Equation (before formed in *Prop.* 3.) *viz.*

$$aaa \left.\begin{matrix}-b\\-c\\-d\end{matrix}\right\}aa \quad \left.\begin{matrix}+bc\\+bd\\+cd\end{matrix}\right\}a - bcd = 0.$$

That is, $aaa - 9aa + 26a - 24 = 0.$

Firſt, the Sum of 2, 3, and 4, (that is, of b, c, d) the three Roots of that Equation is 9, which is the known Number of the ſecond Term $-9aa$. Secondly, the Sum of the Products of every two of the ſaid Roots multiplied one by the other is 26, that is, $+bc+bd+cd$, which is the known Coefficient of the third Term $+26a$, or $\overline{+bc+bd+cd}$ into a. And laſtly, the Product of all the three Roots multiplied one into another is 24, or bcd, to which prefixing $-$ it makes -24, or $-bcd$, the laſt Term of the Equation propos'd.

The like Properties enſue when the Sum of the Numbers of Multitude of Affirmative and negative Roots is equal to the number of Dimenſions in the firſt Term of an Equation; ſaving that here in ſumming up all the Roots which compoſe thoſe known Quantities in the ſecond Term, and likewiſe the Products which compoſe the known Quantities in the following Terms, reſpect muſt be had to the Rules of Addition of $+$ and $-$ in ſuch manner as the Equation propoſed if it be found altogether by Letters will direct; as you may eaſily perceive by the Equation formed in *Prop.* 4. of the preceding *Sect.* 5.

VII. *How to free an Equation from Fractions, when 'tis incumbered therewith in the ſecond, third, or any of the following Terms. Which work is by* Vieta *called* Iſomœria.

The Rules in *Chap.* 12. *Book* 1. ſhew how to reduce an Equation ſo, as that the firſt Term may have no Coefficient but Unity; but if after any Equation is ſo reduced there happens to be any Fraction in the ſecond, third, or any of the following Terms, ſuch Equation may be reduced to another whoſe Terms ſhall be all Integers, by the Method in the five Examples next following.

Example 1.

1. Let this Equation be propos'd to be reduced to another in Integers, *viz.* . . . . $aaa + \frac{3}{2}a = 225$

Operation.

2. Suppoſe $e = 2a$, ($2a$, becauſe 2 is the Denominator of the Fraction $\frac{3}{2}$). . . . . $e = 2a$

3. Then divide each part of the laſt Equation by 2, (the Denominator aforeſaid) and there ariſes . . . . . . $\frac{e}{2} = a$

4. And by multiplying each part of the Equation in the third ſtep cubically, there comes forth . . . . $\frac{eee}{8} = aaa$

5. Again, by multiplying each part of the Equation in the third ſtep by $\frac{3}{2}$, (the Fraction in the ſecond Term of the Equation firſt propoſed) it makes . . . . $\frac{3e}{4} = \frac{3}{2}a$

6. Then add the two laſt Equations into one, and the Sum is $\frac{eee}{8}+\frac{3e}{4} = aaa+\frac{3}{2}a$
7. But by ſuppoſition in the firſt ſtep $225 = aaa+\frac{3}{2}a$
8. Therefore from the two laſt Equations (by 1 *Axiom.* 1 *Elem. Euclid.*) $\frac{eee}{8}+\frac{3e}{4} = 225$
9. Which laſt Equation being reduced to Integers (by *Sect.* 2. *Chap.* 12. *Book* 1.) gives $eee+6e = 1800$

Therefore an Equation is found out, which is altogether expreſs'd by Integers; and when the value of *e* in the laſt Equation is diſcovered, the value of *a* in the Equation propos'd is conſequently known; for by the third ſtep $a=\frac{1}{2}e$, therefore if *e* be 12, then *a* ſhall be 6.

Example 2.

Again, if this Equation be propos'd, $aaa+\frac{3}{2}a = \frac{265}{2}$
It may be reduced in like manner as before in *Example* 1. to this, *viz.* $eee+6e = 1060$
And if *e* be 10, then *a* ſhall be 5.

Example 3.

So likewiſe this Equation $aaa+\frac{3}{2}aa = \frac{325}{2}$
May be reduced to this $eee+3ee = 1300$
And if *e* be 10, then *a* is 5.

Example 4.

1. Again, let there be propoſed $aaa+\frac{11}{12}a = \frac{19}{4}$

Operation.

2. Suppoſe $e=12a$, (12*a* becauſe 12 is the Denominator of the Fraction $\frac{11}{12}$ in the ſecond Term) $e = 12a$
3. Then divide each part of the laſt Equation by 12, (the Denominator aforeſaid) and there ariſes $\frac{e}{12} = a$
4. And by multiplying cubically the laſt Equation, it produces $\frac{eee}{1728} = aaa$
5. And by multiplying the Equation in the third ſtep by $\frac{11}{12}$, it makes $\frac{11e}{144} = \frac{11}{12}a$
6. And by adding the two laſt Equations into one, the Sum makes $\frac{eee}{1728}+\frac{11e}{144} = aaa+\frac{11}{12}a$
7. But by the Equation propoſed $\frac{19}{4} = aaa+\frac{11}{12}a$
8. Therefore from the two laſt Equations (by 1. *Axiom.* 1. *Elem. Euclid.*) $\frac{cee}{1728}+\frac{11e}{144} = \frac{12}{4}$

Which Equation reduced to Integers gives $eee+132e = 8208$

Thus an Equation is found out in Integers; and when the value of *e* is diſcovered, the value of *a* in the Equation propos'd is conſequently known; for by ſuppoſition in the ſecond ſtep *e* is to *a* as 12 to 1; therefore if *e* be 18, then *a* ſhall be $1\frac{1}{2}$.

Example 5.

1. Again, let there be propoſed $aaaa-10aaa+45\frac{5}{6}aa-104\frac{1}{6}a+89 = 0$:

Operation.

2. Suppoſe $e=6a$, (6*a* becauſe 6 is the Denominator of the Fraction $\frac{5}{6}$) $e = 6a$
3. Then by dividing each part of the laſt Equation by 6, there ariſes $\frac{e}{6} = a$
4. And by ſquaring the laſt Equation it makes $\frac{ee}{36} = aa$
5. Likewiſe by ſquaring each part of the laſt Equation, there will be produced $\frac{eeee}{1296} = aaaa$
6. And by multiplying the Equation in the fourth ſtep by that in the third, the Product is $\frac{eee}{216} = aaa$

7. And by multiplying the laſt Equation by 10, it gives this, *viz.* . . . . $\frac{10eee}{216} = 10aaa$

8. And by multiplying the Equation in the fourth ſtep by $45\frac{5}{6}$ it produces . . . . $\frac{275ee}{216} = 45\frac{5}{6}aa$

9. And by multiplying the Equation in the third ſtep by $104\frac{1}{6}$, the Product will be . . . . $\frac{625e}{36} = 104\frac{1}{6}a$

10 Then by connecting the Quantities which ſtand in the firſt Parts of the Equations in the fifth, ſeventh, eighth, and ninth ſteps, together with 89, by the ſame Signs which reſpectively belong to each Term of the Equation propoſed, the Sum ſhall be equal to the Sum of the ſame Equation, and conſequently equal to nothing; hence this Equation ariſes, *viz.*

$$\frac{eeee}{1296} - \frac{10eee}{216} + \frac{275ee}{216} - \frac{625e}{36} + 89 = 0.$$

11. Which Equation being reduced to Integers (by *Sect.* 7. *Chap.* 11. *Book* 1.) gives

$$eeee - 60eee + 1650ee - 22500e + 115344 = 0.$$

Thus an Equation is found out whoſe Terms are all Integers; and the value of the Root e in this Equation is to the value of the Root a in the Equation propoſed as 6 to 1; (for by ſuppoſition in the ſecond ſtep $e=6a$:) and therefore if e be 12, then a ſhall be 2; or if e be 18, then a ſhall be 3.

VIII. *How to take away the ſecond Term of a Compound Equation.*

The Rule is this; Divide the Coefficient (that is, the known Quantity) multiplied into the ſecond Term of an Equation propoſed, by the Index (or Number of Dimenſions) of the Power which is the firſt Term. Then if the Signs of the firſt and ſecond Terms be unlike, (*viz.* if one be + and the other —) ſubtract the Quotient from the Affirmative Root ſought; but if the Signs be like, (that is, both + or both —) add the ſaid Quotient to the Affirmative Root; then Equate the ſaid Sum or Remainder to ſome Letter to repreſent an unknown quantity, and proceed according to the Method in the following Examples; ſo at length a new Equation will ariſe, wherein the ſecond Term is wanting.

Example 1.

1. Let there be propoſed this Equation . . . . $aa-6a = 72$

2. That is, ... $aa-6a-72 = 0$

3. Here the number of Dimenſions in the firſt Term aa is 2, and the known Number multiplied into a making the ſecond Term $6a$ is 6; this divided by the ſaid 2 gives 3, which ſubtracted from the Root a (becauſe the Signs of the firſt and ſecond Terms are unlike) leaves $a-3$, which is equal to ſome unknown number, let it be e; then,

4. By ſuppoſition . . . . $a-3 = e$

5. And conſequently by adding 3 to each part of that Equation there ariſes . . . . $a = e+3$

6. And by ſquaring each part of the laſt Equation there comes forth . . . . $aa = ee+6e+9$

7. And by multiplying each part of the Equation in the fifth ſtep by the Coefficient 6 in the propoſed Equation, it makes . . . . $6a = 6e+18$

8. Then by ſubtracting the laſt Equation from that in the ſixth ſtep, there remains . . . . $aa-6a = ee-9$

9. And laſtly, by ſubtracting 72 (the laſt Term of the Equation propos'd) from the Equation in the eighth ſtep, there remains . . . . $aa-6a-72 = ee-81=0$

Thus you ſee an Equation is found out, to wit, $ee-81=0$, which is equal to the Equation propoſed, and it wants the ſecond Term; (for there is not any number of e in the Equation found out.) Now if the value of e be made known, then the value of a is conſequently known; but the Equation found out, to wit, $ee-81=0$, that is, $ee=81$ gives $e=9$, and by the fifth ſtep $a=e+3$, therefore $a=12$.

Example 2.

1. Again, let there be propoſed this Equation, *viz.* . . . $aa+6a=216$
2. That is, ... $aa+6a-216=0$
3. Here (as before) I divide 6, the Coefficient in the ſecond Term $6a$, by 2, which denotes the Number of Dimenſions in the firſt Term aa, and the Quotient 3 I add to the Root a, (becauſe the firſt and ſecond Terms of the Equation have the ſame Sign $+$) and the Sum $a+3$ is equal to ſome unknown Number, let it be e; then,
4. By ſuppoſition . . . . $a+3=e$
5. Therefore by ſubtracting 3 from each part of that Equation, there ariſes . . . . $a=e-3$
6. And by ſquaring the laſt Equation there comes forth . . . . $aa=ee-6e+9$
7. And by multiplying the Equation in the fifth ſtep by 6, it produces . . . . $6a=6e-18$
8. Then by adding the two laſt Equations into one, the ſum is . . . . $aa+6a=ee-9$
9. And by ſubtracting 216 (the laſt Term of the Equation propos'd) from each part of the Equation in the eighth ſtep, there remains . . . . $aa+6a-216=ee-225=0$

Thus an Equation is found out, to wit, $ee-225=0$, which wants a ſecond Term, (for there is no Number of e in that Equation;) and when the value of e is made known, the value of a in the Equation propos'd is known alſo; but the Equation $ee-225=0$, that is, $ee=225$ gives $e=15$, and by the fifth ſtep $a=e-3$; therefore $a=12$, that is, $15-3$.

Example 3.

1. Again, let this Equation be propos'd . . . $aaa-18aa-7a+696=0$
2. According to the Rule before given, I divide 18 the known Number of the ſecond Term $-18aa$ by 3, which denotes the number of Dimenſions in the firſt Term aaa, and the Quotient is 6, this I ſubtract from the Root a, (becauſe the Signs of the firſt and ſecond Terms are unlike) and the Remainder is $a-6$, which is equal to ſome unknown Number, ſuppoſe it be e; then,
3. By ſuppoſition . . . . $a-6=e$
4. Therefore by adding 6 to each part of that Equation there ariſes . . . . $a=e+6$
5. And by ſquaring the laſt Equation it makes . . . . $aa=ee+12e+36$
6. And by multiplying the two laſt Equations one by the other, the Product is . . . . $aaa=eee+18ee+108e+216$
7. And by multiplying the Equation in the fifth ſtep by 18, (the Coefficient in the ſecond Term of the Equation propos'd) it makes . . . . $18aa=18ee+216e+648$
8. Likewiſe the Equation in the fourth ſtep being multiplied by 7, (the Coefficient in the third Term of the Equation propos'd) produces . . . . $7a=7e+42$
9. Then to the Equation in the ſixth ſtep adding 696, (to wit, the laſt Term of the Equation propos'd) the Sum is
$$aaa+696=eee+18ee+108e+912$$
10. Likewiſe, by adding the eighth Equation to the ſeventh, it makes
$$18aa+7a=18ee+223e+690$$
11. Laſtly, by ſubtracting the Equation in the tenth ſtep from that in the ninth, this following Equation remains, *viz.*
$$aaa-18aa-7a+696=eee-115e+222=0.$$

Thus an Equation is found out, to wit, $eee-115e+222=0$, which wants the ſecond Term, (to wit, the Power ee;) and when the value of the Root e is made known, the value of the Root a ſhall be known alſo, for by the fourth ſtep $a=e+6$; therefore if e be 2, then a ſhall be 8; and if e be equal to $\sqrt{112}-1$, then a ſhall be equal to $\sqrt{112}+5$.

Example 4.

1. Again, let there be propoſed . . . . $aaaa+6aaa+11aa+6a-100=0$
2. According to the Rule before given, I divide 6 the Coefficient in the ſecond Term $+6aaa$ by 4, which denotes the number of Dimenſions in the firſt Term $aaaa$, and the Quotient is $\frac{3}{2}$, which I add to the Root a, (becauſe the Signs of the firſt and ſecond Terms are like) and the ſum is $a+\frac{3}{2}$, which is equal to ſome unknown Number, let it be e, then,
3. By ſuppoſition . . . . $a+\frac{3}{2}=e$
4. Therefore . . . . $a=e-\frac{3}{2}$
5. The Square of the laſt Equation is . . . $aa=ee-3e+\frac{9}{4}$
6. And the two laſt Equations multiplied one by the other make . . . . . . $aaa=eee-\frac{9}{2}ee+\frac{27}{4}e-\frac{27}{4}$
7. And the Equation in the fixth ſtep being multiplied by that in the fourth ſtep, will produce . . . . $aaaa=eeee-6eee+\frac{27}{2}ee-\frac{27}{2}e+\frac{81}{16}$
8. And the Equation in the fixth ſtep multiplied by 6 produces . . . . . . $6aaa=6eee-\frac{54}{2}ee+\frac{162}{4}e-\frac{162}{8}$
9. And the Equation in the fifth ſtep multiplied by 11 produces . . . . . . $11aa=11ee-33e+\frac{99}{4}$
10. And the Equation in the fourth ſtep multiplied by 6 gives . . . . . $6a=6e-9$
11. Now 'tis manifeſt, that if from the Sum of the firſt Parts of the four laſt Equations there be ſubtracted 100, the Remainder will be equal to the Sum of the Equation firſt propos'd equal to 0; therefore alſo if 100 be ſubtracted from the Sum of the latter parts of the ſaid four Equations the Remainder ſhall be equal to 0, *viz.*
$$eeee-\tfrac{5}{2}ee-99\tfrac{7}{16}=0.$$
12. In which laſt Equation the ſecond Term, to wit, the Power eee, is wanting, as was deſired. And when the value of e is made known, the value of the Root a in the Equation propoſed ſhall be known alſo; for by the fourth ſtep $a=e-\frac{3}{2}$, but (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) the value of e in the Equation in the eleventh ſtep will be found $\sqrt{:1\frac{1}{4}+\sqrt{101}:}$ and therefore $a=\sqrt{:1\frac{1}{4}+\sqrt{101}:}-\frac{3}{2}$.

IX. *The uſe of the preceding Rules of this Chapter, in the Reſolution of all manner of Compound Equations in Numbers.*

After an adfected or compound Equation different from any of the three Forms in *Sect.* 1. *Chap.* 15. *Book* 1. is prepared for Reſolution by the Rules of *Chap.* 12. *Book* 1. and reduced (if need be) to Integers, and the ſum of all the Terms made equal to 0 (or nothing) according to *Sect.* 7. and 2. of this *Chap.* ſearch out (by the Rules of *Chap.* 8. of this *Book*) all the juſt Diviſors to the laſt Term (that is, the known abſolute Number of the Equation ſo reduced.) Then try whether any one of thoſe Diviſors connected to the unknown Root a by — or +, will divide the total Sum of the ſaid reduced Equation without leaving a Remainder; for when ſuch Diviſion ſucceeds, either the known part of the ſaid Binomial Diviſor is the deſired value of the Root a, or at leaſt the Quotient gives an Equation, whoſe firſt Term has fewer Dimenſions by one than the Equation divided; and then the Root of this new Equation, if its firſt Term be a Square, may be found out by ſome of the Canons in *Sect.* 6,8,10. of *Chap.* 15. *Book* 1. But if the firſt Term contains three or more Dimenſions, let this Equation be examined by Diviſion, (as before,) and if none of thoſe Diviſions work off juſt without a Fraction, then by taking away the ſecond Term, (by the Rule in *Sect.* 8. of this *Chap.*) another Equation more ſimple, and ſuch as may be reſolved by ſome of the Canons in *Sect.* 6, 8, 10. *Chap.* 15. *Book* 1. will ſometimes ariſe. But if none of thoſe ways prove effectual, you may by the general Method in the foregoing *Chap.* 10. find out one affirmative Root very near a true Root, and then joyning this Root found out to the unknown Root a by the Sign —, you may by this Binomial divide the Equation, and proceed to find out the reſt of the Roots very near the truth. All which will be made manifeſt by the following Queſtions.

QUEST-

QUESTION. I.

$$\left.\begin{array}{lrl}\text{If} \;.\;.\;. & aaa-9aa+26a & =24 \\ \text{That is, if} \;.\;.\;. & aaa-9aa+26a-24 & =0\end{array}\right\}\text{What is the Number } a\,?$$

RESOLUTION.

Firſt, (by the Method in *Sect.* 5. *Chap.* 8. of this *Book*) I ſearch out all the Numbers that will ſeverally divide the laſt Term 24 without a Remainder, and find them to be theſe, *viz.* 1,2,3,4,6,8,12,24. Then by examining in order whether the total ſum of the Equation propos'd may be divided by $a-1$ or $a+1$, by $a-2$ or $a+2$, *&c.* I find it may be exactly divided by $a-2$ without a Remainder, and the Quotient is $aa-7a+12$, as you ſee by this following Diviſion.

$$\begin{array}{rlrl} a-2\;) & aaa-9aa+26a-24 & (& aa-7a+12 \\ & aaa-2aa & & \\ \hline & -7aa+26a & & \\ & -7aa+14a & & \\ \hline & +12a-24 & & \\ & +12a-24 & & \\ \hline & 0 \quad 0 & & \end{array}$$

Therefore 2 the known Number in the Diviſor $a-2$ is one real or affirmative Root of the Equation propoſed; for as well the Diviſor as the Dividend was ſuppoſed equal to nothing, *viz.* $a-2=0$, whence $a=2$; the Quotient alſo is conſequently equal to 0, *viz.* $aa-7a+12=0$, that is, $7a-aa=12$; hence (by the Canon in *Sect.* 10. *Chap.* 15. *Book* I.) two other affirmative values of the Root a will be diſcovered, to wit, 4 and 3. So that three real values of a, to wit, 2, 3, and 4, are found out, by every one of which the Equation propos'd may be expounded, as the Proof will eaſily ſhew.

QUESTION 2.

$$\left.\begin{array}{lrl}\text{If} \;.\;.\;. & aaa-22aa+157a & =360 \\ \text{That is, if} \;.\;.\;. & aaa-22aa+157a-360 & =0\end{array}\right\}\text{What is } a=?$$

RESOLUTION.

Firſt, the Diviſor of the laſt Term 360 will be found theſe, 1,2,3,4,5,6,8,9,10, 12,15,18,20,24,30,36,40,45,60,72,90,120,180, and 360; then by examining in order whether the ſum of the Equation propos'd may be divided by $a-1$ or $a+1$, by $a-2$ or $a+2$, by $a-3$ or $a+3$, *&c.* I find that $a-5$ will preciſely divide the ſaid Sum without a Fraction, and therefore 5 is one affirmative Root or Value of a; then the Quotient $aa-17a+72a=0$, that is, $17a-aa=72$ affords two other affirmative values of a, to wit, 8 and 9. Thus you ſee three real values of a, to wit, 5, 8, and 9, are found out; by every one of which the Equation propoſed, to wit, $aaa-22aa+157a=360$ may be expounded, as will appear by the Proof.

QUESTION. 3.

$$\left.\begin{array}{lrl}\text{If} \;.\;.\;. & 91a-aaa & =330 \\ \text{That is, if} \;.\;.\;. & aaa-91a+330 & =0\end{array}\right\}\text{What is } a=?$$

RESOLUTION.

Firſt, the Diviſors of the laſt Term 330 will be found 1,2,3,5,6,10,11,15,22,30, 55,66,110,165, and 330; then by examining in order whether the ſum of the Equation propos'd, to wit, $aaa-91a+330$ may be divided by $a-1$ or $a+1$, by $a-2$ or $a+2$, *&c.* I find it may be divided by $a-5$ and leave no Remainder; therefore $a-5=0$ gives $a=5$, which is one affirmative Root of the Equation propos'd, and the Quotient $aa+5a-66=0$, that is, $aa+5a=66$ affords another affirmative value of a, to wit 6. So that two real values of a are found out, by each of which the Equation propos'd may be expounded; for if $a=5$, or $a=6$, from either ſuppoſition it follows that $91a-aaa=330$.

QUESTION 4.

To find two Numbers whoſe Sum ſhall be 5, and that if the Sum of their Squares be multiplied by the Sum of their Cubes, the Product may be 455.

RESOLUTION.

This Queſtion may be ſolved by the Canon of *Queſt.* 13. *Chap.* 16. *Book* I. but that Canon being raiſed from Poſitions that lie out of the common Road, I ſhall here ſolve the Queſtion in the ordinary way, and ſo it will exerciſe the preceding Rules of this Chapter. Firſt then,

1. For one of the Numbers ſought put . . . . a
2. Therefore the other Number is . . . . . $5-a$
3. The ſquare of the firſt Number is . . . . aa
4. The Square of the ſecond is . . . . $aa-10a+25$
5. The Sum of thoſe Squares is . . . . $2aa-10a+25$
6. The Cube of the firſt Number is . . . . aaa
7. The Cube of the ſecond is . . . . . . $-aaa+15aa-75a+125$
8. Therefore the Sum of thoſe Cubes is . . . . $+15aa-75a+125$
9. Which Sum being multiplied by the Sum of the Squares in the fifth ſtep gives this Product, *viz.* $30aaaa-300aaa+1375aa-3125a+3125$.
10. But according to the Queſtion, the Product in the laſt ſtep muſt be equal to the given Product 455, hence this Equation ariſes,
$$30aaaa-300aaa+1375aa-3125a+3125=455.$$
11. And by ſubtracting 455 from each part of the laſt Equation this ariſes,
$$30aaaa-300aaa+1375aa-3125a+2670=0.$$
12. And by dividing every Term in the laſt Equation by 30 this ariſes.
$$aaaa-10aaa+45\tfrac{5}{6}aa-104\tfrac{1}{6}a+89=0$$
13. Then by ſuppoſing $e=6a$, and proceeding according to the *Example* 5. in *Sect.* 7. of this *Chap.* to free the Equation in the preceding twelfth ſtep from Fractions, this will be produced, *viz.*
$$eeee-60eee+1650ee-22500e+115344=0.$$
14. Now the Diviſors of the laſt Term 115344 will be found 1,2,3,4,6,8,9,12,18, 24,27, *&c.* and after tryals made by Diviſion, (like as in the three laſt preceding Queſtions) I find that $e-12=0$ will preciſely divide the ſum of the Equation in the thirteenth ſtep, and therefore 12 is one true value of e. Again, the Quotient of that Diviſion being $eee-48ee+1074e-9612$, I ſeek the Diviſors of the laſt Term 9612, and find them to be 1,2,3,4,6,9,12,18,27,36, *&c.* Then after tryals made (as before) I find that $e-18$ will exactly divide the ſaid $eee-48ee+1074e-9612$, and therefore 18 is one other affirmative value of e; and becauſe the Quotient of the laſt mentioned Diviſion, to wit, $ee-30e+534=0$, that is, $30e-ee=534$, is an impoſſible Equation, (as is evident by the Determination in *Sect* 9. *Queſt.* I. *Chap* 15. *Book* I.) I conclude that the Equation in the thirteenth ſtep has no other Root or Value of e beſides 12 and 18 before found. But becauſe by ſuppoſition in the thirteenth ſtep $e=6a$, $\frac{1}{6}$ of 12 and likewiſe of 18, that is, 2 and 3, ſhall be the true values of a to ſolve the Queſtion, for their ſum is 5; and if 13 the ſum of their Squares be multiplied by 35 the ſum of their Cubes, the Product is 455, as was deſired.

Sometimes the taking away of the ſecond Term of an Equation (by the Rule in *Sect.* 8. of this *Chap.*) will be an Expedient to find out an Equation reſolvable by ſome of the Canons in *Sect.* 6, 8, and 10. *Chap.* 15. *Book* I. when tryals by Diviſion (as before) will be in vain, as will appear by the following fifth Queſtion, which I find reſolved two manner of ways in *Pag.* 319. of *Carteſius* his Geometry, ſet forth with Comments by *Fran. van. Schooten*, and Printed at *Amſterdam* 1659.

QUESTION. 5.

To find four Numbers in Arithmetical Progreſſion continued, ſuch that their common Difference may be Unity, and the Product made by their continual Multiplication 100.

RESOLUTION.

1. For the firſt Number put . . . . a
2. Then the ſecond ſhall be . . . . $a+1$
3. The third . . . . $a+2$
4. And the fourth . . . . $a+3$
5. Therefore the Product of their continual Multiplication is . . . . $aaaa+6aaa+11aa+6a$

6. Which Product muſt be equal to 100, therefore . . . . $\Big\}\ aaaa+6aaa+11aa+6a=100$
7. That is, ... $aaaa+6aaa+11aa+6a-100=0$
8. Of which Equation the laſt Term 100 may be divided by 1,2,4,5,10,20,25,50, and 100; but Diviſion being tried by $a-$ or $+1$, by $a-$ or $+2$, by $a-$ or $+4$, *&c.* it proves ineffectual. Then by taking away the ſecond Term, (as in *Example* 4. *Sect.* 8. of this *Chap.*) this Equation ariſes, *viz.* $eeee-2\frac{1}{2}ee-99\frac{7}{16}=0$, in which the Root e (by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1.) will be found equal to $\sqrt{:1\frac{1}{4}+\sqrt{101}:}$ but in taking away the ſecond Term a was put equal to $e-\frac{3}{2}$, and therefore $a=\sqrt{:1\frac{1}{4}+\sqrt{101}:}-\frac{3}{2}$; and conſequently from the firſt, ſecond, third, and fourth ſteps,

$$\text{The four Numbers ſought are theſe,}\ \begin{cases}\sqrt{:1\frac{1}{4}+\sqrt{101}:}-\frac{3}{2}\\ \sqrt{:1\frac{1}{4}+\sqrt{101}:}-\frac{1}{2}\\ \sqrt{:1\frac{1}{4}+\sqrt{101}:}+\frac{1}{2}\\ \sqrt{:1\frac{1}{4}+\sqrt{101}:}+\frac{3}{2}\end{cases}$$

Which Numbers exceed one another by Unity, and the Product of their Multiplication is 100, as before has been proved in *Queſt* 3. *Sect.* 17. *Chap.* 9. of this Book.

Another way of Reſolving Queſt. 5.

For the firſt number put $a-1\frac{1}{2}$, for the ſecond $a-\frac{1}{2}$, for the third $a+\frac{1}{2}$, and for the fourth $a+1\frac{1}{2}$; then by multiplying theſe four Numbers one into another, and comparing the Product to 100, this Equation ariſes, *viz.* $aaaa-2\frac{1}{2}aa=99\frac{7}{16}$; whence the four Numbers ſought will be found the ſame as before.

QUESTION 6.

1. . . If . . . $8a^3+63aa-a^4-341a=1304$
2. That is, if . . . $a^4-8a^3-63aa+341a+1304=0$;
   What is the Number a?

RESOLUTION.

3. The Diviſors of the laſt Term 1304 are 1,2,4,8,163,326, and 1304; then after tryals made by Diviſion, (as in the preceding Queſtions) I find $a-8=0$ will exactly divide the ſum of the Equation propoſed without any Remainder, and therefore 8 is one affirmative value of the Root a. Again, becauſe the Diviſors of 163 the laſt Term of this Equation $aaa-63a-163=0$, (which was the Quotient of the ſaid Diviſion) are only Unity and 163, I try to divide the Equation laſt mentioned by $a-1$ and $a+1$, likewiſe by $a-163$ and $a+163$; but none of theſe Diviſions working off juſt without a Fraction, and there being no ſecond Term to be taken away, I ſearch out one affirmative value of a out of the ſaid Equation $aaa-63a-163=0$, (that is, $aaa-63a=163$) by the general Method in the foregoing *Chap.* 10. and thereby diſcover $a=9.0055$, *&c.* then I divide the ſaid Cubic Equation $aaa-63a-163=0$, by $a-9.0055=0$, and the Quotient (the Remainder after the Diviſion is ended being neglected) is $aa+9.0055a+18.09903025=0$; but this Equation cannot poſſibly have any affirmative Root, and therefore I conclude that the Equation firſt propos'd to be reſolved has only two affirmative Roots or Values of a, to wit, 8 and 9.0055, *&c.* found out as above.

By the like Operation it will appear, that this Equation $a^4-17a^3-212aa+4979a-21131=0$ may be expounded by every one of theſe three Roots or Values of a, to wit 11, 7.1125, *&c.* and 15.8874, *&c.* but by no other affirmative Root.

When the Index of the Power of the unknown Quantity in every Term of an Equation is an even number, the Reſolution of ſuch Equation will admit of a Contraction, which will be made manifeſt by this following

QUESTION. 7.

1. If . . . . . $a^6-29a^4+244a^2-576=0$; What is $a=$?

RESOLUTION.

2. Here becauſe the Indices of the unknown Powers are even Numbers, to wit, 6, 4, and 2, put . . . . $\Big\}\ e=a^2$

3. Then for . . . . : . . . . : . . $\left\{\begin{array}{l}+a^6\\-29a^4\\+244a^2\end{array}\right\}$ write $\left\{\begin{array}{l}+e^3\\-29e^2\\+244e\end{array}\right.$

4. To which Powers of e joyn -576, the laſt Term of the given Equation, and it makes

$$e^3-29e^2+244e-576 = 0.$$

5. Which laſt Equation being reſolved by Diviſion, (in like manner as in the preceding Examples of this Section) there will be found three affirmative values of the Root e, *viz.* 4, 9, and 16; then becauſe e was put equal to aa, the ſquare Root of 4, 9, and 16, that is, 2, 3, and 4, ſhall be three Roots or Values of a in the Equation firſt propoſed, to wit, $a^6-29a^4+244a^2-576=0$, as may eaſily be proved.

I might here ſhew how to reduce a Biquadratic Equation, not falling under any of the three Forms in *Sect.* 1. *Chap.* 15. *Book.* 1. to a Cubic Equation, and ſometimes into two Quadratic Equations, but I ſhall ſpare that labour for theſe Reaſons: Firſt, that Reduction being ſubject to many caſes, is very tedious and troubleſom. Secondly, ſuch a Biquadratic Equation is ſeldom capable of being reduced into two Quadratic Equations; and when 'tis reduced to a Cubic Equation, this may happen to be ſuch as its Root or Roots in Numbers cannot be perfectly found out by any Rules hitherto publiſh'd by any Author. Thirdly, by the Method in this ninth Section all the Roots of any Cubic, Biquadratic, or other Equation of higher degrees, may be found out in Numbers, either exactly if they be Rational, or as near the truth if they be Irrational, as ſhall be needful for any practical uſe. And laſtly, my undertaking (as I have before hinted) is not to handle all, but the moſt uſeful Rules only in this profound Art.

Note. The Reſolutions of the preceding Queſtions of this ninth Section do clearly ſhew, that there is no ſmall labour in making tryals with the Diviſors of the laſt Term of an Equation to find its Root or Roots; and therefore to leſſen that work, firſt, it will be convenient to make ſome tryals by the general Method in the foregoing *Chap.* 10. to find out Limits within which the Root or Roots of an Equation do fall, or to argue the ſame from ſome things given in a Queſtion producing the ſaid Equation, and then to make tryals only with ſuch Diviſors of the laſt Term as fall within thoſe Limits; but when all Contractions are uſed, the work is ſufficiently laborious, ſo that one chief Scope of an Analyſt in reſolving a knotty Queſtion muſt be to frame his Poſitions with ſuch artifice, that the Reſolution may end in as ſimple an Equation as is poſſible. And altho one way of Reſolution may produce an Equation compoſed of high Powers, yet oftentimes by another way you may come to a more ſimple Equation, as may partly appear by the foregoing fourth and fifth Queſtions of this Section; but the skill of finding out the moſt ſimple and facil ways of Reſolution, is not attainable (as I conceive) by any certain or conſtant Method, but rather by much uſe and exerciſe in the ſolving of Queſtions.

Sect. X. *Concerning the Reſolution of certain Cubic Equations in Numbers by two Rules, the Invention whereof* Cardanus *attributes to* Scipio Ferreus.

1. All Cubic Equations, after the ſecond Term is taken away, when there happens to be any, (by the Rule in *Sect.* 8. of this *Chap.*) are reducible to theſe three following Forms, in which a repreſents the Root or Quantity ſought, but p and q known Quantities.

$$\begin{array}{l|l} aaa = -6a+20 & aaa = -pa+q \\ aaa = +6a+40 & aaa = +pa+q \\ aaa = +91a-330 & aaa = +pa-q \end{array}$$

2. Now let it be required to reſolve the firſt of thoſe Equations, *viz.*
   If $aaa = -6a+20$, or $aaa = -pa+q$;
   What is the value of a?

Preparation.

3. Suppoſe . . . . $a = e-y$
4. Suppoſe alſo . . . . $20 = eee-yyy$
5. And . . . . $6 = 3ey$
6. Then by multiplying each part of the Equation in the third ſtep into it ſelf Cubically, this is produced, *viz.* $aaa = eee-3eey+3eyy-yyy$

7. And by multiplying the Equations in the third and fifth ſteps one into the other, it makes . . . . . $6a = 3eey - 3eyy$

8. And by ſubtracting the Equation in the ſeventh ſtep from that in the fourth, there remains . . . . $20 - 6a = eee - 3eey + 3eyy - yyy$

9. Therefore by the ſixth and eighth ſteps 'tis manifeſt that . . . . $aaa = eee - 3eey + 3eyy - yyy = 20 - 6a$

10. From the premiſſes it's evident, that if in the Equation propos'd to be reſolv'd to wit, $aaa = -6a + 20$, or $aaa = -pa + q$, we ſuppoſe the Root a ſought to be equal to the difference of two unknown numbers e and y; alſo the abſolute number 20 (or q) to be equal to the difference of the Cubes of the ſame two numbers, and the Co-efficient 6 (or p) to be equal to the triple Product of their Multiplication: then as well aaa as $20 - 6a$ (that is, $q - pa$) ſhall be equal to the Cube of the difference of thoſe two numbers, *viz.* to the Cube of $e - y$; and therefore when two ſuch numbers are found out; their difference ſhall be the Root or number a ſought. But to find out the ſaid two numbers (e and y) there is given the Product of their Multiplication, to wit 2, (or $\frac{1}{3}p$) that is, one third part of the Co-efficient, as alſo 20 (or q) the difference of the Cubes of the ſame two numbers. And therefore the numbers themſelves ſhall be given ſeverally by the Canon of *Queſt.* 15. *Chap.* 16. *Book* 1. and conſequently the Root a ſought ſhall be given alſo, as will be made manifeſt by this following

	Operation.	
11. To the Square of half the given Abſolute number 20 (or q) *viz.* to . .	100	$\frac{1}{4}qq$
12. Add the Cube of 2 (or $\frac{1}{3}p$) *viz.* the Cube of $\frac{1}{3}$ of the Co-efficient 6 (or p) which Cube is :	8	$\frac{1}{27}ppp$
13. The Sum is	108	$\frac{1}{4}qq + \frac{1}{27}ppp$
14. The Square Root of that Sum is . . .	$\sqrt{108}$	$\sqrt{:\frac{1}{4}qq + \frac{1}{27}ppp:}$
15. To that Square Root add half the Abſolute number 20 (or q) and the Sum is	$10 + \sqrt{108}$	$\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}$
16. The Cube Root of that Sum is the greater number e ſought, *viz.* . .	$\sqrt[3]{:10 + \sqrt{108}:}$	$\sqrt[3]{:\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}:}$
17. Again, from the ſquare Root in the fourteenth ſtep ſubtract half the abſolute number 20 (or q) and the Remainder is .	$-10 + \sqrt{108}$	$-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}$
18. Then the Cubic Root of that Remainder ſhall be the leſſer number y ſought, *viz.* . . .	$\sqrt[3]{:-10 + \sqrt{108}:}$	$\sqrt[3]{:-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}:}$

19. And then the difference of the two Cubic Roots found out in the ſixteenth and eighteenth ſteps ſhall be the value of the Root a in the Equation propoſed, *viz.*

$a = \sqrt[3]{:1 + \sqrt{108}:} - \sqrt[3]{:-10 + \sqrt{108}:}$ that is,

$a = \sqrt[3]{:\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}:} - \sqrt[3]{:-\frac{1}{2}q + \sqrt{\frac{1}{4}qq + \frac{1}{27}ppp}:}$

20. It remains to make tryal whether the Binomial $10 + \sqrt{108}$ has a perfect Cubic Root or not; ſo by the Rule in *Sect.* 18. *Chap.* 9. of this ſecond Book, it will appear that $1 + \sqrt{3}$ is the Cubic Root of $10 + \sqrt{108}$, and $\sqrt{3} - 1$ is the Cubic Root of $\sqrt{108} - 10$; and conſequently the value of the Root a before found out in the nineteenth ſtep is expreſſible by a rational number; for if $\sqrt{3} - 1$ be ſubtracted

from $1+\sqrt{3}$, the Remainder 2 is the deſired value of a in the Equation propoſed; for if $a=2$, then $aaa=20-6a$, or $aaa+6a=20$.

21. In like manner by the Canon in the foregoing nineteeth ſtep the value of a in this Equation $aaa+27a=64$, will be found this that follows, *viz.*

$$a = \sqrt[3]{32+\sqrt{1753}} - \sqrt[3]{-32+\sqrt{1753}}$$

But this value of a cannot be expreſt by any rational number, becauſe the Binomial $32+\sqrt{1753}$ has not a perfect Cubic Root, and therefore the ſaid value muſt either reſt in that Surd Form, or elſe be expreſt by ſome rational number near the true value, which will be found 2. 05, &c. that is, $2\frac{5}{100}$, &c.

22. In the next place let it be required to reſolve a Cubic Equation of the ſecond of the three Forms before mentioned, *viz.*

If . . . . . $aaa = 6a+40$; or, $aaa = pa+q$;
What is the value of a?

Preparation.

23. Suppoſe . . . . $a = e+y$
24. Suppoſe alſo, ... $40 = eee+yyy$
25. And . . . . $6 = 3ey$
26. Then by multiplying each part of the Equation in the twenty third ſtep into itſelf cubically, this is produced, $aaa = eee+3eey+3eyy+yyy$
27. And the Equations in the twenty third and twenty fifth ſteps being mutually multiplied one by the other will produce . . . . $6a = 3eey+3eyy$
28. And the Sum of the Equation in the twenty fourth and twenty ſeventh ſteps makes . . . . $6a+40 = eee+3eey+3eyy+yyy$
29. Therefore by the twenty ſixth and twenty eight ſteps 'tis evident that . $aaa = eee+3eey+3eyy+yyy=6a+40$

30. By the eight laſt preceding ſteps 'tis manifeſt, That if in the Equation propos'd to be reſolved, to wit, $aaa=6a+40$, or $aaa=pa+q$, we ſuppoſe the Root a ſought to be equal to the ſum of two unknown numbers, e and y, alſo the abſolute number 40 (or q) to be equal to the ſum of the Cubes of the ſame two numbers, and the Co-efficient 6 (or p) to be equal to the triple Product of their Multiplication, then as well aaa as $6a+40$ (that is, $pa+q$) ſhall be equal to the Cube of $e+y$; and therefore when two ſuch numbers are found out, their ſum ſhall be the Root or number a ſought. But to find out the ſaid two numbers (e and y) there is given the Product of their Multiplication, to wit 2 (or $\frac{1}{3}p$) that is, $\frac{1}{3}$ part of the Co-efficient, as alſo 40 (or q) the ſum of the Cubes of the ſame two numbers, and therefore the numbers ſhall be given ſeverally by the Canon of *Qu.* 14. *Ch.* 16. *Book* 1. and conſequently the Root a ſought ſhall be given alſo. All which will be made manifeſt by this following

Operation.

31. From the Square of half the given abſolute number 40 (or q) *viz.* from	400	$\frac{1}{4}qq$
32. Subtract the Cube of 2 (or $\frac{1}{3}p$) *viz.* the Cube of $\frac{1}{3}$ of the Co-efficient, which Cube is	8	$\frac{1}{27}ppp$
33. The Remainder is	392	$\frac{1}{4}qq-\frac{1}{27}ppp$
34. The ſquare Root of that Remaind. is	$\sqrt{392}$	$\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}$
35. Which ſquare root added to half the abſolute number 40 (or q) makes the ſum	$20+\sqrt{392}$	$\frac{1}{2}q+\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}$
36. The Cubic Root of the ſum in the laſt ſtep is the value of e .	$\sqrt[3]{20+\sqrt{392}}$	$\sqrt[3]{\frac{1}{2}q+\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}}$
37. The ſquare root in the thirty fourth ſtep being ſubtracted from half the abſolute number 40 (or q) leaves	$20-\sqrt{392}$	$\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}$
38. The Cubic Root of that Remainder is the value of y	$\sqrt[3]{20-\sqrt{392}}$	$\sqrt[3]{\frac{1}{2}q-\sqrt{\frac{1}{4}qq-\frac{1}{27}ppp}}$

36 Then

39. Then the ſum of the two Cubic Roots found out in the thirty ſixth and thirty eighth ſteps ſhall be the value of the Root a in the Equation propos'd to be reſolved

$$a = \surd(3)\overline{:2+\surd 392:}+\surd(3)\overline{:20+\surd 392:}\ \text{that is,}$$

$$a = \surd(3)\overline{:\tfrac{1}{2}q+\surd\tfrac{1}{4}qq+\tfrac{1}{27}ppp:}+\surd(3)\overline{:-\tfrac{1}{2}q+\surd\tfrac{1}{4}qq+\tfrac{1}{27}ppp:}$$

40. It remains to make tryal whether the Binomial $20+\surd 392$ has a perfect Cubic Root or not; ſo by the Rule in *Sect.* 18. *Chap.* 9. of this ſecond Book, you will find $2+\surd 2$ to be the Cubic Root of $20+\surd 392$, and $2-\surd 2$ the Cubic Root of $20-\surd 392$, and conſequently the value of the Root a before found out in the thirty ninth ſtep is expreſſible by a rational Number; for if $2-\surd 2$ be added to $2+\surd 2$, the ſum 4 is the deſired value of a in the Equation propoſed to be reſolved: for if $a=4$, then $aaa=6a+40$.

41. Another Example of reſolving a Cubic Equation of the ſecond Form may be this, *viz.* Let it be required to find the value of a in this Equation $aaa=12a+18$, that is, $aaa=pa+q$, then the Cannon expreſt by the Literal Equation in the thirty ninth ſtep will give

$$a=\surd(3)\overline{:9+\surd 17:}+\surd(3)\overline{:9-\surd 17:}$$

But this value of a is inexpreſſible by any rational number, becauſe the Binomial $9+\surd 17$ has not a perfect Cubic Root, and therefore the ſaid value muſt either reſt in that Surd Form, or elſe be expreſs'd by ſome rational number near the true value, which will be found 4. 05, &c. that is, $4\frac{5}{100}$, &c.

The premiſſes do clearly ſhew the riſe of two Rules delivered by *Cardanus* in his Algebraical Treatiſe entituled *Ars magna*, which Rules are mentioned in divers Authors, and the Subſtance of them is contained in the two literal Equations in the foregoing nineteenth and thirty ninth ſteps; the former of which Equations is a Canon to find out the Root of any Cubic Equation in Numbers, which falls under the firſt of the three Forms before mentioned, and to expreſs the ſame perfectly either by ſome rational or irrational Number; and the later of thoſe literal Equations finds out the like exact Root of any Cubic Equation of the ſecond Form, except in one caſe only, *viz.* when the Square of half the abſolute Number (q) which is the laſt term of the Equation, is leſs than the Cube of one third part of the known Co-efficient (p). But no Author that I have met with, gives a certain Rule, either to find out the Root in that caſe if it be an irrational number; or the two affirmative Roots of a Cubic Equation of the third Form, if each of theſe alſo be irrational. *Huddenius* indeed ſaith in *pag.* 503. of *Carteſius* his Geometry before mentioned, he had a Rule (which he intended to publiſh) by which all irrational Roots, as well of numeral as of literal Equations, may be found out; but that much deſired Rule is not yet come to light. But when a Cubic Equation of what kind ſoever has one Root expreſſible by a rational Number, both that and the reſt of the Roots, when the Equation is capable of more than one, may be exactly found out by the help of the Diviſors of the laſt term, according to *Sect.* 9 of this *Chap.*

CHAP. XII.

Of the Method of reſolving Queſtions wherein many Quantities are ſought, by aſſuming different Letters to repreſent the ſaid Quantities ſeverally.

I. Hitherto in the Algebraical Reſolution of a Queſtion, wherein two or more Quantities have been ſought, I have aſſumed only one letter, as a or e, to repreſent ſome one of the unknown Quantities, and formed the Poſitions for the reſt by the help of that letter and the Quantities given in the Queſtion. But many Queſtions may be more eaſily reſolved by aſſuming a peculiar letter to repreſent every one of the Quantities ſought; as a for one unknown Quantity, e for a ſecond, y for a third, &c. By this Method alſo thoſe intricate and obſcure ways of reſolving Queſtions by ſecond Roots, or (as *Simon Stevin* calls them) poſtpoſed Quantities, will be avoided.

In handling the following Method I ſhall give three principal Rules, and explain them by Examples; but to preſcribe Rules for all Caſes, is (as I conceive) an impoſſible Work.

RULE I.

When many Quantities are ſought by a Queſtion, firſt let them be ſeverally repreſented by different Letters; then after you have well conſidered the Condition in the Queſtion, abſtract it from words, and expreſs the Tenor thereof by Equations; that done by the help of tranſpoſition find what the firſt, that is, any ſingle Letter repreſenting a number or quantity ſought in the firſt Equation is equal to; then wheresoever that firſt Letter is found in the other Equations, take inſtead of it thoſe Quantities to which the ſaid firſt Letter was found equal: ſo ſuch firſt Letter will quite vaniſh out thoſe other Equations. Again, by Tranſpoſition ſet a ſecond Letter alone in one of thoſe Equations out of which the firſt Letter was expell'd; and proceed as before; ſo at length one of the numbers ſought will be made known, by the help whereof the reſt will eaſily be diſcovered. This work will be better underſtood by Examples than many Words, and therefore I ſhall proceed to Queſtions.

QUESTION 1.

A Factor exchanged 6 *French* Crowns and 2 Dollars for 45 Shillings of *Engliſh* Money; alſo at another time he exchanged 9 *French* Crowns and 5 Dollars (each of theſe being of the ſame value with the former) for 76 Shillings: I demand the value of a *French* Crown, and alſo of a Dollar, in *Engliſh* Money?

Let a repreſent the deſired value of a Crown, and e the value of a Dollar, then the Queſtion being abſtracted from Words may be ſtated thus.

1. If . . . . $6a + 2e = 45$
2. And . . . . $9a + 5e = 76$

What are the Numbers a and e? ‖

RESOLUTION.

3. By Tranſpoſition of $2e$ in the firſt Equation this ariſes . $6a = 45 - 2e$
4. And by dividing each part of the third Equation by 6, it gives . . . . $a = \frac{45 - 2e}{6}$
5. The fourth Equation multiplied by 9 produces . $9a = \frac{405 - 18e}{6}$
6. Then if inſtead of $9a$ in the ſecond Equation you take the later Part of the fifth, this will ariſe . . . $\frac{405 - 18e}{6} + 5e = 76$
7. The ſixth Equation, after due Reduction, diſcovers the value of a Dollar, *viz.* . . . . $e = 4\frac{1}{4}$
8. The ſeventh Equation multiplied by 2 gives . . . $2e = 8\frac{1}{2}$
9. And by ſetting the later part of the eighth Equation in the place of $2e$ in the firſt, this Equation ariſes . $6a + 8\frac{1}{2} = 45$
10. From which laſt Equation, after due Reduction, the value of a or one *French* Crown is diſcovered, *viz.* . . $a = 6\frac{1}{12}$

Thus by the ſeventh and tenth Equations it is found that a Dollar was valued at 4 *s.* 3 *d.* and a *French* Crown at 6 *s.* 1 *d.* which numbers will ſatisfie the Conditions in the Queſtion, as may eaſily be proved.

QUESTION 2.

Three Men had every one of them a certain number of Pounds in his Purſe; the ſum of the firſt and ſecond mans Money was 5 (or b) Pounds, the Sum of the ſecond and third mans Money was 12 (or c) Pounds, and the Sum of the third and firſt mans Money was 11 (or d) Pounds: How many Pounds had every one in his Purſe?

Let the three numbers of Pounds ſought be repreſented by a, e, and y; then reſpect being had to the numbers given, the Queſtion may be ſtated thus, *viz.*

1. If . . . . $a + e = b \; (= 5)$
2. And . . . . $e + y = c \; (= 12)$
3. And . . . . $y + a = d \; (= 11)$

What are the Numbers a, e, and y? ‖

RE-

RESOLUTION.

4. By Tranſpoſition of *a* in the firſt Equation there will ariſe $e = b - x$
5. Then by taking the later part of the fourth Equation inſtead of *e* in the ſecond, this Equation ariſes . . $b - a + y = c$
6. And by Tranſpoſition of $b - a$ in the 5th Equation it gives $y = c - b + a$
7. And by taking the later part of the ſixth Equation inſtead of *y* in the third, this ariſes . . . . $c - b + a + a = d$
8. From which ſeventh Equation, after due Reduction, the Number *a* will be made known, *viz.* . . . . . $a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$
9. Again, if inſtead of *a* in the firſt Equation we take the later part of the eighth, this ariſes . . . . $\frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c + e = d$
10. Then from the ninth, after due Reduction, the Number *e* will be made known, *viz.* . . . . $e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d$
11. Again, if inſtead of *a* in the third Equation we take the later part of the eighth, this ariſes . . . . . $y + \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c = d$
12. Laſtly, from the eleventh Equation, after due Reduction, the Number *y* will be made known, *viz.* . . . $y = \frac{1}{2}d + \frac{1}{2}c - \frac{1}{2}b$

The eighth, tenth, and twelfth Equations give this

CANON.

From the ſum of every two of the three Numbers given ſubtract the remaining number, then the halves of the three remainders ſhall be the numbers ſought. Whence the numbers ſought, to wit, *a*, *e*, and *y*, will be found 2, 3, and 9; for $2+3=5$, alſo $3+9=12$, and $9+2=11$, as was required.

The foregoing *Reſolution* of this *Queſt.* 2. is formed according to *Rule* 1. but the ſame Canon may be more expeditiouſly diſcovered by this following *Reſolution*, *viz.*

The Sum of the firſt, ſecond, and third Equations which ſtate the Queſtion is . . . . $2a + 2e + 2y = b + c + d$

The half of that Sum is . . . . $a + e + y = \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}d$

Then from that half ſum ſubtract the firſt Equation, and the Remainder will be . . . . $y = \frac{1}{2}c + \frac{1}{2}d - \frac{1}{2}b$

Again, from the ſaid half ſum ſubtract the ſecond Equation, and the Remainder is . . . . $a = \frac{1}{2}b + \frac{1}{2}d - \frac{1}{2}c$

Laſtly, from the ſaid half ſum ſubtract the third Equation, and the Remainder gives . . . . $e = \frac{1}{2}b + \frac{1}{2}c - \frac{1}{2}d.$

Which three laſt Equations do manifeſtly give the ſame values of *a*, *e*, and *y*, as were found out by the former Reſolution.

QUESTION. 3.

Three men diſcourſing of their Moneys in this manner; the firſt ſays to the other two, if 100 *l.* were added to his Money, the ſum would be equal to both their Moneys; the ſecond ſays to the other two, if 100 *l.* were added to his Money, the ſum would be equal to the double of both their Moneys; the third ſays to the other two, if 100 *l.* were added to his Money, the ſum would be equal to the triple of both their Moneys: The Queſtion is, to find how many Pounds each Man had.

Let the three numbers of Pounds ſought be repreſented by *a*, *e*, and *y*; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $a + 100 = e + y$
2. And . . . . $e + 100 = 2a + 2y$
3. And . . . . $y + 100 = 3a + 3e$

What are the Numbers *a*, *e*, and *y* ? ||

RESOLUTION.

4. From the firſt Equation by Tranſpoſition of *y*, this ariſes, ... $a + 100 - y$? || e
5. Then if inſtead of *e* in the ſecond Equation there be taken that which is equal to *e*, to wit, the firſt part of the fourth, this will ariſe, $a + 100 - y + 100 = 2a + 2y$
6. That is, after due Reduction, ... $200 = a + 3y$

7. Again, if inſtead of $3e$ in the third Equation there be taken the triple of the firſt part of fourth Equation, this will ariſe, to wit. . $y+100 = 3a+3a+300-3y$

8. Which laſt Equation after due Reduction gives $y = \frac{3}{2}a+50$

9. Then if inſtead of $3y$ in the ſixth Equation, there be ſet the triple of the latter part of the eighth, this will come forth, *viz.* . . . $200 = a+\frac{9}{2}a+150$

10. From the ninth Equation, after due Reduction the number a will be diſcovered, *viz.* . $a = 9\frac{1}{11}$

11. Again, if inſtead of a in the ſixth Equation, there be taken $9\frac{1}{11}$, to wit, the value of a found out in the tenth, it will give . . . $200 = 9\frac{1}{11}+3y$

12. The eleventh Equation duly reduced diſcovers the Number y, *viz.* . . . . $y = 63\frac{7}{11}$

13. From the fourth, tenth, and twelfth Equations by exchange of equal Quantities this Equation ariſes, *viz.* . . . . $9\frac{1}{11}+100-63\frac{7}{11} = e$

14. The thirteenth reduced gives : . . . . . $e = 45\frac{5}{11}$

From the 10th. 14th. and 12th. Equations the three numbers ſought a, e and y are diſcovered, *viz.* the firſt man had $9\frac{1}{11}$ *l.* the ſecond $45\frac{5}{11}$ *l.* and the third $63\frac{7}{11}$ *l.* which numbers will ſatisfie the Queſtion, as may eaſily be proved.

If 121 be given inſtead of 100 in this third Queſtion, then the three numbers ſought will be whole Numbers, to wit, 11, 55, 77.

RULE II.

When the ſame Quantity, ſuppoſe a, is found in two ſeveral Equations, and equal numbers are prefixed to thoſe Quantities, then if their ſigns be both + or both —, ſubtract the leſſer Equation from the greater; but if one of the ſigns be +, and the other —, add thoſe two Equations together; ſo the ſaid Quantity a will quite vaniſh, as will appear by the Reſolution of the following Queſtion.

QUESTION 4.

The ſum of two Numbers being given 12 (or b) and their difference 8 (or c) to find the Numbers.

Let a be put for the greater Number, and e for the leſſer, and the Queſtion may be ſtated thus:

1. If . . . . $a+e = b\ (= 12)$
2. And . . . . $a-e = c\ (= 8)$.

What are the Numbers a, and e?

RESOLUTION.

3. For as much as a or $+1a$ is found in each of the Equations propoſed, therefore (according to *Rule* 2.) I ſubtract the leſſer Equation from the greater; whence the letter a quite vaniſhes, and there remains $2e = b-c\ (= 4)$

4. Then by dividing each part of the third Equation by 2, the number e is made known, *viz.* . . $e = \frac{1}{2}b-\frac{1}{2}c\ (= 2)$

5. And by taking the latter part of the fourth Equation inſtead of e in the firſt, there remains . . $a+\frac{1}{2}b-\frac{1}{2}c=b\ (= 12)$

6. Laſtly, the fifth Equation duly reduced diſcovers the number a, *viz.* . . . . $a = \frac{1}{2}b+\frac{1}{2}c\ (= 10)$

The 6th. and 4th. Equations diſcover a Canon to find out the numbers ſought; which in this Example are 10 and 2, and the Canon is the ſame with that before found in *Queſt.* 1. *Chap.* 14. *Book* 1.

Otherwiſe thus.

5. For as much as $a+e$ is found in the firſt Equation, and $-e$ in the ſecond, therefore by adding thoſe two Equations together; (according to *Rule* 2.) the letter e vaniſhes, and the ſum is . . . . . $2a = b+c\ (= 20)$

8. Therefore by dividing each part of the ſeventh Equation by 2, there ariſes the ſame value of a, which was before found in the ſixth Equation, *viz.* $a = \frac{1}{2}b+\frac{1}{2}c \ (= 10)$

9. And by ſetting the latter part of the eighth Equation in the place of a in the firſt, this ariſes, . . $\frac{1}{2}b+\frac{1}{2}c+e = b \ (= 12)$

10. Which laſt Equation reduced diſcovers the ſame value of e, which was before found in the fourth Equation, *viz.* . . . . $e = \frac{1}{2}b-\frac{1}{2}c \ (= 2)$

RULE III.

When the ſame quantity, ſuppoſe a, is found in two ſeveral Equations, but the numbers prefix'd to thoſe equal quantities are unequal, thoſe two Equations may be reduced into two others which ſhall have equal numbers prefix'd to the ſaid Quantity a, by this Rule, *viz.* Multiply all the quantities in the firſt Equation by the number which is prefix'd to the ſaid quantity a in the ſecond; multiply likewiſe all the quantities in the ſecond Equation by the number which is prefix'd before the ſame quantity a in the firſt; ſo by ſuch alternate Multiplication two new Equations will be produced, wherein the numbers prefix'd to the ſaid quantity a will be equal to one another: and then by adding or ſubtracting, according to the import of *Rule* 2. of this *Chap.* that quantity a will quite vaniſh. That done, renew the like work to expel the ſame quantity out of the reſt of the Equations; and proceed in like manner with a ſecond quantity, until at length the value of ſome one quantity be made known. This I ſhall make plain by the Reſolution of five *Queſtions* next following.

QUESTION. 5.

To find two Numbers that if the Quadruple of the greater be increaſed with the triple of the leſs it may make 36; but if the triple of the greater be leſſened by the double of the leſs, the remainder may be 10.

Put a for the greater number, and e for the leſſer, then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $4a+3e = 36$
2. And . . . . $3a-2e = 10$

What are the Numbers a and e?

RESOLUTION.

3. The firſt Equation multiplied by 3, which is prefix'd to a in the ſecond, produces . . . . $12a+9e = 108$
4. The ſecond Equation multiplied by 4, which is prefix'd to a in the firſt, makes . . . . $12a-8e = 40$
5. Now for as much as the quantity $12a$ is found both in the fourth and third Equations, and is affirmative in each, therefore according to *Rule* 2. I ſubtract the leſſer Equation from the greater, ſo the quantity $12a$ vaniſhes, and this Equation remains $9e+8e = 68$
6. The fifth Equation after due Reduction diſcovers the number e, *viz.* . . . . $e = 4$
7. Then I ſet 12 (which by the ſixth Equation is the value of $3e$) in the place of $3e$ in the firſt, and this Equation ariſes . . $4a+12 = 36$
8. Laſtly, the ſeventh Equation duly reduced diſcovers the number a, *viz.* . . . . $a = 6$

From the 8th. and 6th. Equations the two numbers ſought are found 6 and 4, which will ſolve the Queſtion; for four times 6 with thrice 4 make 36; and thrice 6, to wit 18, leſſened by twice 4 gives 10, as was required.

QUESTION 6.

1. If . . . . $2a+3e-2y = 50$
2. And . . . . $5a-2e+5y = 240$
3. And . . . . $a+5e-3y = 10$

What are the numbers a, e, and y?

RE-

RESOLUTION.

4. The firſt Equation multiplied by 5, which is prefix'd to *a* in the ſecond, produces $10a+15e-10y=250$
5. Likewiſe the ſecond Equation multiplied by 2, which is prefix'd to *a* in the firſt, makes $10a-4e+10y=480$
6. Then (according to *Rule* 2.) by ſubtracting the fourth Equation from the fifth, the quantity $10a$ vaniſhes, and this Equation ariſes $-19e+20y=230$
7. Again, the third Equation multiplied by 5, which is prefix'd to *a* in the ſecond produces $-5a+25e-15y=50$
8. And the ſecond Equation multiplied by 1, which is ſuppos'd to be prefix'd to *a* in the third, gives the ſame ſecond Equation without alteration, *viz.* $+5a-2e+5y=240$
9. Then becauſe $+5a$ and $-5a$ by Addition will deſtroy one another, therefore (according to *Rule* 2.) I add the ſeventh and eight Equations together, ſo the letter *a* vaniſhes, and this Equation ariſes, $+23e-10y=290$
10. Again, I proceed with the ſixth and ninth Equations according to *Rule* 3. *viz.* I multiply the ſixth Equation by 23, (which is prefix'd to *e* in the ninth) and it makes $-437e+460y=5290$
11. Alſo the ninth Equation multiplied by 19 (which is prefix'd to *e* in the ſixth) produces $+437e-190y=5510$
12. Then (according to *Rule* 2.) by adding the tenth and eleventh Equations together, the Letter *e* vaniſhes, and this Equation ariſes, *viz.* $+270y=10800$
13. And by dividing each part of the twelfth Equation by 270, the number *y* is diſcovered, *viz.* $y=40$
14. Then inſtead of $10y$ in the ninth Equation taking ten times 40, that is 400, (which by the thirteenth Equation is equal to $10y$) the ninth will be reduced to this, $+23e-400=290$
15. And from the fourteenth Equation, after due Reduction, the number *e* will be diſcovered, *viz.* $e=30$
16. Then inſtead of $3e-2y$ in the firſt Equation, I take $90-80$, (which by the fifteenth and thirteenth Equations will be found equal to $3e-2y$) ſo the firſt Equation will be converted into this, *viz.* $2a+90-80=50$
17. Laſtly, the ſixteenth Equation duly reduced diſcovers the number *a*, *viz* $a=20$

From the 17th. 15th and 13th. Equations the 3 deſired numbers *a*, *e*, *y*, are 20, 30, and 40, which will conſtitute the 3 Equations firſt propoſed, as may eaſily be proved.

QUESTION 7.

Three Men diſcourſe of their Moneys in this manner; the firſt ſaith to the other two, if you give me 100. Pounds, my Money will be made equal to both your remaining Moneys: the ſecond ſaith to the other two, if ye give me 100 Pounds, my Money will be made equal to the double of both your remaining Moneys: laſtly, the third ſaith to the other two, if ye give me 100 Pounds, my Money will be equal to the triple of both your remaining Moneys. I demand how many Pounds each Man had?

Let a Letter be aſſumed to repreſent each Mans Money, as *a* for the firſt, *e* for the ſecond, and *y* for the third; then the Queſtion may be ſtated thus, *viz.*

1. If $a+100=e+y-100$
2. And $e+100=2a+2y-200$
3. And $y+100=3a+3e-300$

What are the numbers *a*, *e*, and *y*? ||

RESOLUTION.

4. The firſt Equation by tranſpoſition will be reduced to this $-a+e+y=200$

5. Likewiſe the ſecond Equation by tranſpoſition gives . . . . . . $+2a-e+2y = 300$

6. And the 3*d* Equation by tranſpoſition produces $+3a+3e-y = 400$

7. Then I proceed with the fourth and fifth Equations according to *Rule* 3. *viz.* I multiply the fourth Equation by 2, (which is prefix'd to *a* in the fifth,) and it produces . $-2a+2e+2y = 400$

8. The Sum of the fifth and ſeventh Equations gives . . $e+4y = 700$

9. Again, I proceed with the fifth and ſixth Equations according to *Rule* 3. *viz.* multiplying the fifth Equation by 3, (which is prefix'd to *a* in the ſixth,) it gives . . . $6a-3e+6y = 900$

10. Alſo the ſixth Equation multiplied by 2, (which is prefix'd to *a* in the fifth) produces $6a+6e-2y = 800$

11. Then by ſubtracting the tenth Equation from the ninth, the Remainder is . . . $-9e+8y = 100$

12. Again, I proceed with the eighth and eleventh Equations according to *Rule* 3. *viz.* multiplying the eighth Equation by 9, (which is prefix'd to *e* in the eleventh,) it makes . $+9e+36y = 6300$

13. Then (according to *Rule* 2.) the eleventh and twelfth Equations added together make . . $44y = 6400$

14. And by dividing the thirteenth Equation by 44, the number *y* is made known, *viz.* . . . $y = 145\frac{5}{11}$

15. From the eighth and fourteenth, by exchange of equal Quantities, this ariſes, *viz.* $e+581\frac{9}{11} = 700$

16. And from the fifteenth, by ſubtraction of $581\frac{9}{11}$ from each part, the number *e* is diſcovered, *viz.* . . . . $e = 118\frac{2}{11}$

17. From the firſt, fourteenth and ſixteenth Equations, by exchange of equal Quantities, this Equation ariſes, *viz.* . . . . . . $a+100=118\frac{2}{11}+145\frac{5}{11}-100$

18. Laſtly, the ſeventeenth Equation, after due Reduction, diſcovers the number *a*, *viz.* . . . $a = 63\frac{7}{11}$

Thus, by the 18*th*, 16*th* and 14*th* Equations it is found that the firſt Man had $63\frac{7}{11}$ *l.* the ſecond $118\frac{2}{11}$ *l.* and the third $145\frac{5}{11}$ *l.* which three Numbers will ſatisfie the Queſtion, as may eaſily be proved.

QUESTION. 8.

1. If . . . . $a+\frac{2}{3}e+\frac{2}{3}y+\frac{2}{3}u = 112$
2. And . . . . $e+\frac{3}{4}a+\frac{3}{4}y+\frac{3}{4}u = 114$
3. And . . . . $y+\frac{4}{5}a+\frac{4}{5}e+\frac{4}{5}u = 125\frac{3}{5}$
4. And . . . . $u+\frac{5}{6}a+\frac{5}{6}e+\frac{5}{6}y = 133\frac{1}{3}$

What are the numbers *a*, *e*, *y* and *u*? ||

RESOLUTION.

5. The firſt Equation multiplied by 3, (the Denominator of the Fraction $\frac{2}{3}$) produces this Equation in Integers, to wit, ... $3a+2e+2y+2u = 336$

6. Likewiſe the ſecond Equation multiplied by 4, produces . . . . $3a+4e+3y+3u = 456$

7. And the third Equation multiplied by 5 gives $4a+4e+5y+4u = 628$

8. Alſo the fourth Equation multiplied by 6 produces . . . . $5a+5e+5y+6u = 800$

9. Foraſmuch as 3*a* is found in the fifth, and alſo in the ſixth Equation, I ſubtract the leſſer from the greater, ſo 3*a* quite vaniſhes, and this Equation ariſes, ... $2e+y+u = 120$

10. Then I proceed with the fifth and ſeventh Equations according to *Rule* 3. *viz.* I multiply the fifth Equation by 4, (which is prefix'd to a in the ſeventh,) and there comes forth . . . . . $12a+8e+8y+8u=1344$

11. Alſo I multiply the ſeventh Equation by 3, (which is prefix'd to a in the fifth,) and it produces $12a+12e+15y+12u=1884$

12. Then by ſubtracting the tenth Equation from the eleventh, the quantity $12a$ quite vaniſhes, and this Equation ariſes, to wit, ... $4e+7y+4u=540$

13. The ninth Equation multiplied by 2, produces . . . $4e+2y+2u=240$

14. Then by ſubtracting the thirteenth Equation from the twelfth, this ariſes to wit, ... $5y+2u=300$

15. Again, I proceed with the fifth and eighth Equations according to *Rule* 3. *viz.* I multiply the fifth Equation by 5, (which is prefix'd to a in the eighth,) and it produces . . . . $15a+10e+10y+10u=1680$

16. Likewiſe the eighth Equation multiplied by 3, (which is prefix'd to a in the fifth,) produces . $15a+15e+15y+18u=2400$

17. Then by ſubtracting the fifteenth Equation from the ſixteenth, this ariſes, *viz.* . . . . $5e+5y+8u=720$

18. Again, I proceed with the ninth and ſeventeenth Equations according to *Rule* 3. *viz.* I multiply the ninth Equation by 5, (which is prefix'd to e in the ſeventeenth,) and it produces . . . . . . $10e+5y+5u=600$

19. And the ſeventeenth Equation multiplied by 2, (which is prefix'd to e in the ninth,) produces . . $10e+10y+16u=1440$

20. Then by ſubtracting the eighteenth Equation from the nineteenth, there remains . . . . . . $5y+11u=840$

21. And by ſubtracting the 14*th* Equation from the 20*th*, (for ſince $5y$ is found in each of thoſe Equations, they need no Reduction according to *Rule* 3.) there remains . . . . $9u=540$

22. Which twenty firſt Equation divided by 9 diſcovers the number u, *viz.* . . . . $u=60$

23. From the 20*th* and 22*d* Equations, by ſetting eleven times 60, to wit, 660 in the place of $11u$ in the 20*th*, there ariſes . . . . $5y+660=840$

24. Therefore from the twenty third Equation, after due Reduction, the number y is diſcovered, *viz.* . . . . $y=36$

25. And from the 9*th*, 24*th*, and 22*d* Equations, this ariſes, . . $2e+36+60=120$

26. The 25*th* duly reduced diſcovers the number e, *viz.* . . . $e=12$

27. From the 5*th*, 26*th*, 24*th*, and 22*d* Equations, by exchange of equal Quantities, this Equation ariſes, $3a+24+72+120=336$

28. Laſtly, from the 27*th*, after due Reduction, the number a is diſcovered, *viz.* . . . . $a=40$

Thus by the 28*th*, 26*th*, 24*th* and 22*d* Equations the four numbers ſought, (to wit, a,e,y,u,) are found 40, 12, 36 and 60, which will conſtitute the four Equations in *Queſt.* 8.

QUEST. 9.

A Maid being at the Market is offer'd 10 Apples for a penny, and 25 Pears for two pence; now if at thoſe rates ſhe would lay out $9\frac{1}{2}$ pence to buy 100 Apples and Pears together, how many Apples, and how many Pears ought ſhe to have?

1. For the number of Apples ſought put . . . . a

2. And for the number of Pears ſought put . . . . e

3. Then ſearch out the coſt of the number of Apples in the firſt ſtep, and ſay, If 10 . 1 :: a . ($\frac{a}{10}$. ſo the coſt of the number of Apples ſought is . . . . $\frac{a}{10}$

4. Search out alſo the coſt of the number of Pears in the ſecond ſtep, and ſay, If 25 . 2 :: e . $(\frac{2e}{25}$. ſo the coſt of the number of Pears ſought is found . . . . $\frac{2e}{25}$

5. Then (according to the Queſtion) the Money laid out for all the Apples and Pears ſought muſt be equal to $9\frac{1}{2}$ Pence; hence this Equation, ... $\frac{a}{10}+\frac{2e}{25}=9\frac{1}{2}$

6. But the number of Apples, together with the number of Pears bought muſt make 100, therefore . . . . . . $a+e=100$

7. Then the Equation in the fifth ſtep, after due Reduction, will give this Equation in Integers, to wit, ... $50a+40e=4750$

8. And the Equation in the ſixth ſtep being multiplied by 50 produces . . . . $50a+50e=5000$

9. Then by ſubtracting the Equation in the ſeventh ſtep from that in the eighth, there ariſes . . . . $10e=250$

10. And the Equation in the ninth ſtep divided by 10, diſcovers the number e, *viz.* . . . . $e=25$

11. Laſtly, from the ſixth and tenth ſteps, the number a is alſo made known, *viz.* . . . . $a=75$

By the firſt, ſecond, eleventh and tenth ſteps it appears that there might be bought 75 Apples, and 25 Pears; which numbers will ſolve the Queſtion, as may eaſily be proved.

QUESTION 10.

To divide 90 into four ſuch Numbers, that if the firſt be increaſed with 2; the ſecond leſſened by 2; the third multiplied by 2; and the fourth divided by 2; the Sum, Remainder, Product and Quotient may be equal between themſelves.

Let b and d be put for the two given Numbers, 90 and 2; alſo a, e, y and u for the four numbers ſought, then the Queſtion may be ſtated thus;

1. If . . . . $a+e+y+u=b$
2. And . . . . $a+d=e-d$
3. And . . . . $a+d=dy$
4. And . . . . $a+d=\frac{u}{d}$

What are the numbers a, e, y and u?

RESOLUTION.

5. The firſt Number ſought is equal to it ſelf, *viz.* . . . $a=a$

6. From the ſecond Equation, by tranſpoſition of $-d$ this ariſes, ... $a+2d=e$

7. And by dividing each part of the third Equation by d, this ariſes . . . . $\frac{a+d}{d}=y$

8. And the fourth Equation multiplied by d produces $da+dd=u$

9. The Sum of the four laſt Equations gives

$$2a+2d+\frac{a+d}{d}+da+dd=a+e+y+u=b$$

10. Which laſt Equation, after due Reduction, gives . $a=\frac{bd-ddd-2dd-d}{dd+2d+1}$

11. Then from the tenth and ſixth Equations, by exchange of equal Quantities, ... $e=\frac{bd+ddd+2dd+d}{dd+2d+1}$

12. And from the tenth and ſeventh Equations . . . $y=\frac{b}{dd+2d+1}$

13. And from the tenth and eighth Equations, . . . $u=\frac{ddd}{dd+2d+1}$

The four laſt Equations give a Canon to find out the four numbers ſought, which are 18, 22, 10 and 40, which will ſolve the Queſtion. For, firſt, their ſum is 90; then if the firſt number 18 be increaſed with the given number 2, it makes 20; and if the ſecond number 22 be leſſened by 2, the Remainder is alſo 20: Moreover, if the third number 10 be multiplied by 2, it likewiſe produces 20: Laſtly, if the fourth number 40 be divided by 2, the Quotient is alſo 20. Therefore the conditions in the Queſtion are ſatisfied.

But the Numerator of the Fraction in the latter part of the tenth Equation ſhews, That the Numbers b and d muſt not be given at random, but ſo, that $ddd+2dd+d$ may be ſubtracted from bd and leave a Remainder greater than nothing; therefore bd muſt be greater than $ddd+2dd+d$, and conſequently b muſt be greater than $dd+2d+1$. Therefore, to the end the Queſtion may be poſſible, the numbers given muſt be ſubject to this,

Determination.

The number given to be divided (b) muſt be greater than the Square of ($d+1$) the ſum of the other number given and Unity.

QUEST. 11.

There are two numbers whoſe Sum is equal to the difference of their Squares; and if the Sum of the Squares of thoſe two numbers be ſubtracted from the Square of their Sum, the Remainder will be 60: what are the two numbers?

Put b for the given number 60, alſo a for the greater number ſought, and e for the leſſer; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $aa-ee = a+e$
2. And . . . . $aa+ee+2ae-aa-ee = b$

What are the numbers a and e?

RESOLUTION.

3. The ſecond Equation after its firſt part is duly contracted is . . . . $2ae = b$
4. And the third Equation divided by 2 gives . . . $ae = \frac{1}{2}b$
5. And if each part of the firſt Equation be divided by $a+e$ it will give . . . . $a-e = \frac{a+e}{a+e} = 1$
6. From the fifth Equation, by tranſpoſition of e, there ariſes . . . . $a = e+1$
7. The ſixth Equation multiplied by e produces . $ae = ee+e$
8. From the fourth and ſeventh Equations, by exchanging equal Quantities . . . . $ee+e = \frac{1}{2}b$
9. Then the eighth Equation being reſolved by the Canon in *Sect.* 6. *Chap.* 15. *Book* 1. the leſſer number ſought will be made known, *viz.* . $e = \sqrt{:\frac{1}{4}+\frac{1}{2}b:}-\frac{1}{2}=5$
10. And from the ninth and ſixth Equations the greater number ſought will alſo be made known *viz.* . . . . $a = \sqrt{:\frac{1}{4}+\frac{1}{2}b:}+\frac{1}{2}=6$

The two laſt Equations give a Canon to find out the two numbers ſought, which are 6 and 5; as may eaſily be proved.

QUEST. 12.

There are two numbers, ſuch, that if their Sum be ſubtracted from the Sum of their Squares, the Remainder is 42; but if the Sum of the ſaid two numbers be added to the Product of their Multiplication, it makes 34: what are the numbers?

Let a and e repreſent the two numbers ſought, then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $aa+ee-a-e = 42$
2. And . . . . $ae+a+e = 34$

What are the Numbers a and e?

RESOLUTION.

3. By adding the firſt and ſecond Equations together, the Sum will be . . . . $aa+ee+ae = 76$
4. And by adding the ſecond Equation to the third, the ſum will be . . . . $aa+ee+2ae+a+e = 110$
5. Suppoſe . . . . $y = a+e$

6. Then by ſquaring each part of the fifth Equation, this ariſes $yy = aa+ee+2ae$

7. The Sum of the two laſt Equations makes $yy+y = aa+ee+2ae+a+e$

8. And from the ſeventh and fourth Equations, by exchange of equal quantities, this Equation ariſes, $yy+y = 110$

9. Which eighth Equation being reſolved by the Canon in *Sect. 6. Chap.* 15. *Book* 1. the number y, to wit, $a+e$ will be made known, *viz.* $y\ (= a+e) = 10$

10. Then by ſetting 10 (the value of $a+e$) in the place of $a+e$ in the ſecond Equation, there ariſes $ae+10 = 34$

11. And by ſubtracting 10 from each part of the tenth Equation, there remains $ae = 24$

12. And from the ninth Equation, by tranſpoſition of a, there ariſes $e = 10-a$

13. And if a in the eleventh be multiplied by $10-a$ inſtead of e, the ſaid eleventh Equation will be reduced to this, $10a-aa=24$

14. Wherefore the laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers ſought will be diſcovered, *viz.* $\left\{\begin{array}{l} a=6 \\ e=4 \end{array}\right.$

Thus 6 and 4 are found out, which will ſolve the Queſtion propoſed, as will be evident by the Proof.

QUEST. 13.

There are two numbers, ſuch, that the Sum of their Squares make 100, and if the Sum of the two numbers be added to the Product of their Multiplication, it makes 62; what are the numbers?

Let a and e be put for the two numbers ſought, then the Queſtion may be ſtated thus, *viz.*

1. If $aa+ee = 100$
2. And $ae+a+e = 62$

What are the Numbers a and e?

RESOLUTION.

3. The ſecond Equation multiplied by 2 produces $2ae+2a+2e = 124$

4. The ſum of the firſt and third Equations gives $aa+ee+2ae+2a+2e = 224$

5. Suppoſe $y = a+e$

6. Then by ſquaring each part of the fifth Equation this is produced, *viz.* $yy = aa+ee+2ae$

7. And by adding the double of the fifth Equation to the ſixth, it gives $yy+2y = aa+ee+2ae+2a+2e$

8. And from the ſeventh and fourth Equations, by exchange of equal quantities, this Equation ariſes $yy+2y = 224$

9. Which laſt Equation being reſolved by the Canon in *Sect. 6. Chap.* 15. *Book* 1. the number y, to wit $a+e$, will be made known, *viz.* $y = a+e = 14$

10. Then from the ninth and ſecond Equations, by taking 14 inſtead of $a+e$, the ſecond Equation will be reduced to this, *viz.* $ae+14= 62$

11. Which laſt Equation, by equal ſubtraction of 14, gives $ae = 48$

12. The ninth Equation by tranſpoſition of a gives $e = 14-a$

13. Then by multiplying a in the eleventh Equation by $14-a$ inſtead of e, this Equation is produced, to wit, $14a-aa = 48$

14. Wherefore the laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers ſought will be diſcovered, *viz.* $\left\{\begin{array}{l} a = 8 \\ e = 6 \end{array}\right.$

So the numbers ſought are found 8 and 6, which will ſolve the Queſtion, as will appear by the Proof.

QUESTION 14.

There are two numbers, ſuch, that their ſum is equal to the Product of their multiplication; and if the Product or ſum of the ſaid Numbers be added to the ſum of their Squares, it makes $15\frac{3}{4}$: What are the Numbers?

Let a and e be put for the two numbers ſought, then the Queſt. may be ſtated thus, *viz.*

1. If . . . . $ae = a+e$

2. And . . . . $aa+ee+ae = 15\frac{3}{4}$

What are the Numbers a and e?

RESOLUTION.

3. The Sum of the firſt and ſecond Equations is . . $aa+ee+2ae=a+e+15\frac{3}{4}$

4. And from the third Equation, by tranſpoſition of $a+e$, there ariſes . . . . . . $aa+ee+2ae-a-e=15\frac{3}{4}$

5. Suppoſe . . . . $y = a+e$

6. Then by ſquaring each part of the fifth Equation . . $yy = aa+ee+2ac$

7. And by ſubtracting the fifth Equation from the ſixth, there remains . . . . . . $yy-y = aa+ee+2ae-a-e$

8. And from the fourth and ſeventh Equations, by exchange of equal Quantities, there will ariſe . $yy-y = 15\frac{3}{4}$

9. Which laſt Equation being reſolved by the Canon in *Sect.* 8. *Chap.* 15. *Book* 1. the number y, to wit, $a+e$ will be made known, *viz.* . . . . $y = a+e = 4\frac{1}{2}$

10. Therefore from the firſt and ninth Equations, . . $a+e = ae = 4\frac{1}{2}$

11. From the ninth Equation by tranſpoſition of a . . . . . $e = 4\frac{1}{2}-a$

12. The eleventh Equation multiplied by a, pro-produces . . . . $ae = 4\frac{1}{2}a-aa$

13. And from the tenth and twelfth Equations, by exchange of equal Quantities, ... $4\frac{1}{2}a-aa = 4\frac{1}{2}$

14. Wherefore the laſt Equation being reſolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the two numbers ſought will be diſcovered, *viz.* . . . $\begin{cases} a = 3 \\ e = 1\frac{1}{2} \end{cases}$

So the numbers ſought are found 3 and $1\frac{1}{2}$, which will ſolve the Queſtion; for their Sum is equal to the Product of their Multiplication, and if their Sum $4\frac{1}{2}$ be added to $11\frac{1}{4}$ the Sum of their Squares, it makes $15\frac{3}{4}$, as the Queſtion requires.

QUESTION. 15.

There are two Numbers, ſuch, that the Square of their difference is equal to the Product of their Multiplication, and the Sum of their Squares makes 20: what are the Numbers?

Let a and e be put for the two Numbers ſought, and let a be the greater; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $aa-2ae+ee = ae$

2. And . . . . $aa+ee = 20$

What are the Numbers a and e?

RESOLUTION.

3. From the firſt Equation by tranſpoſition of $-2ae$, this ariſes . . . . . . $aa+ee = 3ae$

4. Therefore from the ſecond and third Equations . $3ae = 20$

5. And the third Equation divided by 3, gives . . $ae = \frac{20}{3}$

6. And by adding the double of the fifth Equation to the ſecond, it makes . . . . . $aa+ec+2ac = \frac{100}{3}$

7. Therefore by extracting the Square Root of each part of the ſixth Equation, the ſum of the two numbers ſought will be made known, *viz.* $a+e = \sqrt{\frac{100}{3}}$

8. From the ſeventh Equation, by tranſpoſition of a, this ariſes . . . . $e = \sqrt{\frac{100}{3}}-a$

9. The eighth Equation multiplied by a, produces . . $ae = \sqrt{\frac{100}{3}}\times a, -aa$

10. And from the fifth and ninth Equations this ariſes, . $\sqrt{\frac{100}{4}} \times a, - aa = \frac{20}{3}$

11. Wherefore the laſt Equation being reſolved by the Canon in *Sect* 10. *Chap.* 15. *Book.* 1. the two numbers ſought will be diſcovered, *viz.* . . . . $\left\{\begin{array}{l} a = \sqrt{8\frac{1}{3}} + \sqrt{1\frac{2}{3}} \\ e = \sqrt{8\frac{1}{3}} - \sqrt{1\frac{2}{3}} \end{array}\right.$

The Proof.

The difference of the two numbers in the eleventh ſtep is . $\sqrt{1\frac{2}{3}} + \sqrt{1\frac{2}{3}} = \sqrt{\frac{20}{3}}$

The ſquare of the ſaid difference is . . . . $\frac{20}{3}$

And (by the laſt of the three Rules in *Sect.* 10. *Chap.* 9. of this *Book*) the Product of the Multiplication of the ſame two numbers is alſo . . . . $\frac{20}{3}$

Laſtly, (by the firſt and ſecond of the ſaid three Rules) the ſum of the Squares of the ſaid two numbers is 20

QUEST. 16.

There are two numbers, ſuch, that if their ſum be multiplied by their difference, the Product is 21; but if the ſum of the Squares of thoſe two numbers be multiplied by the difference of their Squares, the Product is 609: what are the numbers?

Let *a* and *e* be put for the two numbers ſought, and let *a* repreſent the greater; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $\overline{a+e} \times \overline{a-e}$, that is, $aa-ee$, $= 21$
2. And . . . . $\overline{aa+ee} \times \overline{aa-ee}$, that is, $aaaa-eeee$, $= 609$

What are the numbers *a* and *e*?

RESOLUTION.

3. By ſuppoſition in the firſt Equation, ... $aa-ee = 21$
4. Therefore (by tranſpoſition of $-ee$) . . . . $aa = ee + 21$
5. And by ſquaring each part of the fourth Equation this ariſes, ... $aaaa = eeee + 42ee + 441$
6. And by taking the latter part of the fifth Equation inſtead of *aaaa* in the ſecond, the ſaid ſecond Equation will be reduced to this, ... $eeee + 42ee + 441 - eeee = 609$
7. The ſixth Equation, after due Reduction, gives . . . . $ee = 4$
8. Therefore by extracting the ſquare Root out of each part of the ſeventh Equation, the leſſer number ſought is diſcovered, *viz.* . . . . $e = 2$
9. Then from the fourth and ſeventh Equations this ariſes, ... $aa = 4 + 21 = 25$
10. Therefore by extracting the ſquare Root out of each part of the laſt Equation, the greater number ſought is alſo made known, *viz.* . . . . . $a = 5$

So the numbers ſought are found 5 and 2, which will ſolve the Queſtion, as will be evident by the Proof.

QUEST. 17.

There are two numbers, ſuch, that if their ſum be multiplied by the ſum of their Squares, the Product is 272; but if the difference of the ſame two numbers be multiplied by the difference of their Squares the Product is 32: what are the numbers?

Put *a* for the greater number ſought, and *e* for the leſſer; then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $\overline{a+e} \times \overline{aa+ee} = 272$
2. And . . . . $a-e \times aa-ee = 32$

What are the numbers *a* and *e*?

RESOLUTION.

3. By multiplying $a+e$ into $aa+ee$, the firſt Equation will be reduced to this, . . . $aaa + aae + aee + eee = 272$

4. Likewiſe by multiplying $a-e$ into $aa-ee$, the ſecond Equation will be reduced to this, . . $aaa-aae-aee+eee=32$

5. The ſum of the third and fourth Equations gives $2aaa+2eee=304$

6. The half of the fifth Equation is, . . . $aaa+eee=152$

7. The fourth Equation ſubtracted from the third leaves . . . . $2aae+2aee=240$

8. The half of the ſeventh Equation is . . . . $aae+aee=120$

9. The Sum of the ſeventh and eighth Equations is . . $3aae+3aee=360$

10. The ſum of the ſixth and ninth Equations is $aaa+3aae+3aee+eee=512$

11. The Cubic Root of the tenth being extracted, there ariſes . . . . $a+e=8$

12. By dividing each part of the firſt Equation by the reſpective part of the eleventh, there will ariſe . . $aa+ee=34$

By the two laſt Equations, the ſum of the two numbers ſought is found 8, and the ſum of their Squares 34; therefore by the Canon of *Queſt.* 7. *Chap.* 16. *Book* 1. the numbers themſelves will be found 5 and 3, which will ſolve the Queſtion, as may eaſily be proved.

QUEST. 18.

To divide a given number 14 (or b) into three continual Proportionals, ſuch, that if the ſaid given number be divided ſeverally by every one of the ſaid three Proportionals, the ſum of the three Quotients may be equal to $12\frac{1}{4}$ (or d) a number given.

RESOLUTION.

1. For the firſt (or leaſt) of the three Proportionals ſought put . . . . e

2. For the ſecond (or mean) Proportional put . a

3. Then the ſquare of the mean Proportional being divided by the firſt gives the third, to wit, $\frac{aa}{e}$

4. Therefore the Sum of the three Proportionals is $e+a+\frac{aa}{e}$

5. Which ſum muſt be equal to the given number 14, (or b,) whence this Equation ariſes, *viz.* $e+a+\frac{aa}{e}=b$

6. Then by reducing that Equation to Integers, this ariſes $ee+ae+aa=be$

7. Again, (according to the Queſtion) let the given number b be divided by every one of the three Proportionals in the fourth ſtep, ſo the three Quotients added together will give . . $\frac{b}{e}+\frac{b}{a}+\frac{be}{aa}$

8. But the ſum of the three Quotients in the ſeventh ſtep muſt be equal to the given ſum $12\frac{1}{4}$, (or d,) hence this Equation ariſes, . . . $\frac{b}{e}+\frac{b}{a}+\frac{be}{aa}=d$

9. Which laſt Equation reduced to Integers will produce . . . . $baaa+baae+baee=daaae$

10. And by dividing every Term of the Equation in the ninth ſtep by a, this ariſes . . . . $baa+bae+bee=daae$

11. The ſixth Equation multiplied by b, produces $baa+bae+bee=bbe$

12. And from the tenth and eleventh Equations, (where each of two Quantities is found equal to a common third) this ariſes, *viz.* . . . $daae=bbe$

13. The twelfth Equation divided by e gives . . . $daa=bb$

14. And the thirteenth Equation divided by d gives . . $aa=\frac{bb}{d}$

15. Therefore by extracting the ſquare Root out of each part of the fourteenth Equation, the mean Proportional ſought will be made known, *viz.* $a=\sqrt{\frac{bb}{d}}=4$

16. And becauſe a is now known, to wit, 4; and $b=14$; therefore the Equation in the ſixth ſtep may be reduced into this, *viz.* . . . $ee+4e+16=14e$

17. Which laſt Equation, after due Reduction, will give . $10e - ee = 16$

18. Laſtly, the Equation in the ſeventeenth ſtep being reſolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. the firſt and third Proportionals will be diſcovered, *viz.* . . $e = \left\{\begin{array}{l}2\\8\end{array}\right.$

Thus the three Proportionals ſought are found 2, 4, 8, which will ſatisfie the conditions in the Queſtion: For firſt, 2, 4 and 8 are manifeſtly in continual proportion; ſecondly, their ſum is 14; thirdly, if 14 be divided by 2, 4 and 8 ſeverally, the ſum of the Quotients 7, $3\frac{1}{2}$ and $1\frac{3}{4}$ is $12\frac{1}{4}$; as was preſcribed in the Queſtion.

It may alſo be obſerved, that thoſe three Quotients are continual Proportionals, as will be manifeſt from the ſeventh ſtep of the Reſolution, where they are repreſented by $\frac{b}{e}$, $\frac{b}{a}$ and $\frac{be}{aa}$; for the Product made by the Multiplication of the two extremes, to wit, the Product $\frac{bbe}{aae}$, that is, $\frac{bb}{aa}$, is equal to the Square of the mean Proportional $\frac{b}{a}$.

QUEST 19.

To find three numbers in Arithmetical Progreſſion, ſuch that if the firſt be multiplied by 1, the ſecond by 2, the third by 3, the ſum of the Products may be 62; and that the ſum of the ſquares of the three numbers may make 275.

Let the three numbers ſought be repreſented by *a*, *e*, *y*, and ſuppoſe *a* to be the ſmalleſt and firſt Term, then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $e - a = y - e$
2. And . . . . $a + 2e + 3y = 62$
3. And . . . . $aa + ee + yy = 275$

What are the numbers *a*, *e*, *y*?

RESOLUTION.

4. By ſuppoſition in the firſt ſtep . . . . . . $e - a = y - e$
5. Therefore by Tranſpoſition of $-a$ and $-e$, there ariſes . . . . . . $a + y = 2e$
6. And by dividing each part of the laſt Equation by 2, it gives . . . . . . $\frac{1}{2}a + \frac{1}{2}y = e$
7. And by ſquaring the Equation in the ſixth ſtep there comes forth . . . . . . $\frac{1}{4}aa + \frac{1}{2}ay + \frac{1}{4}yy = ee$
8. Then if inſtead of $2e$ in the ſecond Equation, there be taken the firſt part of the fifth, the ſecond will be converted into this, *viz.* . . $a + a + y + 3y = 62$
9. That is, ... $2a + 4y = 62$
10. The half of the laſt Equation is . . . . . . $a + 2y = 31$
11. And by tranſpoſition of Quantities in the tenth Equation this ariſes, *viz.* . . . $31 - 2y = a$
12. And by ſquaring the eleventh Equation, there comes forth . . . . . . $961 - 124y + 4yy = aa$
13. From the ſeventh, eleventh and twelfth Equations this ariſes, ... $\frac{961}{4} - \frac{31}{2}y + \frac{1}{4}yy = ee$
14. It is evident that . . . . $yy = yy$
15. And by adding the twelfth, thirteenth and fourteenth Equations into one ſum, it makes . $\frac{21}{4}yy - \frac{279}{2}y + \frac{4805}{4} = aa + ee + yy$
16. But by ſuppoſition in the third ſtep, ... $275 = aa + ee + yy$
17. Therefore from the fifteenth and ſixteenth Equations, by Exchange of equal Quantities, . $\frac{21}{4}yy - \frac{279}{2}y + \frac{4805}{4} = 275$
18. And after due Reduction the Equation in the ſeventeenth ſtep gives . . . . $y\frac{184}{7} - yy = \frac{1235}{7}$
19. Therefore by reſolving the Equation in the 18 ſtep, (according to the Canon in *Sect.* 10. *Chap.* 15. *Book* 1.) two values of *y* will be diſcovered, *viz.* . . . . . . $y = 13$, or $13\frac{4}{7}$
20. And from the 19*th* and 11*th* Equations . . . . . . $a = 5$, or $3\frac{6}{7}$
21. Laſtly, from the 20*th*, 19*th* and 6*th* Equations . . . . . . $e = 9$, or $8\frac{5}{7}$

From the three laſt Equations 'tis evident, that the three deſired Numbers a, e, y may be either 5, 9, 13, or $3\frac{6}{7}$, $8\frac{5}{7}$, and $13\frac{4}{7}$: For firſt, 5, 9, 13 are in Arithmetical Progreſſion; and if 5 be multiplied by 1, 9 by 2, and 13 by 3, the ſum of the three Products is 62; moreover, the ſum of the Squares of 5, 9, 13 makes 275, as was required. The like may be proved by $3\frac{6}{7}$, $8\frac{5}{7}$ and $13\frac{4}{7}$.

QUEST. 20.

To find three ſuch numbers, that the Square of the firſt being added to the Product of the firſt multiplied into the ſecond may make the ſum 48; alſo, that the Square of the firſt being ſubtracted from the Product of the firſt multiplied into the third the Remainder may be 32; and that the Sum of the Squares of the firſt and third, may have the ſame Proportion to the ſquare of the ſecond as 5 to 2.

Let the three Numbers ſought be repreſented by a, e, y, and then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $aa + ae = 48$
2. And . . . . $ay - aa = 32$
3. And . . . . $aa + yy \,.\, ee :: 5 \,.\, 2$

What are the numbers a, e, y? ||

RESOLUTION.

4. From the firſt Equation by tranſpoſition of aa, this ariſes, *viz.* . . . $ae = 48 - aa$
5. And by dividing each part of the laſt Equation by a, it gives . . . $e = \frac{48 - aa}{a}$
6. And by tranſpoſition of $-aa$ in the ſecond Equation, it makes . . . $ay = aa + 32$
7. And by dividing the ſixth Equation by a, there ariſes . . . $y = \frac{aa + 32}{a}$
8. From the Analogy in the third ſtep, by comparing the Product of the extremes to the Product of the means, this Equation ariſes . . . $5ee = 2aa + 2yy$
9. The Square of the ſeventh Equation is . . . $yy = \frac{1024 + 64aa + a^4}{aa}$
10. The double of the ninth Equation is . . . $2yy = \frac{2048 + 128aa + 2a^4}{aa}$
11. If inſtead of $2yy$ in the later part of the eighth Equation there be taken the later part of the tenth, the eighth will be converted into this, *viz.* . . . $5ee = \frac{2048 + 128aa + 4a^4}{aa}$
12. The Square of the fifth Equation is . . . $ee = \frac{2304 - 96aa + a^4}{aa}$
13. The twelfth Equation multiplied by 5 gives . . . $5ee = \frac{11520 - 480aa + 5a^4}{aa}$
14. From the eleventh and thirteenth Equations, by comparing their later parts one to the other, and reducing the Equation thereby reſulting, this Equation ariſes, *viz.* . . . $608aa - a^4 = 9472$
15. Which Equation in the 14*th* ſtep being reſolved by the Canon in *Sect.* 10. *Chap.* 15. *Book* 1. will diſcover two values of a, *viz.* . . . $a = \sqrt{592}$, or 4
16. But the leſſer of thoſe two values of a, to wit, 4, is the firſt number ſought by the Queſtion, for the Square of the greater value $\sqrt{592}$ exceeds 48, but according to the ſuppoſition in the firſt ſtep it ought to be leſs than 48; ſuppoſing then $a = 4$, it follows from the fifth ſtep, that . . . $e = 8$
17. Laſtly, from the 15*th* and 7*th* Equations, ... $y = 12$

So three numbers are found out, to wit, 4, 8 and 12; which will ſatisfie the Queſtion, as may eaſily be proved.

QUEST 21.

To find three ſuch numbers, that the ſquare of the firſt, together with the Product of the firſt multiplied by the ſecond may make 10; alſo, that the Square of the ſecond with the Product of the ſecond into the third may make 21; and laſtly, that the Square of the third, with the Product of the third into the firſt may make 24.

Let the three numbers ſought be repreſented by a, e, y, and then the Queſtion may be ſtated thus;

$$\left.\begin{array}{lr} \text{1. If} \ldots\ldots\ldots & aa+ae=10 \\ \text{2. And} \ldots\ldots\ldots & ee+ey=21 \\ \text{3. And} \ldots\ldots\ldots & yy+ya=24 \end{array}\right\}\text{What are the Numbers } a, e, y\,?$$

RESOLUTION.

4. By tranſpoſition of aa in the firſt Equation this ariſes, ... $ae = 10 - aa$

5. And by dividing each part of the fourth Equation a, it gives . . $e = \dfrac{10 - aa}{a}$

6. And by ſquaring the fifth Equation it makes . . . . $ee = \dfrac{a^4 - 20aa + 100}{aa}$

7. And from the ſecond, fifth and ſixth Equations this ariſes, . . $\dfrac{a^4 - 20aa + 100}{aa} + \dfrac{10 - aa}{a}y = 21$

8. And by ſubtracting $\dfrac{a^4 - 20aa + 100}{aa}$ from each part of the ſeventh Equation this remains . . . . $\dfrac{10 - aa}{a}y = \dfrac{41aa - 100 - a^4}{aa}$

9. And by dividing each part of the 8*th* Equation by $\dfrac{10 - aa}{a}$ this ariſes $y = \dfrac{41aa - 100 - a^4}{10a - aaa}$

10. And by ſquaring the ninth Equation it makes . . . . $yy = \dfrac{a^8 - 82a^6 + 1881a^4 - 8200aa + 10000}{100aa - 20a^4 + a^6}$

11. And by multiplying the ninth Equation by a, it produces . . $ya = \dfrac{41aaa - 100a - a^5}{10a - aaa}$

12. And by adding the eleventh Equation to the tenth, the ſum makes

$$yy + ya = \frac{2a^8 - 133a^6 + 2391a^4 - 9300aa + 10000}{100aa - 20a^4 + a^6}$$

13. Therefore from the third and twelfth Equations this ariſes,

$$\frac{2a^8 - 133a^6 + 2391a^4 - 9200aa + 10000}{100aa - 20a^4 + a^6} = 24$$

14. Which laſt preceding Equation, after due Reduction, gives this that follows, *viz.*

$$-a^8 + 78\tfrac{1}{2}a^6 - 1435\tfrac{1}{2}a^4 + 5800aa = 5000.$$

15. That is, after Tranſpoſition of 5000,

$$-a^8 + 78\tfrac{1}{2}a^6 - 1435\tfrac{1}{2}a^4 + 5800aa - 5000 = 0.$$

16. Then by ſuppoſing $u = 2a$, and proceeding according to the Rule in *Sect.* 7. *Ch.* 11. of this ſecond *Book*, the Equation laſt above written will be reduced to this following Equation in Integers, *viz.*

$$-u^8 + 314u^6 - 22968u^4 + 371200uu - 1280000 = 0$$

17. And by ſuppoſing $x = uu$ we may inſtead of $-u^8$ in the laſt preceding Equation write $-x^4$, and inſtead of $+314u^6$ we may ſet $314x^3$, alſo $-22968xx$ in the place of $-22968u^4$, and $+371200x$ inſtead of $+371200uu$, and laſt of all the Abſolute number -1280000: whence this following Equation ariſes; and then after x is made known, its ſquare Root be the number u; (for by ſuppoſition $x = uu$,)

$$-x^4 + 314x^3 - 22968xx + 371200x - 1280000 = 0.$$

18. Now becauſe the laſt Term -1280000 in the Equation laſt above written has many Diviſors which will be uſeleſs in the finding of the value of x, it will be convenient before they be found out, to ſearch out limits, within which ſuch a value of the Root x doth fall as will produce a value of a capable of ſolving the Queſtion propoſed; to which end I proceed thus, *viz.*

19. By the latter part of the fourth Equation it's manifeſt that $a \sqsupset \sqrt{10}$

20. And by the ſecond Equation, after tranſpoſition of ee, it will likewiſe appear that . . . . $e \sqsupset \sqrt{21}$

21. Now ſuppoſe . . . . $e = \sqrt{21}$

22. Then by multiplying $\sqrt{21}$ inſtead of e by a in the firſt Equation, it will be reduced to this, *viz.* . . . . . $aa + \sqrt{21} \times a = 10$

23. Which laſt Equation being reſolved by rhe Canon in *Sect.* 6. *Chap.* 15. *Book* 1. gives . . . . $a = 1\frac{61}{100}$, *&c.*

24. And becauſe when e is ſuppoſed to be equal to $\sqrt{21}$, the Equation in the twenty ſecond ſtep gives $a = 1\frac{61}{100}$, it may eaſily be conceived that when e is leſs than $\sqrt{21}$, (as it ought to be) then the firſt Equation, to wit $aa + ea = 10$ will neceſſarily give . . . . $a \sqsubset 1\frac{61}{100}$, *&c.*

25. Therefore by doubling each part of the nineteenth and twenty fourth ſteps, it is manifeſt that . . . . . . $2a \begin{cases} \sqsupset \sqrt{40} \\ \sqsubset 3\frac{22}{100}, \text{ \&c.} \end{cases}$

26. And by ſquaring each part in the twenty fifth ſtep, it follows that . . . . $4aa \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ \&c.} \end{cases}$

27. But by ſuppoſition in the ſixteenth ſtep $u = 2a$, and conſequently . . . . $uu = 4aa$

28. Therefore from the two laſt precedent ſteps it's evident that $uu \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{200}, \text{ \&c.} \end{cases}$

29. And becauſe by ſuppoſition in the ſeventeenth ſtep, . . $x = uu$

30. Therefore from the twenty eighth and twenty ninth ſteps it follows that . . . . $x \begin{cases} \sqsupset 40 \\ \sqsubset 10\frac{36}{100}, \text{ \&c.} \end{cases}$

31. Having found that ſuch a value of x in the Equation in the ſeventeenth ſtep as is capable of producing a true value of the deſired firſt number a, muſt be leſs than 40, but greater than $10\frac{36}{100}$; it is manifeſt that among the Diviſors of 1280000, the laſt Term of that Equation, theſe three only, to wit, 16, 20, 32, are neceſſary to make tryals in finding out the ſaid value of x, and conſequently of a; and therefore (according to the *Rule* in *Sect.* 9. *Chap.* 11, of this *Book*) I firſt divide the ſaid Equation in the ſeventeenth ſtep, to wit, $-x^4 + 314x^3 - 22968xx + 371200x - 1280000 = 0$ by $a - 16$, and the Quotient is exactly $-x^3 + 298xx - 18200x + 80000$, wherefore 16 ſhall be a true value of x in that Equation: And becauſe by ſuppoſition $x = uu = 4aa$, it follows that $\sqrt{16}$ (that is, $\sqrt{x}) = u = 2a$, and conſequently $2 = a$ the firſt number ſought.

32. Now ſince 2 is found equal to a, the firſt Equation, to wit, $aa + ae = 10$ will be reduced to this, *viz.* . . . . $4 + 2e = 10$

33. Whence the ſecond number e is diſcovered, *viz.* . . . $e = 3$

34. And conſequently the ſecond Equation will be reduced to this . . . . $9 + 3y = 21$

35. Whence the third number y is diſcovered, *viz.* . . . . . $y = 4$

Thus the three numbers ſought (to wit, a, e, y, are found 2, 3, 4, which will ſolve the Queſtion: For the Square of the firſt with the Product of the firſt and ſecond makes 10; alſo the Square of the ſecond with the Product of the ſecond and third makes 21; and the Square of the third with the Product of the third and firſt makes 24, as was required.

Note, That the Quotient found out in the thirty firſt ſtep, to wit, the Equation $-x^3 + 298xx - 18200x + 80000 = 0$ has three Affirmative Roots, whoſe values (by the *Rule* in *Sect.* 9. *C.* 11. of this ſecond *Book*) will be found very near equal to $4\frac{76}{100}$, $78\frac{12}{100}$, and $215\frac{12}{100}$; but theſe are without the limits of x diſcovered in the thirtieth ſtep, and therefore although the Equation in the fifteenth ſtep may be expounded by four Affirmative values of a, yet only one of them, to wit, 2, is capable of ſolving the Queſtion propoſed.

Note alſo, That if none of thoſe Diviſors which were diſcovered to be within the limits for the finding of a due value of x had produced an exact Quotient without a Remainder, and conſequently in ſuch caſe the number a had been Irrational, yet a Rational number, near the true value of x, and conſequently of a might be found out by the help of the General Method in *Chap.* 10. of this ſecond *Book.*

CHAP. XIII.

Concerning the Reſolution of ſuch Arithmetical Queſtions as are capable of innumerable Anſwers.

I. AFter a Queſtion is ſtated by Equations in ſuch manner as has been ſhewn in the foregoing twelfth *Chapter*, if thoſe Equations be equal in multitude to the Quantities ſought, then the Queſtion has a certain determinable number of Anſwers; but whenſoever a Queſtion affords not as many given Equations, not mutually depending upon one another, as there be Quantities required, it is capable of innumerable Anſwers. Queſtions of this latter kind are very pleaſant and delightful, but oftentimes exceeding hard to be reſolved, eſpecially when all the Anſwers in whole numbers that a Queſtion is capable of are deſired; and therefore I ſuppoſe it will not be unacceptable to the Learner, if in this *Chapter* I give him a taſte of that vaſt Skill, by expounding three Propoſitions found out by Monſieur *Bachet*; the two firſt of which contain the ſubſtance of the eighteenth and twenty firſt in his ingenious little Book, entituled *Problemes plaiſans & delectables, qui ſe font par les Nombres*, (Printed at *Lyons* in 1624;) but his Method of ſolving and demonſtrating the ſame being very tedious and obſcure, I ſhall wave it, and deliver two ways of my own finding out, which are both intelligible and demonſtrative. The third Propoſition (which is handled by the ſame Author in his Comment upon 41 *Prop.* of the Fourth Book of *Diophantus*,) I ſhall alſo explain at large by various Queſtions.

PROP. I.

Two whole numbers prime between themſelves being given, to find out two others, ſuppoſe a and b; that if a be multiplied by the greater of the two given numbers, and to the Product there be added a given whole number, the ſum ſhall be equal to the Product of b multiplied by the leſſer of the two numbers firſt given. Moreover to find out all the whole numbers a and b that are capable of producing the ſame effect.

Explication.

1. Numbers prime between themſelves are ſuch as have only Unity for their common Diviſor; (*per Defin.* 12. *Elem.* 7. *Euclid.*) ſo 12 and 5 are ſaid to be Prime between themſelves, becauſe they have no common Diviſor but 1, to divide them ſeverally, ſo as to leave no Remainder; the like may be ſaid of 20 and 21, 7 and 3, *&c.*
2. I call a number the *Multiple* of another when it exactly contains that other twice, thrice, or more times, without any Remainder: As, 6 is a Multiple of 3, becauſe it contains 3 exactly twice; likewiſe 18 is a Multiple of 6, becauſe it contains 6 juſt thrice without any Remainder. Moreover I take the Liberty to call a number the Multiple of it ſelf, becauſe it contains it ſelf juſt once. Theſe things premiſed, I ſhall proceed to ſhew two ways of ſolving the preceding *Prop.* 1. and explain the ſame by Queſtions.

Sect. II. *The firſt Method of ſolving the foregoing* Prop. 1.

QUEST. 1.

To find out all the values of a and b in whole numbers that may make $9a+6=7b$, *viz.* that nine times the whole number a with 6 added may make ſeven times the whole number b.

The Equation propoſed . . $9a+6=7b$,

The Reſolution, . . .	1	15	14	2
	2	24	21	3
	3	33	28	4
	4	42	35	5
	5	51	42	6
	6	60		
	7	69		

Explication.

1. To the number 9 prefixt to a I add 6, (to wit, $+6$ which follows $9a$) and it makes 15, to this I add again 9 and the ſum is 24, to which I add again 9, and it gives 33: and in like manner I continue the addition of 9 to every next preceding ſum until I have found out theſe ſeven numbers, 15, 24, 33, 42, 51, 60, 69, which ſtand (as you ſee in the Example) under $9a$, and on the left hand of thoſe numbers I ſet 1, 2, 3, 4, 5, 6, 7. Theſe two Columels of numbers do ſhew that if 1 be taken for the value of a, then $9a+6$ makes 15; but if $2=a$, then $9a+6=24$; if $3=a$, then $9a+6=33$; and ſo of the reſt. The addition aforeſaid is in this Example continued only to the ſeventh ſum incluſive, becauſe (as hereafter will appear) the ſmalleſt whole number that can expreſs the value of a, never exceeds the number prefix'd to b in the Equation propos'd.

2. Then under $7b$ I ſet the Multiples of 7 orderly one under another, *viz.* 14. (to wit twice 7,) 21, 28, *&c.* until I have found out a number equal to one of the ſeven numbers 15, 24, 33, *&c.* ſo at length among the Multiples of 7, I find 42, that is, ſix times 7, to be equal to 42 that ſtands among the numbers in the ſecond Columel, which later 42 (by the conſtruction aforeſaid) is compos'd of 6 and four times 9. Whence 'tis manifeſt that if 4 be taken for the value of a, and 6 for the value of b, then $9a+6=7b$ $(=42)$ *viz.* nine times 4 together with 6 is equal to ſeven times 6, and therefore one Anſwer to the Queſtion is diſcovered.

Note 1. When the given whole number prefix'd to b in the Equation propos'd is a ſingle figure, or ſome ſmall number of two places, then this firſt Method will readily diſcover the ſmalleſt values of a and b in whole numbers; for the ſmalleſt whole number a never exceeds the given number prefix'd to b, as hereafter will be made manifeſt: But if the number prefix'd to b be large, then the work by this firſt Method will be intolerably tedious, eſpecially in the ſolving of *Prop.* 2.

Note 2. If the two given whole numbers which are prefix'd to a and b in the Equation propos'd be not prime between themſelves, then it will ſometimes be impoſſible to find out any whole numbers for the values of a and b, to ſolve the Propoſition: as, if two whole numbers a and b be deſired that may make $6a+3=2b$, it may eaſily be ſhewn that 'tis impoſſible to find out two ſuch whole numbers; for the whole number a muſt be either even or odd, but whither it be even or odd, if it be multiplied by the even number 6 the Product ſhall be even; (by *Prop.* 21, *&* 28 *Elem.* 9. *Euclid*) to which adding 3 the ſum will be odd, (for odd, added to even makes odd,) which ſum muſt be equal to $2b$, and conſequently the half of that ſum is the number b; but the half of an odd number cannot be a whole Number, and therefore b in the Equation propos'd cannot be a whole number: But if the given whole numbers which are prefix'd to a and b be Prime to one another, then whatever whole number be given to be added to the deſired Multiple of a, innumerable whole numbers may be found out for the values of a and b, as hereafter will be ſhewn.

3. After the two ſmalleſt whole numbers are found out for the values of a and b to conſtitute the Equation propoſed, all other pairs of whole numbers that are capable of producing the ſame effect, may be orderly enumerated into two Arithmetical Progreſſions thus formed; *viz.* Having found 4 for the ſmalleſt whole number a, and 6 for the ſmalleſt whole number b to conſtitute the Equation before propoſed, to wit, $9a+6=7b$, let the ſaid 4 be made the firſt Term, and 7, which is prefix'd to b, the common difference of the Terms of the firſt Progreſſion; then let 6, the ſmalleſt whole number b, be the firſt Term. and 9 which is prefix'd to a in the ſaid Equation, the common difference of the Terms of the latter Progreſſion, ſo the Terms of thoſe Progreſſions will be theſe, *viz.*

Values of a; 4, 11, 18, 25, 32, 39, 46, 53, *&c.*
Values of b; 6, 15, 24, 33, 42, 51, 60, 69, *&c.*

4. Now out of the firſt of thoſe Progreſſions you may take any Term for the value of a, as 11, (the ſecond Term,) and then the correſpondent Term in the latter Progreſſion, to wit, 15, ſhall be the value of b; by which two numbers 11 and 15 the Equation $9a+6=7b$ may be expounded, *viz.* nine times 11 with 6 added is equal to ſeven times 15. Likewiſe 18 and 24, alſo 25 and 23, and every pair of correſpondent Terms in thoſe two Progreſſions will cauſe the ſame effect, as I ſhall now demonſtrate.

Preparation.

5. Let c and n repreſent two whole numbers Prime between themſelves, and a, b, d, three other whole numbers, ſuch that all five will make this Equation, *viz.* . . . . } $ca+d=nb$

6. Let an Arithmetical Progreſſion be ſo formed that a may be the firſt and leaſt Term, and n the common difference of the Terms, as . . . . } $a,\ a+n,\ a+2n,$ &c.

7. Let another Arithmetical Progreſſion be formed from b the firſt and leaſt Term, and c the common difference of the Terms, as, ... } $b,\ b+c,\ b+2c,$ &c.

8. I ſay, if you multiply c by $a+n$ (the ſecond Term of the firſt Progreſſion,) inſtead of a in the Equation in the fifth ſtep, and to the Product add d, the ſum ſhall be equal to a Multiple of n, to wit, the Product of n multiplied into $b+c$, (the ſecond Term of the later Progreſſion;) and the like may be affirmed of every following Term in each Progreſſion.

Demonſtration.

9. By ſuppoſition in the fifth ſtep, ... $ca+d=nb$

10. And by adding cn to each part of that Equation, this ariſes, ... } . $ca+cn+d=nb+cn$

11. Therefore from the laſt Equation, ... $c\times\overline{a+n},\ +d=n\times\overline{b+c}$
Which was to be ſhewn.

12. Again, if to each part of the Equation firſt granted in the ninth ſtep you add $2cn$, it makes . . } . $ca+2cn+d=nb+2cn$

13. That is, ... $c\times\overline{a+2n},\ +d=n\times\overline{b+2c}$

14. After the ſame manner it may be ſhewn that . . $c\times\overline{a+3n},\ +d=n\times\overline{b+3c}$
And ſo forwards. Which was to be proved.

15. Now ſuppoſing a and b to expreſs the ſmalleſt whole numbers that are capable of conſtituting the Equation in the fifth ſtep, to wit, $ca+d=nb$, I muſt demonſtrate that no other whole numbers beſides the Terms which follow a and b in the two Progreſſions formed in the ſixth and ſeventh ſteps, can be taken inſtead of a and b to produce the ſame effect: If it be poſſible, let $a+$ ſome whole number f, viz. $a+f$ be taken inſtead of a; and let $b+$ ſome whole number g, viz. $b+g$ be taken inſtead of b; then c multiplied by $a+f$ makes $ca+cf$, to which adding d, the ſum is $ca+cf+d$, which muſt be equal to the Product of n multiplied by $b+g$, to wit, $nb+ng$, whence . . . . $ca+cf+d=nb+ng$

16. And by ſuppoſition in the fifth ſtep, ... $ca+d=nb$

17. Therefore by ſubtracting the laſt Equation from the laſt but one, this remains, . . . } . . . . . . $cf=ng$

18. And by reſolving the laſt Equation into Proportionals, this Analogy ariſes, *viz.* . . . } . $n\ .\ c\ ::\ f\ .\ g$

19. Whence it is manifeſt that the whole numbers f and g are in the ſame Reaſon (or Proportion) as the whole numbers n and c, and conſequently, ſince n and c are by ſuppoſition whole Numbers Prime between themſelves, f muſt neceſſarily be equal either to n, or $2n$, or $3n$, &c. and g muſt be equal to c, or $2c$, or $3c$, &c. Wherefore $a+n$, $a+2n$, $a+3n$, &c. *viz.* the Terms which follow a in the Progreſſion in the ſixth ſtep, and $b+c$, $b+2c$, $b+3c$, &c. *viz.* the Terms which follow b in the Progreſſion in the ſeventh ſtep, are the only whole numbers that can be taken inſtead of a and b, the leaſt whole numbers to conſtitute the Equation propoſed, to wit, $ca+d=nb$. Which was to be ſhewn.

20 If there be two whole numbers a, and b, given or found out, which will conſtitute the Equation before, propoſed or ſuch like, and thoſe two numbers be not the ſmalleſt values of a and b, you may by the help of thoſe given find out the ſmalleſt, by this Rule; *viz.* Divide the given whole number a, by the given number which is prefixt to b in the Equation propoſed, then after the Diviſion is finiſh'd there will remain either a number or nothing; if a number remain, it ſhall be the ſmalleſt value of a but if o remain, then the number prefixt to b is the ſmalleſt value of a, and conſequently the correſpondent value of b is eaſily diſcovered by the Equation. The reaſon of this *Rule* is manifeſt by *S.9.C.17.B.1.* For if any Term greater than the leaſt of an Arithmetical

Progreſſion be given, as alſo the common Difference, the leaſt Term ſhall be given alſo, either by a continual ſubtraction of the common Difference, or by the *Rule* above expreſt.

As for Example, If in the former of the two Arithmetical Progreſſions in the third ſtep, which expreſs values of a and b to conſtitute the Equation $9a+6+7b$, there be given 32 for the value of a, I divide 32 by 7 which is prefix'd to b, and find 7 contain'd four times in 32, and there remains 4; now this Remainder 4 is the ſmalleſt value of a, whence the correſpondent whole number b, is eaſily diſcovered; for if $a=4$, then $9a+6=42=7b$; Therefore 42 divided by 7 gives 6 for the whole number b.

Again, if $a=20$, and $b=26$, then this will be a true Equation, *viz.* $5a+4-4b$; now if you deſire the ſmalleſt whole numbers a and b to conſtitute that Equation, divide 20 the given value of a by 4 which is prefix'd to b, and there remains 0, therefore (according to the *Rule* before given) the ſaid 4 ſhall be the ſmalleſt value of a; whence $5a+4=24=4b$, and conſequently $6=b$.

Laſtly, from what has been ſaid in the third ſtep, all the values of a and b in whole numbers that are capable of conſtituting the ſaid Equation $5a+4=4b$ are the Terms of theſe two Arithmetical Progreſſions, *viz.*

Values of a; 4, 8, 12, 16, 20, 24, 28, 32, &c.
Values of b; 6, 11, 16, 21, 26, 31, 36, 41, &c.

Sect. III *Another way of ſolving the foregoing* Prop. 1.

In this later Method there are four principal Caſes, which I ſhall firſt explain by Queſtions, and then ſhew how the Reſolution of the Propoſition will always run into one of thoſe four Caſes.

QUEST. 2.

To find all the whole numbers a and b that are capable of conſtituting this Equation *viz.* $8a+97=5b$.

The Equation propoſed, ...	1	$8a+97=5b$
The Reſolution	2	$8+97=105$
	3	$\frac{105}{5}=21=b$
	4	$1=a$

Explication.

Firſt I add 97 (to wit, +97 in the Equation propoſed) to 8, which is prefix'd to a, and it makes 105, this I divide by 5 the number prefix'd to b; and becauſe the Quotient 21 happens to be exactly a whole number without any remainder, it ſhall be the ſmalleſt whole number b ſought, and the whole number a in this caſe is always 1. The Reaſon is evident, for if $a=1$, then $8a+97=8+97$; and if this ſum happens to be a Multiple of the given number prefix'd to b, then b is neceſſarily a whole number. This is the firſt of the four Caſes above mentioned.

Then after 1 and 21, the ſmalleſt whole numbers a and b to conſtitute the Equation propos'd, are found out, all the other values of a and b in whole numbers will be found in theſe two following Arithmetical Progreſſions formed according to the *Rule* in the third ſtep of the foregoing *Sect.* 2. *viz.*

Values of a; 1, 6, 11, 16, 21, 26, &c.
Values of b; 21, 29, 37, 45, 53, 61, &c.

I ſay every two correſpondent numbers in thoſe Progreſſions may be taken for values of a and b in this Equation, $8a+97=5b$; as for Example, if 11 be taken for a, and 37 for b, then eight times 11, with 97 added ſhall be equal to five times 37, *viz.* $185=185$. And ſo of the reſt.

QUEST. 3.

To find all the whole numbers a and b that are capable of conſtituting this Equation, *viz.* $49a+6=13b$

The

$$\begin{array}{ll|l}
\text{The Equation propoſed, . . .} & 1 & 49a+6 = 13b \\
\hline
 & 2 & 55 = 65-10 \\
 & 3 & 49 = 39+10 \\
\cline{3-3}
\text{The Reſolution, ...} \left\{ \begin{array}{l} \\ \\ \\ \\ \\ \\ \end{array} \right. & 4 & 104 = 104 \\
 & 5 & \dfrac{104}{13} = 8 = b \\
 & 6 & \dfrac{104-6}{46} = 2 = a \\
\hline
\end{array}$$

Explication.

Firſt, I add 6 (to wit, $+6$ in the Equation propoſed) to 49 which is prefix'd to a, and it makes 55; now if this 55 were exactly diviſible by 13 which is prefix'd to b, the Quotient would be the whole number b ſought, and 1 the number a, (as in *Queſt.* 2.) But 55 not being a Multiple of 13, I proceed thus, *viz.* I ſeek the Multiple of 13 which is next greater than 55, by dividing 55 by 13, ſo I find that four times 13 is leſs than 55, but five times 13, that is, 65, exceeds 55, by 10; therefore 55 is equal to 65 wanting 10, *viz.* $55=65-10$. This is the ſecond Equation in the Example.

2. Then I divide 49 which is prefix'd to a, by 13 which is prefix'd to b, ſo I find that three times 13, that is, 39, is the greateſt Multiple of 13 contained in 49, and there remains 10; therefore $49=39+10$: which is the third Equation.

3. Now becauſe $+10$ is found in the third Equation, and -10 in the ſecond, I add thoſe Equations together, ſo the ſaid 10 vaniſhes, and there ariſes $104=104$; which is the fourth Equation.

4. Then I divide 104, that is, either part of the fourth Equation, by 13 which is prefix'd to b in the Equation propos'd, and the Quotient 8 is the whole number b ſought.

5. Then from the ſaid 104 in the fourth Equation, I ſubtract 6, (to wit, $+6$ in the Equation propos'd) and divide the Remainder 98 by 49 which is prefix'd to a, ſo the Quotient gives 2 for the whole number a ſought.

I ſay $2=a$ and $8=b$ will make $49a+6=13b$, as was required in *Queſt.* 3. and all the values of a and b in whole numbers that are capable of producing the ſame effect, are the Terms of theſe two following Arithmetical Progreſſions whoſe conſtruction has been ſhewn before.

Values of a;	2,	15,	28,	41,	54,	67,	*&c.*
Values of b;	8,	57,	106,	155,	204,	253,	*&c.*

Note, That the manner of forming the ſecond and third Equations in the foregoing Reſolution of *Queſt.* 3. muſt be diligently obſerved, becauſe the like work is conſtantly uſed in the following fourth, fifth, ſixth, ſeventh, eighth and ninth Queſtions: But it's by accident, that the ſame number 10 follows the Signs — and + in the ſaid ſecond and third Equations, and therefore the adding them together to produce the fourth Equation, is an Operation peculiar only to this and the like accident, which I call the ſecond of the four Caſes before mentioned.

But that in this ſecond Caſe, the Reſolution infallibly produces whole Numbers for the values of a and b, I prove thus: Firſt by Conſtruction, $65-10$ (the later part of the ſecond Equation) wants 10 of a Multiple of 13, and $39+10$ (the later part of the third Equation) exceeds a Multiple of 13 by 10; therefore the Sum of the ſaid $65-10$ and $39+10$, to wit, 104 (the later part of the fourth Equation) ſhall be a Multiple of 13; and conſequently 104 divided by 13 will exactly give a whole Number, to wit, 8, for the value of b. Secondly, becauſe 104 (the firſt part of the fourth Equation) is by conſtruction compos'd of a Multiple of 49 together with 6; by ſubtracting 6 from 104, the Remainder 98 ſhall be a Multiple of 49, and conſequently 98 divided by 49 will give the Quotient an exact whole number, to wit, 2, for the value of a. Whence it is manifeſt, that if after the ſecond and third Equations are formed out of the firſt (to wit, the Equation propoſed) according to the preceding Directions for ſolving *Queſt.* 3. it happens that the number following + in the later part of the third Equation, is the ſame with the Number following — in the later part of the ſecond, there will certainly ariſe two whole Numbers for the values of a and b.

QUEST. 4.

To find all the whole Numbers a and b that may make $82a+66=13b$.

The Equation propos'd, . .	1	$82a+66$	$=$	$13b$
The Reſolution, ...	2	148	$=$	$156-8$
	3	82	$=$	$78+4$
	4	164	$=$	$156+8$
	5	312	$=$	312
	6	$\frac{312}{13}$	$=$	$24=b$
	7	$\frac{312-66}{82}$	$=$	$3=a$

Explication.

1. The ſecond and third Equations are formed out of the firſt in ſuch manner as before has been explain'd in the Reſolution of *Queſt.* 3.

2. Becauſe the Number 4 which follows the Sign + in the later part of the third Equation, happens to be an Aliquot Part, to wit, $\frac{1}{2}$ of 8 which follows the Sign — in the later part of the ſecond Equation, I multiply each part of the third Equation by 2 (the Denominator of the ſaid Aliquot Part,) to the end there may be $+8$ in the Equation made by that Multiplication; ſo there is produced $164=156+8$, which is the fourth Equation.

3. Now ſince $+8$ is found in the fourth Equation, and -8 in the ſecond, I add thoſe Equations together, ſo the ſaid 8 vaniſhes, and there ariſes $312=312$; which is the fifth Equation.

4. Then I divide 312, (to wit, either part of the fifth Equation) by 13 which is prefix'd to b in the Equation propoſed, and the Quotient 24 is the whole number b ſought.

5. Laſtly, from the ſaid 312 (in the fifth Equation) I ſubtract 66, to wit, $+66$ in the Equation propos'd, and divide the Remainder 246 by the given number 82, (which is prefix'd to a;) ſo the Quotient 3 is the whole Number a ſought.

I ſay, $3=a$ and $24=b$ will make $82a+66=13b$, as was required in *Queſt.* 4. and all the values of a and b in whole Numbers that are capable of producing that Equation, are the Terms of theſe two Arithmetical Progreſſions, (whoſe Conſtruction has been ſhewn before in the third ſtep of *Sect.* 2.) *viz.*

Values of a; 3, 16, 29, 42, 55, 68, *&c.*
Values of b; 24, 106, 188, 270, 352, 434, *&c.*

Note, That it was by meer chance that the number following the Sign + in the third Equation happened to be an Aliquot Part of the number following the Sign — ſecond, and therefore the multiplying of the third Equation by the Denominator of the Aliquot Part, is an Operation peculiar only to that and the like accident, which is the third of the four Caſes before mentioned. The Reaſon of the Operation in this fourth Queſtion (or third Caſe,) may be eaſily diſcerned by the Demonſtration before given in *Queſt.* 3. but for further illuſtration I ſhall add another Example of *Caſe* 3.

QUEST. 5.

To find all the whole Numbers that may be values of a and b in this Equation, *viz* $601a+9=200b$.

The Equation propoſed . . .	1	$601a+9$	$=$	$200b$
The Reſolution, ...	2	610	$=$	$800-190$
	3	601	$=$	$600+1$
	4	114190	$=$	$114009+190$
	5	114800	$=$	114800
	6	$\frac{114800}{200}$	$=$	$574=b$
	7	$\frac{114800-9}{601}$	$=$	$191=a$

Explication.

The Reſolution of this Queſtion is like that in the foregoing *Queſt.* 4. for ſince + 1 in the later part of the third Equation happens to be an Aliquot part of 190 which follows — in the ſecond Equation, I multiply each part of the third by 190, to the end that + 190 may be found in the Product, as you ſee in the fourth Equation; then by adding the fourth Equation to the ſecond, the Sum makes the fifth, which is free from the Signs + and —; laſtly, from the fifth Equation the whole numbers 574 and 191 expreſſing the values of *b* and *a* are diſcovered, in like manner as in the preceding third and fourth Queſtions; which numbers will conſtitute the Equation propoſed: For 601 times 191 together with 9 is equal to 200 times 574, that is, 114800; and all the reſt of the values of *a* and *b* in whole Numbers to make that Equation will be found in theſe two following Arithmetical Progreſſions formed by the Rule before given in the third ſtep of *Sect.* 2.

Values of *a*; 191, 391, 591, 791, 991, &c.
Values of *b*; 574, 1175, 1776, 2377, 2978, &c.

QUEST. 6.

If	1	$121a + 5$	$= 93b,$		What are *a* and *b* in whole Numbers?
Out of 1.	2	126	$= 186 - 60$		
	3	121	$= 93 + 28$		
Suppoſe	4	$93c + 60$	$= 28d$		$c = ?\ d = ?$
Out of 4.	5	153	$= 168 - 15$		
	6	93	$= 84 + 9$		
Suppoſe	7	$28e + 15$	$= 9f$		$e = ?\ f = ?$
Out of 7.	8	43	$= 45 - 2$		
	9	28	$= 27 + 1$		
Eq. $9 \times \bar{2}$.	10	56	$= 54 + 2$		
Eq. 8+10.	11	99	$= 99$		
Out of 11 and 7.	12	$\frac{99}{9}$	$= 11$	$= f$	Here the Regreſſive work begins.
12, 6 and 5.	13	$11 \times 93 + 153$	$= 1176$		
13 and 4.	14	$\frac{1176}{28} = 42$		$= d$	
14, 3 and 2.	15	$42 \times 121 + 126$	$= 5208$		
15 and 1.	16	$\frac{5208}{93} = 56$		$= b$	
15 and 1.	17	$\frac{5208 - 5}{121} = 43$		$= a$	

Explication.

1. The ſecond and third Equations are formed out of the firſt in like manner as before in the Explication of *Queſt.* 3.

2. But becauſe 28 which follows + in the third Equation, is not equal to, nor an Aliquot part of 60 which follows — in the ſecond, the proceſs cannot be made like that in the third, fourth and fifth Queſtions; ſo that now a fourth Caſe takes riſe, and the ſcope of a new ſearch is to find out a number *d*, ſuch, that if it multiply the ſaid + 28, the Product may exceed a Multiple of 93 (which is prefix'd to *b*) by 60; for then it will be evident, that if the third Equation be multiplied by that number *d*, an Equation will be produced whoſe firſt part ſhall be a Multiple of 121, and the latter part ſhall exceed a Mutiple of 93 by 60, and then the reſt of the work will be like that in Caſe 2. in *Queſt.* 3. In the ſearch therefore of the number *d*, the fourth Equation is aſſumed, to wit, $93c + 60 = 28d$.

3. The fifth and ſixth Equations are formed out of the fourth, in like manner as the ſecond and third out of the firſt.

4. Becauſe 9 which follows + in the ſixth Equation, is neither equal to, nor an Aliquot part of 15 which follows the Sign — in the fifth, the next ſcope (for the like reaſon before

given

given concerning the number d) is to find out a number f, ſuch, that if it multiply the ſaid $+9$, the Product may exceed a Multiple of 28 which is prefix'd to d, by the ſaid 15; to which end the ſeventh Equation is aſſumed, to wit, $28e+15=9f$.

5. The eighth and ninth Equations are formed out of the ſeventh, in like manner as the ſecond and third out of the firſt.

6. Becauſe 1 which follows $+$ in the ninth Equation, is an Aliquot Part of 2 which ſtands next after $-$ in the eighth, the ninth is multiplied by 2 the Denominator of the ſaid part; (according to the Rule in Caſe 3. *Queſt.* 3.) whence the tenth Equation is produced, to wit, $56=54+2$.

7. The eleventh Equation, to wit, $99=99$ is the Sum of the eighth and tenth; and ſince the ſaid eleventh is free from the Signs $+$ and $-$, a Regreſſive work now begins, to find out the whole numbers f, d, b and a; in this manner, *viz.*

8. By dividing either part of the eleventh Equation, to wit, 99, by 9 which is prefix'd to f in the ſeventh, there ariſes $11=f$, as in the twelfth Equation.

9. Then multiplying the number f, to wit, 11, by 93, that is, either part of the ſixth Equation, and to the Product adding 153, that is, either part of the fifth Equation, the Sum makes 1176, (as you ſee in the thirteenth Equation) which 1176 is a Multiple of 28, to wit, that which is repreſented by $28\,d$ in the fourth Equation; Therefore,

10. By dividing the ſaid 1176 by 28, the Quotient 42 is the number d, as in the fourteenth Eqnation.

11. Then multiplying the number d, to wit, 42, by 121, that is, either part of the third Equation, and to the Product adding 126, that is, either part of the ſecond Equation, the Sum makes 5208, as you ſee in the fifteenth Equation, which 5208 is a Multiple of 93, to wit, that which is repreſented by $93\,b$ in the firſt Equation; Therefore,

12. By dividing either part of the fifteenth Equation, to wit, 5208 by 93, the Quotient 56 is the number b ſought.

13. Then from the ſaid 5208 ſubtracting 5, to wit, $+5$ in the firſt Equation, and dividing the Remainder 5203 by 121 which is prefix'd to a in the firſt Equation, the Quotient gives 43 for the number a ſought, as in the ſeventeenth and laſt Equation. Therefore, if 43 be for a, and 56 for b, then $121a+5=93b$, which is the Equation propoſed in *Queſt.* 6. and all the values of a and b in whole Numbers that are capable of conſtituting that Equation are the Terms of theſe two following Arithmetical Progreſſions, whoſe Conſtruction has been ſhewn before in the third ſtep of *Sect.* 2.

Values of a; 43, 136, 229, 322, 415, 508, &c.
Values of b; 56, 177, 298, 419, 540, 661, &c.

14. After the Numbers f and d in the foregoing Reſolution of *Queſt.* 6. are known, the Numbers e and c in the ſeventh and fourth Equations, may eaſily be diſcovered; but there is no need of their help in the finding out of the deſired Numbers a and b.

15. But methinks I hear the Reader make this Objection, *viz.* How does it appear, that from every three whole numbers given in ſuch ſort as before is declared in *Prop.* 1. there may infallibly be found out two whole numbers a and b to ſolve the ſaid Propoſition, by the Operation before explained in the four Caſes before mentioned: For Anſwer to this Objection, I ſhall here ſhew how far the Proceſs need be continued at the fartheſt, to find out an Equation having $+1$ in its later part; for when ſuch Equation ariſes, 'tis manifeſt by the Operation in the third Caſe explain'd in *Queſt.* 4, and 5. that two whole numbers a and b will infallibly be diſcovered to ſatisfie the Propoſition, and conſequently innumerable other pairs of whole numbers to produce the ſame effect. Firſt, then in the foregoing *Queſt.* 6. the given number 121 which is prefix'd to a, being divided by the given number 93 which is prefix'd to b, after the Diviſion is finiſh'd there remains 28, to wit $+28$ in the later part of the third Equation: Secondly, the ſaid Diviſor 93 being divided by the ſaid Remainder 28, after the Diviſion is ended there remains 9, to wit, $+9$ in the later part of the ſixth Equation: Again, the laſt Diviſor 28 being divided by the laſt Remainder 9, after this Diviſion is ended there remains 1, that is, $+1$ in the later part of the ninth Equation, which Remainder 1 you will always infallibly come unto by a continued Diviſion in that manner, becauſe the two given Numbers prefix'd to a and b are (as the Propoſition requires) Prime between themſelves; and that continued Diviſion is nothing elſe but the Method of finding out the greateſt common Diviſor unto two

Numbers; ſo that you may at firſt (if you pleaſe) diſcover unto what Letter at the fartheſt, the proceſs need be continued before you return backward according to the Operation explain'd in *Queſt.* 6. But oftentimes before you come to the ſaid Remainder 1, the Reſolution will run into one of the three Caſes explain'd in *Queſt.* 2, 3, 4, and 5. as will appear by the following ſeventh, eighth, and ninth Queſtions.

QUEST. 7.

If	1	$97a + 1$	$= 26\,b,$	What are a and b in whole Numbers?
Out of 1.	2	98	$= 104 - 6$	
	3	97	$= 78 + 19$	
Suppoſe	4	$26\,c + 6$	$= 19\,d$	$c = ?\ d = ?$
Out of 4.	5	32	$= 38 - 6$	
	6	26	$= 19 + 7$	
Suppoſe	7	$19\,e + 6$	$= 7f$	$e = ?\ f = ?$
Out of 7.	8	25	$= 28 - 3$	
	9	19	$= 14 + 5$	
Suppoſe	10	$7g + 3$	$= 5\,h$	$g = ?\ h = ?$
Out of 10.	11	$7 + 3$	$= 10$	
Out of 10 and 11.	12	$\frac{10}{5}$	$= 2 - h$	Here the Regreſſive work begins.
Out of 12, 9, 8.	13	$2 \times 19, + 25$	$= 63$	
13, and 7.	14	$\frac{63}{7} = 9$	$= f$	
14, 6 and 5.	15	$9 \times 26, + 32$	$= 266$	
15 and 4	16	$\frac{266}{19} = 14$	$= d$	
16, 3 and 2.	17	$14 \times 97, + 98$	$= 1456$	
17 and 1.	18	$\frac{1456}{26} = 56$	$= b$	
17 and 1.	19	$\frac{1456 - 1}{97} = 15$	$= a$	

Explication.

In this ſeventh Queſtion the proceſs is formed like that in the foregoing ſixth, and the laſt Letter in the work is h, whoſe value is diſcovered in the twelfth Equation by the help of the tenth and eleventh, according to the Operation in *Queſt.* 2. and then by the help of the Number h, the Work returns backward to find out the Numbers f, d, b and a, in like manner as in *Queſt.* 6. But in this ſeventh Queſtion the laſt Letter in the Proceſs, to wit, h, is made known before an Equation ariſes which has $+1$ in its later Part; aud the like effect happens in the following eighth and ninth Queſtions.

Now in Anſwer to this ſeventh Queſtion, all the values of a and b in whole Numbers that are capable of conſtituting the Equation propoſed, to wit, $97a + 1 = 26\,b$, are the Terms of the two following Arithmetical Progreſſions, which are deduced from the two ſmalleſt values of a and b, (to wit, 15 and 56 found out as above,) according to the Rule in the third ſtep of *Sect.* 2.

Values of a; 15, 41, 67, 93, 119, 145, &c.
Values of b; 56, 153, 250, 347, 444, 541, &c.

QUEST. 8.

If	1	$119a+6 = 57b,$	What are the whole numbers a and b?
Out of 1.	2	$125 = 171-46$	
	3	$119 = 114+5$	
Suppoſe	4	$57c+46 = 5d$	$c=?\ d=?$
Out of 4.	5	$103 = 105-2$	
	6	$57 = 55+2$	
5 + 6.	7	$160 = 160$	
7, 4.	8	$\frac{160}{5} = 32 = d$	Regreſs.
8, 3, 2.	9	$32 \times 119, + 125 = 3933$	
9, 1.	10	$\frac{3933}{57} = 69 = b$	
9, 1.	11	$\frac{3933-6}{119} = 33 = a$	

Values of a; 33, 90, 147, 204, 261, 318, &c.
Values of b; 69, 188, 307, 426, 545, 664, &c.

In which Progreſſions, every two correſpondent Terms may be taken for values of a and b to conſtitute the Equation in *Queſt.* 8.

QUEST. 9.

If	1	$173a+1 = 71b,$	What are the whole numbers a and b?
Out of 1.	2	$174 = 213-39$	
	3	$173 = 142+31$	
Suppoſe	4	$71c+39 = 31d$	$c=?\ d=?$
Out of 4.	5	$110 = 124-14$	
	6	$71 = 62+9$	
Suppoſe	7	$31e+14 = 9f$	$e=?\ f=?$
Out of 7.	8	$31+14 = 45$	
8, and 7.	9	$\frac{45}{9} = 5 = f$	Regreſs.
9, 6, 5.	10	$5 \times 71, +110 = 465$	
10, 4.	11	$\frac{565}{31} = 15 = d$	
11, 3, 2.	12	$15 \times 173, +174 = 2769$	
12, 1.	13	$\frac{2769}{71} = 39 = b$	
12, 1.	14	$\frac{2769-1}{173} = 16 = a$	

Values of a; 16, 87, 158, 229, 300, 371, &c.
Values of b; 39, 212, 385, 558, 731, 904, &c.

Sect. 4. *PROP.* II.

Two whole numbers Prime between themſelves being given, to find out two others, ſuppoſe a and b, that if a be multiplied by the leſſer of thoſe two numbers given, and to the Product there be added a whole number given, the ſum ſhall be equal to the Product of b multiplied by the greater of the two numbers firſt given. Moreover, to diſcover all the whole numbers a and b that are capable of producing the ſame effect.

When each of the two given numbers which are Prime between themſelves is a ſingle Figure, or ſome ſmall number conſiſting of two Characters, then the firſt of the two ways of ſolving the foregoing *Prop.* 1. will readily ſolve this ſecond; but waving that Method I ſhall ſhew two other ways by the help of the later of thoſe two Methods.

The firſt Method of ſolving Prop. 2.

QUEST. 10.

If	1	$71a+3 = 173b,$	What are a and b in whole Numbers?
Out of 1.	2	$145 = 173-28$	
By *Prop.* 1.	3	$2769 = 2768+1$	
Eq. 3 × 28.	4	$77532 = 77504+28$	
2 + 4.	5	$77677 = 77677$	
Out of 5, 1.	6	$\frac{77677}{173} = 449 = b$	true Values.
5, 1.	7	$\frac{77677-3}{71} = 1094 = a$	
By the Rule in *Sect.* 2. *Num.* 20.	8	$56 = a$	the leaſt Values.
	9	$23 = b$	

Explication.

1. I multiply 71 which is prefix'd to a in the Equation propoſed, by ſuch a Number, that when 3, to wit, $+3$ in the ſame Equation is added to the Product, the Sum may be either equal to, or leſs than ſome Multiple of 173; ſo multiplying 71 by 2, the Product 142 increaſed with 3 makes 145, which is equal to 173 wanting 28, *viz.* $145=173-28$, which is the ſecond Equation.

2. Then by *Prop.* 1. of this *Chap.* I ſeek two ſuch Numbers a and b, that if a be multiplied by 173, and the Product increaſed with $+1$, the Sum may be equal to the Product of b multiplied by 71; *viz.* Suppoſing $173a+1=71b$, and proceeding according to the foregoing *Queſt.* 9. I find 16 for the value of a, and 39 for b; therefore $\overline{173\times 16}, +1 = 71\times 39$; or $71\times 39 = \overline{173\times 16}, +1$; that is, $2769 = 2768+1$, which is the third Equation.

3. Becauſe $+1$ in the later part of the third Equation is an Aliquot Part of 28 in the ſecond, I multiply the third Equation by 28 the Denominator of the ſaid Part, and it makes the fourth Equation, to wit, $77532 = 77504+28$.

4. Then by adding the fourth Equation to the ſecond the Sum gives the fifth, which is free from the Signs $+$ and $-$; and from the fifth Equation the whole Numbers 449 and 1094 are diſcovered for values of b and a, in like manner as in *Queſt.* 4, and 5. and by the help of thoſe the ſmalleſt values of a and b, to wit, 56 and 23 are found out by the Rule in the twentieth ſtep of *Sect.* 2.

5. Laſtly, by the help of the two ſmalleſt values of a and b, and the Rule in the third ſtep of *Sect.* 2, all that are capable of ſolving *Queſt.* 10. will be found in the two following Arithmetical Progreſſions, which may be continued as far as you pleaſe.

Values of a;	56,	229,	402,	575,	748,	921,	1094,	&c.
Values of b;	23,	94,	165,	236,	307,	378,	449,	&c.

QUEST. 11.

If	1	$22a+5000 = 65b,$	What are a and b in whole Numbers?
Out of 1.	2	$5022 = 5070-48$	
By *Prop.* 1.	3	$66 = 65+1$	
Eq. 3 × 48.	4	$3168 = 3120+48$	
2 + 4.	5	$8190 = 8190$	
Out of 5, and 1.	6	$\frac{8190}{65} = 126 = b$	true Values.
5, 1.	7	$\frac{8190-5000}{22} = 145 = a$	
By the Rule in *Sect.* 2. *Num.* 20.	8	$15 = a$	the leaſt Values.
	9	$82 = b$	

Explication.

1. I add 22 to 5000 and it makes 5022, which is not exactly diviſible by 65, for 77 times 65 is leſs than 5022, but 78 times 65, that is, 5070, exceeds 5022 by 48; therefore $5022=5070-48$, which is the ſecond Equation.

2. Then by *Prop.* 1. of this *Chap.* I ſeek two ſuch whole numbers a and b, that if a be multiplied by 65, and to the Product there be added 1, the Sum may be equal to the Product of b multiplied by 22; *viz.* Suppoſing $65a+1=22b$, and proceding according to the later Method of reſolving the foregoing *Prop.* 1. I find 1 and 3 to be values of a and b; therefore, $\overline{65\times1},+1=22\times3$; or $22\times3=\overline{65\times1},+1$; that is, $66=65+1$, which is the third Equation.

3. By proſecuting the Work as before in the Explication of *Queſt.* 10. all the deſired values of a and b in whole numbers that are capable of conſtituting the Equation firſt propoſed in this eleventh Queſtion will be found to be the Terms of theſe two following Arithmetical Progreſſions, *viz.*

Values of a;	15,	80,	145,	210,	275,	340,	*&c.*
Values of b;	82,	104,	126,	148,	170,	192,	*&c.*

Another way of ſolving Prop. 2.

QUEST. 12.

If	1	$71a+3 = 173b,$	What are a and b in whole numbers?
Out of 1.	2	$145 = 173-28$	
	3	$213 = 173+40$	
Suppoſe	4	$173c+28 = 40d$	$c=?\ d=?$
Out of 4.	5	$201 = 240-39$	
	6	$173 = 160+13$	
$6\times\overline{3}$.	7	$519 = 480+39$	
$5+7$.	8	$720 = 720$	
8, 4.	9	$\frac{720}{40} = 18 = d$	Regreſs.
9, 3, 2.	10	$18\times213,+145 = 3979$	
10, 1.	11	$\frac{3979}{173} = 23 = b$	
10, 1.	12	$\frac{3979-3}{71} = 56 = a$	

Explication.

1. In this Queſtion, which is the ſame with the foregoing tenth, the ſecond Equation is formed as is there directed.

2. The third Equation is thus formed: Foraſmuch as the given number 71 is leſs than 173 which is prefix'd to b, I multiply 71 by ſuch a Number that the Product may exceed 173, and be alſo Prime to it; ſo multiplying 71 by 3, the Product 213 exceeds 173, alſo 213 and 173 are Prime to one another, then I divide the ſame 213 by 173, and find that 213 contains 173 once, and 40 over and above; therefore $213=173+40$, which is the third Equation.

3. The fourth, fifth, and ſixth Equations here, are formed like the fourth, fifth, and ſixth Equations in the foregoing *Queſt.* 6.

4. Then becauſe 13 which follows $+$ in the ſixth Equation is an Aliquot part of 39 which follows $-$ in the fifth, I multiply the ſixth Equation by 3 the Denominator of the ſaid Part, (for 13 is $\frac{1}{3}$ of 39,) and it produces the ſeventh Equation, to wit, $519=480+39$.

5. The eighth Equation is the Sum of the fifth and ſeventh, (according to the Operation in Caſe 2.) and then in the ninth Equation the Regreſſive Work begins, to find out the values of d, b and a in ſuch manner as has been ſhewn in divers preceding Queſtions of this *Chap.* So at length all the values of a and b in whole numbers to ſolve this twelfth Queſtion will by this later Method be found the ſame as before in *Queſt.* 10.

Sect. 5. *PROP.* III.

To divide a given number into three or more numbers, ſuch, that if every one of them be multiplied by a different number given, the ſum of the Products may be equal to a given number. But the ſum of thoſe Products muſt fall between the two Products made by multiplying the given Dividend into the greateſt and leaſt of the given Multiplicators.

The ſolution of this Problem is explain'd by the following Queſtions of this Chapter, and oftentimes requires the help of the two preceding Propoſitions, as will partly appear by the fifteenth Queſtion.

QUEST. 13.

To divide 24 into three ſuch whole numbers, that if the firſt be multiplied by 36, the ſecond by 24, and the third by 8, the ſum of the three Products may make 516.

Let the numbers ſought be repreſented by a, e & y, then the Queſtion may be ſtated thus:

1. If . . . . $a+e+y=24$
2. And . . . . $36a+24e+8y=516$

What are the whole numbers a, e and y? ||

RESOLUTION.

3. The firſt Equation multiplied by 36, which is prefix'd to a in the ſecond, produces . . . $36a+36e+36y=864$
4. The 2*d* Equation ſubtracted from the third, leaves . . $12e+28y=348$
5. The 4*th* Equation by tranſpoſition of $+28y$, gives . . $12e=348-28y$
6. The fifth Equation divided by 12 gives . . . . $e=29-\frac{7y}{3}$
7. If inſtead of e in the firſt Equation there be taken the later part of the ſixth, this ariſes . . $a+29-\frac{7y}{3}+y=24$
8. That is, ... $a+29-\frac{4y}{3}=24$
9. From the eighth Equation by tranſpoſition of $29-\frac{4y}{3}$ this ariſes, ... $a=24-29+\frac{4y}{3}$
10. That is, ... $a=\frac{4y}{3}-5$
11. By the later part of the tenth Equation 'tis evident that . . . . $\frac{4y}{3}\sqsubset 5$
12. Therefore by multiplying each part in the eleventh ſtep by 3, it follows that . . . . $4y\sqsubset 15$
13. And by dividing each part in the 12*th* ſtep by 4, $y\sqsubset 3\frac{3}{4}$
14. And from the later part of the ſixth Equation, by arguing in like manner as in the eleventh, twelfth and thirteeth ſteps, it will be manifeſt that $y\sqsupset 12\frac{3}{7}$
15. Now if Fractions or mixt numbers were admitted to be the values of a, e, and y, then by the thirteenth, fourteenth, tenth and ſixth ſteps 'tis evident that

$y=$ any number between $3\frac{3}{4}$ and $12\frac{3}{7}$;

$$a=\frac{4y}{3}-5;$$

$$e=29-\frac{7y}{3}$$

16. But to find out whole numbers to ſolve the Queſtion, the limits in the thirteenth & fourteenth ſteps do ſhew that y muſt be ſome whole number greater than 3, but not greater than 12, yet every whole number within thoſe limits will not ſerve the turn, for the values of a and e before diſcovered will not be whole numbers unleſs $\frac{4y}{3}$ and $\frac{7y}{3}$ be whole numbers; but $\frac{4y}{3}$ and $\frac{7y}{3}$ cannot be whole numbers unleſs y be 3, or ſome Multiple of 3, and becauſe 3 is without the limits, y may be 6, or 9, or 12, and conſequently

a	*e*	*y*
3	15	6
7	8	9
11	1	12

from the fifteenth ſtep a ſhall be 3, or 7, or 11; and e, 15, or 8, or 1. Now in anſwer to the Queſtion, 3, 15 and 6, (to wit, a, e and y) are three ſuch whole numbers, that their ſum is 24, and if the firſt be multiplied by 36, the ſecond by 24, and the third by 8, the ſum of the three Products makes 516, as was required. The like may be ſaid of each of the two other Anſwers. But if Fractions or mixt numbers were admitted, innumerable Anſwers might be given to the Queſtion, as before has been ſhewn in the fifteenth ſtep.

Note. When one part of an Equation conſiſts of an Affirmative letter and ſome Negative Abſolute number, a limit may thence be inferr'd, above which the number ſignified by that letter ought to be taken. But if one part of an Equation conſiſts of a Negative letter and of an Affirmative abſolute number, it will give a limit beneath which the number repreſented by that letter muſt be choſen. Sometimes alſo two limits will be diſcovered, (as in this thirteenth Queſtion for the choice of the number y;) and ſometimes but one, (as in divers of the following Queſtions.)

QUEST. 14.

To find three ſuch whole numbers that their ſum may make 100; and that if the firſt be multiplied by 4, the ſecond by 3, and the third by $1\frac{4}{5}$, the ſum of the three Products may make 300.

For the three numbers ſought put a, e and y, then the Queſtion may be ſtated thus;

1. If . . . . $a+e+y=100$

2. And . . . . $4a+3e+1\frac{4}{5}y=300$

What are the whole numbers a, e and y? ||

RESOLUTION.

3. The firſt Equation multiplied by 4, (which is prefix'd to a in the ſecond Equation,) produces . . . . $4a+4e+4y=400$

4. The ſecond Equation ſubtracted from the third, leaves . . $e+\frac{11y}{5}=100$

5. The fourth Equation by tranſpoſition of $+\frac{11y}{5}$ gives . . $e=100-\frac{11y}{5}$

6. If inſtead of e in the firſt Equation there be taken the later part of the fifth, this will ariſe, ... $a+100-\frac{11y}{5}+y=100$

7. That is, after due Reduction, ... $a=\frac{6y}{5}$

8. From the later part of the fifth Equation it's manifeſt that . . . . $\frac{11y}{5}\sqsupset 100$

9. And conſequently by multiplying each part in the eighth ſtep by 5 . . . . $11y\sqsupset 500$

10. And by dividing each part in the ninth ſtep by 11, it follows that . . . . $y\sqsupset 45\frac{5}{11}$

Whence 'tis manifeſt, that if the three numbers ſought were not reſtrained to whole numbers, any number leſs than $45\frac{5}{11}$ might be taken for the number y, and then the numbers a and e would be diſcovered from the ſeventh and fifth ſteps. But to have the Queſtion ſolved by whole numbers, the number y muſt be ſome whole number not greater than 45, and ſuch as may cauſe $\frac{11y}{5}$ and $\frac{6y}{5}$ to be whole Numbers, for otherwiſe the values of e and a in the fifth and ſeventh ſteps will not be expreſſible by whole Numbers; but $\frac{11y}{5}$ and $\frac{6y}{5}$ cannot be whole Numbers unleſs y be 5, or ſome Multiple of 5, and therefore y may be 5, or 10, or 15, or any of the reſt of the numbers in the third Columel of this Table; and conſequently, from the fifth and ſeventh ſteps of the Reſolution, the whole numbers e and a will be ſuch as ſtand under e and a. Thus you ſee that the Queſtion receives nine Anſwers in whole Numbers, which are all that it's capable of: So that if you take 6 for a; 89 for e; and 5 for y, their ſum is 100; and if 6 be multiplied by 4;

a	e	y
6	89	5
12	78	10
18	67	15
24	56	20
30	45	25
36	34	30
42	23	35
48	12	40
54	1	45

by 4; 89 by 3; and 5 by $1\frac{4}{5}$, the ſum of the three Products makes 300, as the Queſtion requires The like may be proved of every one of the other eight Anſwers.

Note. When three numbers are ſought by a Queſtion of this nature that is capable of many Anſwers in whole numbers, all the values of every one of the letters in whole numbers are in Arithmetical Progreſſion, and therefore when two of thoſe Anſwers are found out, all the reſt within the limits diſcovered by the Reſolution are conſequently given by Addition or Subtraction of the common difference in each Rank, as may eaſily be perceived by the values of *a*, *e*, *y* in the Table above-written. But when four numbers are ſought, the values of a letter are oftentimes found in ſeveral Arithmetical Progreſſions, as in the following *Queſt.* 20.

QUEST. 15.

To divide 1533 into three whole numbers, ſuch, that $\frac{1}{8}$ of the firſt, together with $\frac{3}{8}$ of the ſecond and $\frac{2}{113}$ of the third may make 167.

For the three whole numbers ſought put *a*, *e*, and *y*, then the Queſtion may be ſtated thus?

1. If . . . . $a+e+y=1533$
2. And . . . . $\frac{1}{8}a+\frac{3}{8}e+\frac{2}{113}y=167$

What are the whole numbers *a*, *e*, and *y*? ||

RESOLUTION.

3. The firſt Equation multiplied by $\frac{1}{8}$, produces . $\frac{1}{8}a+\frac{1}{8}e+\frac{1}{8}y=\frac{1533}{8}$
4. The fourth Equation ſubtracted from the third, leaves . . . . $-\frac{1}{4}e+\frac{97}{904}y=\frac{197}{8}$
5. The ſecond Equation by tranſpoſirion of $-\frac{1}{4}e$ gives . . . . $\frac{97}{904}y=\frac{197}{8}+\frac{1}{4}e$
6. The fifth Equation divided by $\frac{97}{904}$, gives . $y=\frac{22261}{97}+\frac{226e}{97}$
7. If inſtead of *y* in the firſt Equation there be taken the later part of the ſixth this ariſes, $a+e+\frac{22261}{97}+\frac{226e}{97}=1533$
8. The ſeventh Equation, after due Reduction, gives . . . . $a=\frac{126440}{97}-\frac{323e}{97}$
9. By the eighth Equation it's manifeſt that . $323e \sqsupset 126440$
10. And conſequently by dividing each part of the laſt ſtep by 323, ... $e \sqsupset 391\frac{147}{323}$
11. Now to find out the values of *a*, *e* and *y* in whole numbers, (if there be a poſſibility) I multiply the ſixth Equation by the Denominator 97, and it makes . . . . $97y=22261+226e$
12. That is, ... $226e+22261=97y$
13. Then by the foregoing *Prop.* 1. of this Chapter, I ſearch out all ſuch whole numbers as may be values of *e* and *y* to conſtitute the laſt Equation, that is, $226e+22261=97y$; but with this Condition, *viz.* That the greateſt whole number among thoſe that are found out for the values of *e* may not exceed 391, as the preceding tenth ſtep requires; ſo I find four values of *e*, to wit, 47, 144, 241, 338; and four values of *y*, to wit, 339, 565, 791 and 1017: Then the Sum of every two correſpondent values of *e* and *y* being ſubtracted from 1533 the Number firſt given to be divided, the Remainders ſhall be the deſired values of *a*, to wit, 1147, 824, 501 and 178; ſo there are only four Anſwers to the Queſtion in whole Numbers, to wit, thoſe inſerted in the Table in the Margin.

a	*e*	*y*
1147	47	339
824	144	565
501	241	791
178	338	1017

The Proof of the firſt Anſwer.

The Sum of 1147, and 47 339 is 1533,
$\frac{1}{8}$ of 1147 is . . . . $143\frac{3}{8}$,
$\frac{3}{8}$ of 47 is . . . . $17\frac{5}{8}$,
$\frac{2}{113}$ of 339 is 6,
Laſtly, the ſum of thoſe three Products is 167,

Therefore all the Conditions in the Queſtion are ſatisfied, and the like may be proved by every one of the other three Anſwers in whole Numbers; but if Fractions were admitted, innumerable Anſwers might be given by the tenth, eighth, and ſixth ſteps of the Reſolution.

QUEST. 16.

To find the three Numbers, that their Sum may make 300; and that if the firſt be multiplied by 6, the ſecond by 5, and the third by $2\frac{1}{300}$, the Sum of the three Products may make 1496.

Let a,e,y be put for the three Numbers ſought; then by forming the reſolution in like manner as in the preceding thirteenth, fourteenth and fifteenth Queſt. it will appear that

$y =$ any Number between $1\frac{307}{893}$ and $76\frac{532}{1193}$;

$$e = 304 - \frac{1193y}{300};$$

$$a = \frac{893y}{300} - 4.$$

Whence 'tis evident, that there cannot be three whole numbers found out to ſolve this Queſtion, for 300 is the ſmalleſt whole Number that can be taken for y to cauſe $\frac{1193y}{300}$ and $\frac{893y}{300}$ to be whole Numbers; but 300 exceeds the greater of the two limits above diſcovered for chuſing of the number y.

QUEST. 17.

If one would lay out 98 pence to buy 40 Birds, ſuppoſe Patridges, Larks and Quails; how many of each kind may be bought when Patridges are at 3 pence a piece, Larks at an half penny a piece, and Quails at 4 pence a piece?

Let a repreſent the number of Patridges, e the number of Larks, and y the number of Quails; then according to the Queſtion, $a+e+y=40$; and becauſe the number of all the Patridges multiplied by the price of one of them produces the full coſt of all, it's manifeſt that $3a$ is the full coſt of all the Patridges; and for the like reaſon $\frac{1}{2}e$ ſignifies the full coſt of all the Larks; likewiſe $4y$ the full coſt of the Quails: But thoſe three particular Sums of Money muſt be equal to 98 pence, therefore $3a+\frac{1}{2}e+4y = 98$; ſo that the Queſtion may be ſtated thus;

1. If . . . . $a+e+y = 40$
2. And . . . . $3a+\frac{1}{2}e+4y=98$

What are the whole Numbers a, e and y? ||

RESOLUTION.

3. The firſt Equation multiplied by 3 (which is prefix'd to a in the ſecond,) produces . . . . $3a+3e+3y=120$
4. The ſecond Equation ſubtracted from the third, leaves . $\frac{5e}{2} - y=22$
5. From the fourth Equation, after due Tranſpoſition, this ariſes . . . . $y=\frac{5e}{2}-22$
6. Then inſtead of y in the firſt Equation, if there be ſet the later part of the fifth, the firſt will be reduced to this, $a+e+\frac{5e}{2}-22=40$
7. The ſixth Equation, after due Reduction, gives . . $a=62-\frac{7e}{2}$
8. By the later part of the fifth Equation it's evident that $\frac{5e}{2} \sqsubset 22$
9. And conſequently by multiplying each part in the eighth ſtep by 2, ... $5e \sqsubset 44$
10. Whence by dividing each part by 5, it follows that $e \sqsubset 8\frac{4}{5}$
11. Again, from the later part of the ſeventh Equation, by arguing in like manner as in the eighth, ninth and tenth ſteps, it will appear that . . . . $e \sqsupset 17\frac{5}{7}$

12. Now ſince the nature of this Queſtion requires that the deſired value of a, e and y be whole numbers, it's evident from the fifth and ſeventh ſteps that e muſt be an even number, otherwiſe $\frac{5e}{2}$ and $\frac{7e}{2}$ will not be whole numbers; for if e be an odd number, the Dividends $5e$ and $7e$ will be odd, (for odd multiplied by odd produces odd) and therefore their halves cannot be whole numbers. Since then e muſt be an even number, it's manifeſt by the tenth and eleventh ſteps, that e may be 10, or 12, or 14, or 16, but no other even number whatever; and conſequently from the fifth ſtep y ſhall be 3, or 8, or 13, or 18; and from the ſeventh ſtep, a ſhall be 27, or 20, or 13, or 6. Thus it appears that the Queſtion may be ſolved by four ſeveral Anſwers (and not more) in whole numbers, *viz.* Firſt, 27 Patridges, 10 Larks, and 3 Quails, which are in multitude 40, may be bought for 98 pence at their reſpective prices given in the Queſtion; or 20 Partridges, 12 Larks, and 8 Quails, which are likewiſe 40 in Multitude, and the like may be affirmed of the other two Anſwers inſerted in the Table in the Margin.

Partr. a	Larks. e	Quails. y
27	10	3
20	12	8
13	14	13
6	16	18

But if a Queſtion of the ſame nature be deſired that has but one anſwer in whole numbers, the following Epigram (cited by Monſieur *Bachet* in his Comment upon the one and fortieth Queſtion of the fourth Book of *Diophantus*,) will be ſatisfactory.

QUEST. 18.

Ut tot emantur aves, bis denis utere nummis;
Perdix, Anſer, Anas empta vocetur avis.
Sit ſimplex obolus pretium Perdicis, ematur
Sex obolis Anſer, biſque duobus Anas.
Ut tua procedat in lucem quæſtio, mentem
Conſule, ſic loquiter pectoris arca mihi.
Sint Anates tres atque duæ, ſimplex erit Anſer,
Accipe Perdices quatuor atque decem.

The ſence is this: If the price of a Patridge be an half penny, a Gooſe 3 pence, and a Duck 2 pence; how many of each kind may be bought at thoſe rates, if it be deſired that all the Birds bought may be 20 in number, and coſt 20 pence?

Let a repreſent the number of Patridges, e the number of Geeſe, and y the number of Ducks, then this Queſtion (like the preceding ſeventeenth) may be ſtated thus:

1. If . . . . $a+e+y=20$
2. And . . . . $\frac{1}{2}a+3e+2y=20$

What are the whole Numbers a, e and y?

RESOLUTION.

3. The firſt Equation multiplied by $\frac{1}{2}$, produces . . . $\frac{1}{2}a+\frac{1}{2}e+\frac{1}{2}y=10$
4. The third Equation ſubtracted from the ſecond, leaves . $\frac{5e}{2}+\frac{3y}{2}=10$
5. By tranſpoſition of $\frac{3y}{2}$ in the fourth Equation, this ariſes . $\frac{5e}{2}=10-\frac{3y}{2}$
6. The fifth Equation divided by $\frac{5}{2}$, gives . . . . . $e=4-\frac{3y}{5}$
7. By ſetting the later part of the ſixth Equation in the place of e in the firſt, this ariſes . . . . . . $a+4-\frac{3y}{5}+y=20$
8. Which laſt Equation, after due Reduction, gives . $a=16-\frac{2y}{5}$
9. From the later part of the ſixth Equation it may be inferr'd, (in like manner as in divers of the preceding Queſtions) that . . . . $y \sqsupset 6\frac{2}{3}$

10. But the ſixth and eight ſteps do ſhew, that to the end the values of e and a may be whole numbers, as the nature of this Queſtion requires, it is requiſite that $\frac{3y}{5}$ and $\frac{2y}{5}$

be whole numbers; by $\frac{3y}{5}$ and $\frac{2y}{5}$ cannot be whole numbers, unleſs y be 5 or ſome Multiple of 5; and by the ninth ſtep y muſt be leſs than $6\frac{2}{3}$, therefore 5 is the only whole number that can be taken for y, or the number of Ducks; and conſequently the ſixth ſtep gives 1 for the value of e, that is, 1 Gooſe; and by the eighth ſtep, the value of a is 14, that is, 14 Partridges; which three numbers will ſolve the Queſtion, as may eaſily be proved.

The Reſolutions of the following nineteenth and twentieth Queſtions do ſhew how to find out innumerable Anſwers to any Queſtion belonging to the Rule of Alligation alternate *in vulgar Arithmetic, when three or more things are to be mixed together, according to the import of that Rule.*

QUEST. 19.

A Vintner having three ſorts of Wines, the prices whereof *per* Gallon are 24 pence, 22 pence, and 18 pence, deſires to make a Mixture out of them that may contain 60 Gallons, in ſuch manner, that the total Mixture being ſold at ſome mean price *per* Gallon between 24 pence and 18 pence, ſuppoſe at 20 pence, may make the ſame ſum of Money, as all the particular quantities of Wine in the Mixture at their own prices. The Queſtion is, to find what quantity of each ſort of Wine may be taken to make that Mixture.

For the deſired number of Gallons of the firſt ſort of Wine to make the Mixture, put a; for the number of the ſecond ſort e; and of the third y: Then $a+e+y=60$, (the total number of the Gallons in the Mixture;) and becauſe every Gallon of the mix'd quantity muſt be ſold for 20 pence, the 60 Gallons mix'd are worth 1200 pence, and ſo much alſo muſt all the Products of the particular Quantities of each ſort of Wine multiplied by their peculiar prices amount unto; therefore $24a+22e+18y=1200=60\times20$. So that the Queſtion may be ſtated thus:

1. If . . . . $a+e+y=60$
2. And . . . . $24a+22e+18y=1200$ $(=60\times20)$

What are the numbers a, e, y?

RESOLUTION.

3. The firſt Equation multiplied by 24 (which is prefix'd to a in the ſecond Equation) produces . . . . . $24a+24e+24y=1440$
4. The ſecond Equation ſubtracted from the third, leaves . . . . $2e+6y=240$
5. The fourth Equation by tranſpoſition of $6y$, gives . . . . $2e=140-6y$
6. The fifth Equation divided by 2, gives . . . $e=120-3y$
7. By taking the later part of the ſixth Equation inſtead of e in the firſt, this ariſes, $a+120-3y+y=60$
8. The ſeventh Equation, after due Reduction, diſcovers the value of a, viz. $a=2y-60$
9. From the *8th* Equation it's evident that $y \sqsubset 30$
10. And from the ſixth Equation, . . $y \sqsupset 40$
11. By the *10th*, *9th*, *8th* and *6th* ſteps, it's manifeſt that innumerable Anſwers may be given to the Queſtion propoſed; for ſince Fractions are not here excluded from being Anſwers, you may eſteem . $y=$ any number between 30 and 40;
$a=2y-60$;
$e=120-3y$.
12. Whence nine Anſwers in whole numbers are diſcovered, to wit, thoſe expreſt in this Table. But the Rule of Alligation in Vulgar Arithmetic finds out only one Anſwer to this Queſtion, to wit, the ſixth. And becauſe innumerable Numbers may be taken between 30 and 40 for values of y, you may find out as many Anſwers as you pleaſe in Fractions, (which are not excluded in Queſtions of this Nature;) ſo if for y you take $30\frac{1}{2}$, then $a=1$, $(=2y-60,)$ and $e=28\frac{1}{2}$, $(=120-3y.)$

a	e	y
2	27	31
4	24	32
6	21	33
8	18	34
10	15	35
12	12	36
14	9	37
16	6	38
18	3	39

The Proof of the firſt Anſwer.

Two Gallons of Wine at 24 pence *per* Gallon, together with 27 Gallons at 22 pence *per* Gallon, and 31 Gallons at 18 pence *per* Gallon, amount to 1200 pence; which is alſo the value of 60 Gallons at 20 pence *per* Gallon.

QUEST. 20.

A Vintner having four ſorts of Wines, whoſe prices *per* Quart are 16 pence, 10 pence, 8 pence, and 6 pence, deſires to make a Mixture out of them that may contain 100 Quarts, ſo as this mixt quantity being ſold at ſome mean price *per* Quart between 16 pence and 6 pence, ſuppoſe at 12 pence, may produce the ſame ſum of money, as all the particular quantities of Wine in the Mixture if they were ſold at their own prices. The Queſtion is, to find what quantity of Wine of each ſort may be taken to make that Mixture?

Let a, e, y and u be put for the unknown quantities of Wine that are ſought to make the Mixture; then $a+e+y+u=100$, (the total number of Quarts in the Mixture,) and by multiplying thoſe Quantities ſeverally into their peculiar prices, the ſum of the Products is $16a+10e+8y+6u$; which ſum muſt be equal to the Product of 100 multiplied into 12, that is, 1200 pence; So that the Queſtion may be ſtated thus;

1. If . . . . $a+e+y+u=100$
2. And . . . . $16a+10e+8y+6u=1200$

What are the Numbers a, e, y and u? ||

The given Equations being fewer in multitude than the numbers ſought, it's a ſign that the Queſtion is capable of innumerable Anſwers; now that you may find out as many of them as you pleaſe, the firſt ſcope in the Reſolution muſt be to diſcover limits to direct your choice of ſome one of the numbers ſought, and accordingly, the drift in the eight Equations next following is to ſearch out limits for the firſt number a.

RESOLUTION.

3. From the firſt Equation by tranſpoſition of a, this ariſes, ... $e+y+u=100-a$
4. And from the ſecond Equation by tranſpoſition of $16a$, this ariſes, ... $10e+8y+6u=1200-16a$
5. The third Equation multiplied by 6, to wit, the leaſt of the known numbers which are prefix'd to the letters in the firſt part of the fourth Eqation, produces . . . . $6e+6y+6u=600-6a$
6. Again, the third Equation multiplied by 10, that is, the greateſt of the known numbers which are prefix'd to the letters in the firſt part of the fourth Equation produces . . . . $10e+10y+10u=1000-10a$
7. It is manifeſt that the firſt part of the fifth Equation is leſs than the firſt part of the fourth, therefore alſo the later part of the fifth ſhall be leſs than the later part of the fourth, *viz* . . $600-6a \sqsupset 1200-16a$
8. Therefore from the ſeventh ſtep, after due Reduction, it follows, that . . . . $a \sqsupset 60$
9. Again, for as much as the firſt part of the ſixth Equation is greater than the firſt part of the 4*th*, therefore alſo the later part of the ſixth ſhall be greater than the later part of the fourth, *viz.* . $1000-10a \sqsubset 1200-16a$
10. Therefore from the ninth ſtep, after due Reduction, it follows, that . . . . $a \sqsubset 35\frac{1}{3}$

Now ſince it is found by the eighth and tenth ſteps, that a the number of Quarts ſought of the firſt ſort of Wine to make the Mixture muſt be leſs than 60, but greater than $33\frac{1}{3}$, let ſome number within thoſe limits be taken for the value of a, viz.

11. Suppoſe . . . . $47=a$
12. Then by ſetting 47 in the place of *a* in the firſt Equation, this ariſes . . . . . $47+e+y+u=100$
13. Whence by equal ſubtraction of 47 there remains . . . . $e+y+u=53$
14. And by multiplying the Equation in the eleventh ſtep by 16, (the number prefix'd to *a* in the ſecond,) it gives . . . . . $752=16a$
15. Then by ſetting 752 in the place of $16a$ in the ſecond Equation, this ariſes . . $752+10e+8y+6u=1200$
16. And by ſubtracting 752 from each part of the Equation in the fifteenth ſtep, this remains, *viz.* . . . . $10e+8y+6u=448$
17. The Equation in the thirteenth ſtep multiplied by 10, (which is prefix'd to *e* in the ſixteenth,) produces . . . . $10e+10y+10u=530$
18. Then by ſubtracting the Equation in the ſixteenth ſtep from that in the ſeventeenth, the letter *e* vaniſhes, and this Equation remains, *viz.* . . . . $2y+4u=82$
19. From the eighteenth ſtep by tranſpoſition of $+4u$, this Equation ariſes, ... $2y=82-4u$
20. And by dividing each part of the Equation in the nineteenth ſtep by 2, it gives . . $y=41-2u$
21. Then by ſeting the latter part of the Equation in the twentieth ſtep in the place of *y* in the thirteenth ſtep, it makes . . . . $e+41-2u+u=53$
22. Whence, after due Reduction, ... $e=u+12$
23. By the later part of the Equation in the twentieth ſtep, it's evident that $2u \sqsupset 41$, therefore . . . . $u \sqsupset 20\frac{1}{2}$

And becauſe the known number 12 which follows $+u$ in the twenty ſecond ſtep, (expreſſing the value of *e*) is Affirmative, there is not any limit to ſhew above which the number *u* ought to be taken; and therefore, according to the three and twentieth ſtep, *u* may be any number leſs than $20\frac{1}{2}$: Therefore,

24. Suppoſe . . . . $u=20$
25. Then from the twentieth and twenty fourth ſteps it follows, that . . . . $y=1$, $(=41-2u)$
26. And from the twenty ſecond and twenty fourth ſteps, ... $e=32$, $(=u+12)$

Thus by the eleventh, twenty ſixth, twenty fifth and twenty fourth ſteps; four whole numbers are diſcovered, to wit, 47, 32, 1 and 20 for the values of *a*, *e*, *y*, and *u*, which numbers will ſolve the Queſtion. For if 42 Quarts of the firſt ſort of Wine, 37 Quarts of the ſecond, 1 quart of the third, and 20 of the fourth be mixed together, the ſum makes 100 quarts, which at 12 pence *per* quart yields 1200 pence; and the ſame number of pence will be produced by ſelling 47 quarts at 16 pence *per* Quart, 32 quarts at 10 pence, 1 quart at 8 pence, and 20 quarts at 6 pence; which was required.

But becauſe (by the twenty third ſtep) *u* may be any whole number leſs than $20\frac{1}{2}$, nineteen Anſwers more in whole numbers may be found out by repeating the Proceſs in the twenty fourth, twenty fifth and twenty ſixth ſteps; ſo that 47 being taking for *a*, there will be twenty Anſwers in whole numbers, which are inſerted in the following Table. And by putting *a* equal to every whole number ſeverally between $33\frac{1}{3}$ and 60, which are the limits diſcovered in the eighth and tenth ſteps, for the chuſing of the number *a*, after a due repetition of the Proceſs with every one of thoſe whole numbers, in like manner as before with 47 from the eleventh ſtep to the end of the Reſolution, two hundred ninety four Anſwers more in whole numbers will be diſcovered, which with thoſe twenty in the Table make three hundred and fourteen Anſwers in whole numbers to this twentieth

Queſtion, to which the Rule of *Alligation* in Vulgar Arithmetic gives only one Anſwer, which conſiſts partly of Fractions too; but by the Method above deliver'd, innumerable Anſwers may be found out in Fractions. The Table follows.

a	*e*	*y*	*u*
47	32	1	20
47	31	3	19
47	30	5	18
47	29	7	17
47	28	9	16
47	27	11	15
47	26	13	14
47	25	15	13
47	24	17	12
47	23	19	11
47	22	21	10
47	21	23	9
47	20	25	8
47	19	27	7
47	18	29	6
47	17	31	5
47	16	33	4
47	15	35	3
47	14	37	2
47	13	39	1

QUEST. 21.

Forty-one perſons conſiſting of Men, Women and Children, ſpent in the whole at a Feaſt 40 Shillings; whereof every Man paid 4 Shillings, every Woman 3 Shillings, and every Child 4 pence, or $\frac{1}{3}$ of a Shilling: It's deſired to find the number of Men, likewiſe of the Women and Children.

The Nature of this Queſtion not admitting Fractions in the Anſwer, the ſcope of the Reſolution muſt be to divide 41 into three ſuch whole Numbers, that if the firſt be multiplied by 4, the ſecond by three, and the third by $\frac{1}{3}$, the Sum of the three Products may make 40: To which purpoſe, let *a*, *e* and *y* be put for the deſired numbers of Men, Women and Children, and then the Queſtion may be ſtated thus, *viz.*

1. If . . . . $a + e + y = 41$
2. And . . . . $4a + 3e + \frac{1}{3}y = 40$

What are the whole numbers *a*, *e*, *y*?

RESOLUTION.

3. By forming the Reſolution in like manner as in the foregoing thirteenth, fourteenth and fifteenth Queſtions it will appear, that . . . .

$$\begin{cases} y \sqsubset 31\frac{1}{8} \\ y \sqsupset 33\frac{9}{11}, \\ e = 124 - \frac{11y}{3}, \\ a = \frac{8y}{3} - 83. \end{cases}$$

Whence 'tis manifeſt that 32 and 33 are the only whole Numbers within the Limits for the chuſing of the Number *y*, but this muſt neceſſarily be a Multiple of 3, otherwiſe $\frac{11y}{3}$ and $\frac{8y}{3}$ will not be whole Numbers, and conſequently the values of *e* and *a* above expreſs'd cannot be whole Numbers; therefore 33 is the ſole whole Number that can be taken for the value of *y*, to wit, the number of Children, and conſequently the values of *e* and *a* above expreſs'd will give 3 for the number of Women, and 5 for the number of Men: which three numbers 5, 3 and 33 will ſolve the Queſtion, for their ſum is 41; and if the firſt be multiplied by 4, the ſecond by 3, and the third by $\frac{1}{3}$, the ſum of the three Products is 40, as was required.

QUEST. 22.

Twenty perſons, conſiſting of Men, Women, Boys and Girls ſpent at a Feaſt in the whole 94 Shillings; whereof every Man paid 6 Shillings, every Woman 4 Shillings, every Boy 3 Shillings, and every Girl 1 Shilling: It's deſired to find out the number of Men, likewiſe of Women, Boys and Girls.

The ſcope of this Queſtion is to find out four ſuch whole numbers that their ſum may make 20, and that if the firſt be multiplied by 6, the ſecond by 4, the third by 3, and the fourth by 1, the ſum of the four Products may make 94; therefore by putting *a*, *e*, *y*, *u*, to repreſent thoſe four whole numbers, the Queſtions may be ſtated thus;

1. If . . . . $a+e+y+u = 20$
2. And . . . . $6a+4e+3y+u = 94$

What are the whole Numbers *a*, *e*, *y*, *u*?

RESOLUTION.

The firſt Scope is to ſearch out Limits for the Number *a* in like manner as before in the twentieth Queſtion, *viz.*

3. By tranſpoſition of *a* in the firſt Equation, this ariſes, $e+y+u = 20-a$
4. Likewiſe by tranſpoſition of $6a$ in the ſecond Equation, there comes forth . . . . $4e+3y+u = 94-6a$
5. The third Equation multiplied by 1, (to wit, the ſmalleſt of the Numbers prefix'd to the Letters in the firſt part of the fourth Equation, where 1 is ſuppoſed to be prefix'd to *u*,) does produce the ſame third, *viz.* $e+y+u = 20-a$
6. Again, the third Equation multiplied by 4, to wit, the greateſt of the Numbers prefix'd to the Letters in the firſt part of the fourth Equation, does produce . $4e+4y+4u = 80-4a$
7. It is manifeſt that the firſt part of the fifth Equation is leſs than the firſt part of the fourth, therefore alſo the later part of the fifth ſhall be leſs than the later part of the fourth, *viz.* . . . . $20-a \sqsupset 94-6a$
8. Therefore from the ſeventh ſtep, after due Reduction, it follows that . . . . $a \sqsupset 14\frac{4}{5}$
9. Again, foraſmuch as the firſt part of the ſixth Equation is greater than the firſt part of the fourth, therefore alſo the later part of the ſixth ſhall be greater than the later part of the fourth, *viz.* . . . . . . $80-4a \sqsubset 94-6a$
10. Therefore from the ninth ſtep, after due Reduction, it follows, that . . . . $a \sqsubset 7$

Now ſince 'tis found by the tenth and eigth ſteps, that *a*, (or the number of Men) is greater than 7, but leſs than $14\frac{4}{5}$, let ſome whole number within thoſe Limits be taken for the value of *a*, *viz.*

11. Suppoſe . . . . $12 = a$
12. Then by ſetting 12 in the place of *a* in the firſt Equation, this ariſes, ... $12+e+y+u = 20$
13. Whence by equal ſubtraction of 12, there remains . . $e+y+u = 8$
14. And by multiplying the Equation in the eleventh ſtep by 6, it makes . . . . $72 = 6a$
15. Then by ſetting 72 in the place of $6a$ in the ſecond Equation, it gives . . . . $72+4e+3y+u = 94$
16. And by ſubtracting 72 from each part of the laſt Equation, the Remainder is . . . . $4e+3y+u = 22$
17. The Equation in the thirteenth ſtep being multiplied by 4, (which is prefix'd to *e* in the ſixteenth) gives . . $4e+4y+4u = 32$
18. Then by ſubtracting the Equation in the ſixteenth ſtep from that in the ſeventeenth, the Letter *e* vaniſhes, and this Equation remains, ... $y+3u = 10$

19. Whence by tranſpoſition of $3u$, this Equation ariſes, ... $y = 10 - 3u$

20 Then by ſetting the later part of the Equation in the nineteenth ſtep in the place of y in the thirteenth, this ariſes, ... $e + 10 - 3u + u = 8$

21. Whence, after due Reduction, this Equation ariſes, $e = 2u - 2$

22. From the later part of the nineteenth Equation, it may be infer'd that . . . . $u \sqsupset 3\frac{1}{3}$

23. And from the later part of the twenty firſt Equation, $u \sqsubset 1$

Now ſince by the twenty ſecond and twenty third ſteps, u (or the number of Girls) is found to fall between 1 and $3\frac{1}{3}$ let 2 be taken for the value of u, *viz.*

24. Suppoſe . . . . $u = 2$

25. Then from the nineteenth and twenty-fourth ſteps, $y = 4 \ (= 10 - 3u)$

26. And from the twenty firſt and twenty fourth ſteps, $e = 2 \ (= 2u - 2)$

Thus by the eleventh, twenty ſixth, twenty fifth and twenty fourth ſteps, four whole numbers are diſcovered, to wit, 12, 2, 4 and 2, for the values of a, e, y and u.

Again, by taking 3 for the value of u, (which is within the Limits before diſcovered) the nineteenth and twenty firſt ſteps will diſcover 1 and 4 for the values of y and e, (a being 12, as before. Wherefore two Anſwers to the Queſtion are found out; for the number of Men being put 12, the number of Women will be 2, the number of Boys 4, and the number of Girls 2; or the number of Men being 12 as before, there will be four Women, 1 Boy and 3 Girls. Again, if 11 be put equal to a, (or the number of Men,) and the proceſs be repeated from the eleventh ſtep to the end of the Reſolution, there will be found two Anſwers more in whole numbers. In like manner, if 9, 10 and 13 be ſeverally be put equal to a, three Anſwers more will be diſcovered; But if 8 and 14 be ſeverally put equal to a, altho they be within the Limits in the eighth and tenth ſteps, yet the work being repeated as before will not ſucceed to find e, y and u in whole numbers; ſo that there are only ſeven Anſwers, to wit, thoſe inſerted in the Table; but that every one of them will ſolve the Queſtion may eaſily be proved.

a	e	y	u
9	9	1	1
10	6	3	1
11	5	2	2
11	3	5	1
12	2	4	2
12	4	1	3
13	1	3	3

If a Queſtion of this nature be deſired that has but one Anſwer in whole numbers, let the number of perſons be 60, and 100 the number of Shillings ſpent; alſo let every Man ſpend 2 Shillings, every Woman $\frac{2}{3}$ of a Shilling, every Boy $\frac{3}{5}$ of a Shilling, and every Girl $\frac{1}{2}$ of a Shilling; then by forming the Reſolution as before, the number of Men will be found 46, the number of Women 3, the number of Boys 5, and the number of Girls 6.

QUEST. 23.

To divide 200 into five ſuch whole numbers, that if the firſt be multiplied by 12, the ſecond by 3, the third by 1, the fourth by $\frac{1}{2}$, and the fifth by $\frac{1}{3}$, the Sum of the Products may alſo make 200.

This Queſtion may be reſolved like the foregoing twentieth and twenty ſecond, but I ſhall leave it as an exerciſe to the induſtrious Analyſt, who (if he thinks it to be worth his pains,) may find out 6639 Anſwers to it in whole numbers, (as Monſieur *Bachet*, in the two laſt Pages of his little Book before cited in *Sect.* 1. of this Chapter, does affirm.

Nicholas Tartaglia handling this Queſtion, (which is the laſt of the ſeventeenth Book of the Firſt Part of his Arithmetic,) thought it a great matter that he had found out one ſingle Anſwer to it in theſe five whole numbers, to wit, 6, 12, 34, 52, 96, and aſſerted, That Queſtions of this ſort could not be perfectly ſolved, either by the *Algebraical Art*, or any certain *Rule*; but the Contents of this Chapter do manifeſtly ſhew, that the Imperfection was in the Artiſt, and not in the Art.

The End of the Firſt Volume.

u 67/11642